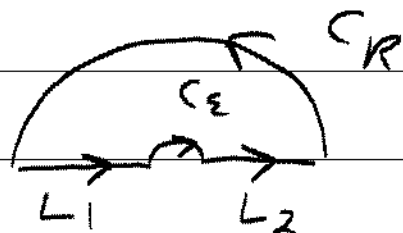


# Solutions to Exam 2

① Let  $\log z = \ln |z| + i\theta$  where  $z = |z|e^{i\theta}$  with  $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$  (i.e., with a branch cut along the negative imaginary axis). Write  $\sqrt{z} = e^{\frac{1}{2}\log z}$ . Notice that

$$|\sqrt{z}| = \sqrt{|z|}$$

$$\text{Let } f(z) = \frac{1}{\sqrt{z}(z^2+4)}$$



$$I_\epsilon^R = \int_\epsilon^R \frac{1}{\sqrt{t}(t^2+4)} dt = \int_{L_2} f(z) dz$$

$$-L_1: z(t) = -t; \quad \epsilon \leq t \leq R$$

$$\int_{L_1} f(z) dz = - \int_{-L_1} f(z) dz = - \int_\epsilon^R \frac{1}{\sqrt{-t}((1-t)^2+4)} (-dt)$$

$$= \frac{1}{i} \int_\epsilon^R \frac{1}{\sqrt{t}(t^2+4)} dt = -i I_\epsilon^R$$

$$\left| \int_{C_R} f(z) dz \right| \leq \left( \frac{1}{\sqrt{R}(R^2-4)} \right) \pi R \rightarrow 0 \text{ as } R \rightarrow \infty$$

$$\left| \int_{C_\epsilon} f(z) dz \right| \leq \frac{1}{\sqrt{\epsilon}(4-\epsilon^2)} \pi \epsilon \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

Now the Residue Theorem yields

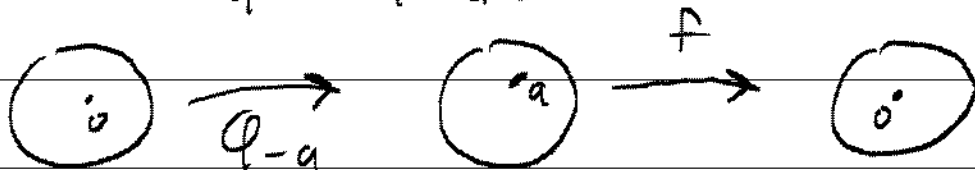
$$\int_{C_\epsilon} f dz + \int_{C_R} f dz + (1-i) I_\epsilon^R = 2\pi i \operatorname{Res}_{2i} f$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$0 + 0 + (1-i) I = 2\pi i \left( \frac{1/\sqrt{2i}}{2(2i)} \right).$$

$$\text{So } I = \frac{1}{(1-i)} 2\pi i \frac{1/[\sqrt{2} e^{i\pi/4}]}{2(2i)}.$$

2. Let  $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$ . Note that  $\varphi_a$  maps  $D_1(0)$  1-1 onto itself and  $(\varphi_a)^{-1} = \varphi_{-a} = \frac{z+a}{1+\bar{a}z}$ .



$f(\varphi_{-a}(z))$  satisfies the hypotheses of the Schwarz Lemma. So  $|f(\varphi_{-a}(z))| \leq |z|$ .

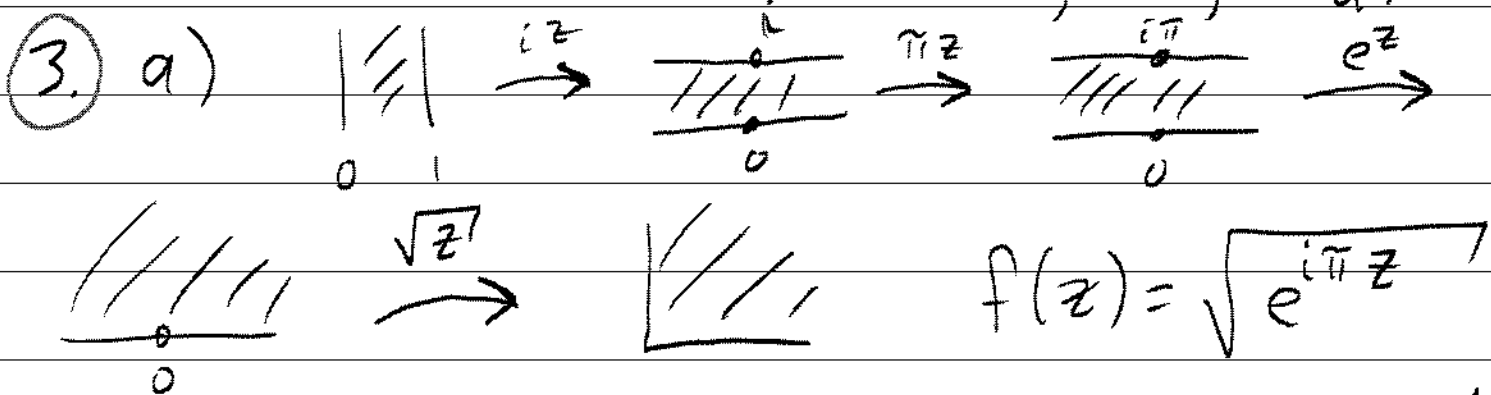
Let  $w = \varphi_{-a}(z)$ . Then  $z = \varphi_a(w)$  and the desired inequality is obtained.

