MATH 530 Exam 1 Solutions

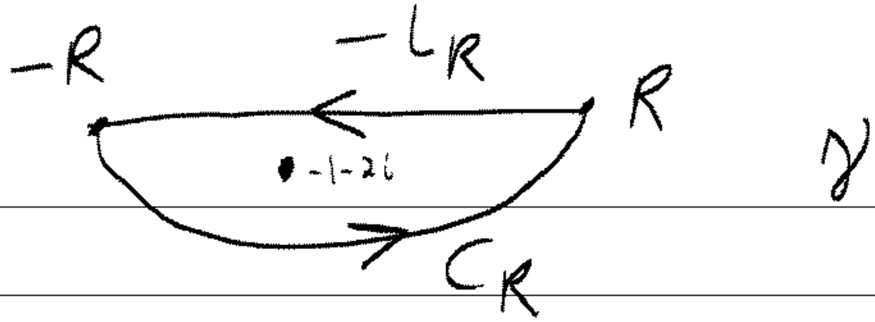
1. $$f(z) = a_1(z-a) + a_2(z-a)^2 + \cdots$$
$$= (z-a)\underbrace{\left[a_1 + a_2(z-a) + \cdots\right]}_{H(z)}$$

Note that $a_n = \dfrac{f^{(n)}(a)}{n!} = \dfrac{H^{(n-1)}(a)}{(n-1)!}$.

Now $$\frac{1}{f(z)^2} = \frac{1}{(z-a)^2} \cdot \frac{1}{H(z)^2}$$
$$= \frac{1}{(z-a)^2}\left[A_0 + A_1(z-a) + \cdots\right]$$
$$= \frac{A_0}{(z-a)^2} + \frac{A_1}{(z-a)} + (\text{Analytic})$$

$$A_1 = \operatorname{Res}_a \frac{1}{f^2} = \frac{\frac{d}{dz}\left[\frac{1}{H(z)^2}\right]\Big|_{z=a}}{1!}$$
$$= -2H(a)^{-3}H'(a) = -2a_1^{-3}a_2$$
$$= -2f'(a)^{-3}\frac{f''(a)}{2!} = -\frac{f''(a)}{f'(a)^3}$$

2.



$$e^{-is(x+iy)} = e^{-isx}e^{sy}$$

$|e^{-isz}| = e^{s\,\mathrm{Im}\,z} \le 1$ on C_R. And

$$|z^2+2z+5| \ge c|z|^2 \quad \text{if } |z| > R_0$$

for some $c>0$ and $R_0>0$.

So $$\left|\int_{C_R} \frac{e^{-isz}}{z^2+2z+5}\,dz\right| \le \left(\operatorname{Max}_{C_R}\left|\frac{e^{-isz}}{z^2+2z+5}\right|\right)\pi R$$

$$\le \frac{1}{cR^2}(\pi R) \quad \text{if } R>R_0 \text{ and}$$

this tends to zero as $R\to\infty$.

$z^2+2z+5 = (z+1)^2+2^2$ has zeroes at $-1\pm 2i$. So $f(z) = \dfrac{e^{-isz}}{z^2+2z+5}$ has only one pole inside γ at $-1-2i$ with residue

$$\frac{e^{-is(-1-2i)}}{2(-1-2i)+2} = \frac{e^{(-2+i)s}}{-4i}.$$ Residue Thm

$$\Rightarrow \int_{-L_R} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i \ \mathrm{Res}_{-1-2i} f$$

$$-\int_{L_R} f\,dz + \int_{C_R} f\,dz = 2\pi i \left(\frac{e^{(-2+i)s}}{-4i} \right)$$

$$\downarrow \qquad\qquad \downarrow$$

$$-\int_{-\infty}^{\infty} \frac{e^{-ist}}{t^2+2t+5}\,dt + 0 = -\frac{\pi e^{(-2+i)s}}{2}$$

3. $zf(z)$ tends to zero as $z \to 0$.

So $F(z) = \begin{cases} zf(z) & z \neq 0 \\ 0 & z = 0 \end{cases}$ is analytic

on $D_r(0)$ by the Riemann Removable Sing. Thm

Since $F(z)$ has a zero at $z=0$, we may write $F(z) = z^N H(z)$ where H is analytic on $D_r(0)$ and $H(0) \neq 0$ and $N \geq 1$. Now we see that $f(z) = z^{N-1} H(z)$ if $z \neq 0$ and it follows that zero is removable.

4. Step 1: f has finitely many zeroes.

Indeed, $c|z|^N \leq |f(z)|$ for $|z| > R$ implies that f has no zeroes outside $\overline{D_R(0)}$. If f had infinitely many zeroes on the compact set $\overline{D_R(0)}$, they must have a limit point in $\overline{D_R(0)}$. This would imply that $f \equiv 0$. $\lightning$.

Step 2: Let $\{R_j\}_{j=1}^N$ be the principal parts of $\frac{1}{f(z)}$ at the N zeroes of f.

$\frac{1}{f(z)} - \sum_1^N R_j$ has removable singularities at the zeroes of f, so it is entire. The inequality for f shows that $\lim_{z \to \infty} \frac{1}{f} = 0$.

The same is true for each R_j. So

$\frac{1}{f} - \sum^N R_j$ is a bounded entire function.

Liouville's $\Rightarrow$ it is constant. Hence f is a rational function. The only rational functions with no singular points in $\mathbb{C}$ are polynomials. If f is a poly of deg M, then $c|z|^N \leq |f(z)| \leq B|z|^M$ for large $|z|$, and this reveals that $M \geq N$.