

Solutions to Exam 2 MA530

1. Pick $z_0 \in \Omega$ and suppose $f(z_0) = n_0$.
 $\exists r > 0$ such that $D_r(z_0) \subset \Omega$. Let $\varepsilon = 1/2$. f continuous $\Rightarrow \exists \delta$ with $0 < \delta < r$ such that $|f(z) - f(z_0)| < 1/2$ if $|z - z_0| < \delta$.
Hence $f(z) \equiv n_0$ for $z \in D_\delta(z_0)$. Let $U_1 = \{z \in \Omega; f(z) = n_0\}$ and let $U_2 = \Omega - U_1$.
The argument above shows that U_1 is open. It also shows that U_2 is open. Ω connected $\Rightarrow U_1 = \emptyset$ or $U_1 = \Omega$. But $z_0 \in U_1$. So $U_1 = \Omega$. \checkmark

2. Define $\log z = \ln|z| + i\theta$ where $\theta = \arg z$ with $-\pi/2 < \theta < 3\pi/2$.
 $z^\alpha = e^{\alpha \log z}$
 $|z^\alpha| = |e^{\alpha(\ln|z| + i\theta)}| = e^{\alpha \ln|z|} = |z|^\alpha$.

$$\left| \int_{C_R} \frac{z^\alpha}{z^2+4} dz \right| \leq \frac{R^\alpha}{R^2-4} \pi R \rightarrow 0$$

as $R \rightarrow \infty$ since $\alpha < 1$.

$$\left| \int_{C_\varepsilon} \frac{z^\alpha}{z^2+4} dz \right| \leq \frac{\varepsilon^\alpha}{4-\varepsilon^2} \pi \varepsilon \rightarrow 0$$

as $\epsilon \rightarrow 0$,

2

$$\int_{\text{RIGHT}} f(z) dz = \int_{\epsilon}^R \frac{t^{\alpha}}{t^2+4} dt$$

$$\int_{\text{LEFT}} f(z) dz = - \int_{\epsilon}^R \frac{(-t)^{\alpha}}{(-t)^2+4} (-1) dt$$

$$= \int_{\epsilon}^R \frac{e^{\alpha(\ln|-t| + i\pi)} dt}{t^2+4}$$

$$= e^{i\alpha\pi} \int_{\epsilon}^R \frac{t^{\alpha}}{t^2+4} dt$$

$$\int_{\gamma} f(z) dz = 2\pi i \operatorname{Res}_{2i} \frac{z^{\alpha}}{z^2+4} = 2\pi i \frac{(2i)^{\alpha}}{2 \cdot (2i)}$$
$$= 2\pi i \frac{e^{\alpha(\ln 2 + i\pi/2)}}{4i} = \frac{\pi}{2} 2^{\alpha} e^{i\alpha\pi/2}$$

$$= \left(\int_{C_C} + \int_{C_R} + \int_{\text{LEFT}} + \int_{\text{RIGHT}} f(z) dz \right)$$

$$0 + 0 + (e^{i\alpha\pi} + 1) I,$$

3

$$\text{So } I = \frac{\pi e^{i\alpha\pi/2}}{2^{1-\alpha} (1 + e^{i\alpha\pi})}.$$

3. Let $\{R_n(z)\}_{n=1}^N$ denote the principal parts of $f(z)$ at its poles $\{a_n\}_{n=1}^N$. Let $H(z) = f(z) - \sum_{n=1}^N R_n(z)$. Since the a_n are removable singularities of H , we may think of H as an entire function. Since $\lim_{z \rightarrow \infty} R_n(z) = 0$ for each n , it

follows that, if f is bounded on $\{z : |z| > R\}$ for some R , then H is a bounded entire function. Liouville's $\Rightarrow H \equiv c$, a constant. So $f(z) \equiv c + \sum_{n=1}^N R_n(z)$ is rational.

4. $u_x - iu_y$ is analytic on Ω and

$u_x - iu_y = v_y + i v_x$ on a set with a limit point in Ω . But v harmonic

$\Rightarrow V_y + \bar{\partial} V_x$ is analytic. 4

$$\begin{cases} \frac{\partial}{\partial x}(V_y) \stackrel{?}{=} \frac{\partial}{\partial y}(V_x) \quad \checkmark & \text{mixed partials} \\ \frac{\partial}{\partial y}(V_y) \stackrel{?}{=} -\frac{\partial}{\partial x}(V_x) \quad \checkmark & \text{harmonic} \end{cases}$$

Hence $u_x - \bar{\partial} u_y \equiv V_y + \bar{\partial} V_x$ on Ω by the identity theorem. So CR eqns hold on Ω and $u + \bar{\partial} v$ is analytic there.

5. Suppose f analytic on Ω and $|f|$ has a local max at $z_0 \in \Omega$. Let $w_0 = f(z_0)$. $\exists r > 0$ such that $D_r(z_0) \subset \Omega$ and $|f(z)| \leq |f(z_0)|$ for $z \in D_r(z_0)$. If f is not constant on Ω , then the Open Mapping Theorem implies that $f(D_r(z_0))$ contains a disc about $f(z_0)$. But there are points in such a disc with modulus $> |w_0|$. \downarrow Hence f must be constant on Ω and we see that the OMT \Rightarrow MP.