Furthermore  $|f'(\varphi_{-a}(0))| |\varphi_{-a}'(0)| \leq 1$ .

Hence  $|f'(a)| \leq \frac{1}{1-|a|^2}$  because

Note:  $\varphi_{-a}'(z) = \frac{1 \cdot (1 + \bar{a}z) - \bar{a}(z+a)}{(1 + \bar{a}z)^2} = \frac{1 - |a|^2}{(1 + \bar{a}z)^2}$

We know the sup is attained by the map in the class that is 1-1 onto, i.e.,  $\varphi_a$ .

3) a) 

where  $\sqrt{z}$  is defined as in problem 1.

b)  $L(-1) = 0$                        $L(0) = -1$   
 $L(1) = \infty$                           $L(\infty) = 1$

$L$  maps line  $l_1$  containing  $\{-1, 0, 1\}$  to the line  $l_2$  containing  $\{0, -1, \infty\}$ .

Since  $L: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  one-to-one onto, we conclude that  $L$  maps  $(-1, 1)$  one-to-one onto  $(-\infty, 0)$ .  
 So  $\Omega = (\mathbb{C} - \{1\}) - (-\infty, 0]$ , which is not simply connected.

4. Use Rouché's.

$$|P(z)| = \left| \underbrace{z^n}_{f(z)} - \underbrace{(-a_{n-1}z^{n-1} - \dots - a_0)}_{g(z)} \right|$$

$$\angle 1 = \underbrace{|z^n|}_{f(z)} \quad \text{on } \{|z|=1\}.$$

Hence  $f$  and  $g$  have same # zeroes,  $n$ .  
 But if an  $n-1$  degree polynomial has at most  $n-1$  zeroes if it is not  $\equiv 0$ . This contradiction implies that there is no such  $g$ , and hence, no such  $P$ .

5. Suppose a harmonic function  $u$  has a local max at a point  $z_0$  in a domain  $\Omega$ . There is an  $r > 0$  such that  $D_r(z_0) \subset \Omega$ . We know that  $u = \operatorname{Re} f$  on  $D_r(z_0)$  for some analytic function  $f$  on  $D_r(z_0)$ . Now  $|e^{f(z)}| = e^{u(z)}$  reveals that  $e^f$  has a local max at  $z_0$ . Max Princ  $\Rightarrow e^f \equiv \text{const}$  on  $D_r(z_0) \Rightarrow f \text{ const on } D_r(z_0) \Rightarrow u = \text{constant on } D_r(z_0)$ . Say  $u \equiv c$  there. Let  $U_1 = \{z \in \Omega : \exists \varepsilon > 0 \text{ such that } D_\varepsilon(z) \subset \Omega \text{ and } u \equiv c \text{ on } D_\varepsilon(z)\}$ .  $U_1$  is open and  $\neq \emptyset$ .  $\Omega - U_1$  is open too, since if not, there would be a point  $z_0 \in \Omega - U_1$ .

and a sequence of discs  $D_{\varepsilon_j}(z_j) \subset \Omega$  with  $z_j \neq z_0$  and  $z_j \rightarrow z_0$ ,  $u \equiv c$  on  $D_{\varepsilon_j}(z_j)$ . If  $D_r(z_0) \subset \Omega$  and  $u = \operatorname{Re} f$  on  $D_r(z_0)$ , then  $f$  would have constant real part on an open subset of  $D_r(z_0)$ , and would therefore be constant. So  $u$  is constant on  $D_r(z_0)$ . The constant must be  $c$  since  $u(z_j) = c$  and  $z_j \rightarrow z_0$ . So  $z_0 \in U_1$ , contrary to assumption. Hence  $\Omega - U_1$  is open too. Connectivity  $\Rightarrow U_1 = \Omega$ .