Math 530

Exam

1. (20 pts) Assume $\epsilon > 0$. Compute

$$\int_\gamma \frac{1}{z}\, dz$$

where γ is any curve in the plane that starts at $1-\epsilon i$ and ends at $1+\epsilon i$ and that avoids zero and the positive real axis.

2. (20 pts) Suppose that u is a real valued continuous function on a domain Ω. Prove that if u has no zeroes on Ω, then u is either always positive or always negative. Prove that a non-vanishing real valued $\mathcal{C}^2$-smooth harmonic function on the whole complex plane must be constant.

3. (30 pts) In this problem, you will compute the real integral $I = \displaystyle\int_0^\infty \frac{\sqrt{x}}{x^5+1}\, dx$ by integrating the complex valued function $f(z) = \dfrac{e^{\frac{1}{2}\operatorname{Log} z}}{z^5+1}$ (where Log denotes the principal branch of the complex log function) around the closed contour γ that follows the real axis from the origin to $R > 0$, then follows the circular arc $Re^{i\theta}$ as θ ranges from zero to $2\pi/5$, then returns to the origin via the line segment joining $Re^{i2\pi/5}$ to the origin, and finally letting $R \to \infty$.

a) Express the integral of $f(z)$ along the part of γ that follows the line from $Re^{i2\pi/5}$ to the origin in terms of real integrals.
b) Show that the integral of $f(z)$ along the circular part of the boundary of γ tends to zero as $R \to \infty$.
c) Compute the residue of f at the point $e^{i\pi/5}$.
d) Use the Residue Theorem and let $R \to \infty$ to get a formula for I. (Do not bother to simplify.)

4. (20 pts) Prove that if the product of finitely many analytic functions on a domain is the zero function, then at least one of the functions must be the zero function.

5. (10 pts) The first part of the Schwarz lemma says that, if f is an analytic function mapping the unit disc into itself such that $f(0) = 0$, then $|f(z)| \le |z|$ for $z \in D_1(0)$ and $|f'(0)| \le 1$. What does the second part of the lemma say?