Math 530
Exam 2

Each problem is worth 25 points

1. Given two distinct points \( a \) and \( b \) in the unit disc, show that there is an automorphism of the unit disc that maps \( a \) to \( b \). We say that the automorphism group of the unit disc is transitive. Show that the automorphism group of the unit square in \( \mathbb{C} \) is transitive.

2. Suppose a sequence of analytic functions \( f_n \) on a disc \( D_R(a) \) converges uniformly on compact subsets to a function \( f(z) \) that has a zero of multiplicity \( m \) at \( a \) and no other zeroes in \( D_R(a) \). Show that, given an \( \epsilon \) with \( 0 < \epsilon < R \), there is an \( N \) such that each \( f_n \) has exactly \( m \) zeroes in \( D_\epsilon(a) \) (counted with multiplicity) if \( n > N \).

3. Carefully state the whole Schwarz Lemma. Prove the first part of the lemma, i.e., the inequality part. (Don’t prove the part about the consequences of equality.)

4. Compute

\[
\int_0^\infty \frac{x^2}{x^7 + 1} \, dx
\]

by integrating \( f(z) = z^2/(z^7 + 1) \) around the contour that follows the real line from zero to \( R \), then follows the circle \( Re^{it} \) from \( t = 0 \) to \( t = 2\pi/7 \), and then follows the line \( te^{2\pi i/7} \) from \( t = R \) back to \( t = 0 \). Use the Residue Theorem and let \( R \to \infty \).