Math 530

Final Exam

Each problem is worth 20 points.

1. Show that the Maximum Principle implies the Fundamental Theorem of Algebra.

2. Suppose f is a function defined on the unit disc. Show that if $f(z)$ and $f(\bar{z})$ are both analytic functions of z, then f must be constant.

3. What are the possible values of

$$\int_\gamma \frac{1}{z}\,dz,$$

where γ is a path that starts at $z = -i$ and ends at $z = 2i$ and that avoids the origin? Explain.

4. Show that if f is an entire function that never takes values along the negative real axis, then f must be a constant function. (You may use any result that was covered in class *except* the Picard Theorems.)

5. First, carefully state the *Schwarz Lemma*. Next, suppose that $f(z)$ is analytic on $D_1(0)$ and that $|f(z)| < 1$ for all $z \in D_1(0)$. Prove that if $f(0) = a \neq 0$, then f has no zeroes in the disc $D_r(0)$ where $r = |a|$.

6. Calculate

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\,dx.$$

(If you say that a limit exists, prove that it does using an estimate.)

7. Prove that $\ln|z|$ does not have a harmonic conjugate on $\{z : 1 < |z| < 2\}$.

8. Suppose that f is a meromorphic functions on a simply connected domain Ω that satisfies the following properties. The zeroes of f, if any, have even multiplicity. The poles of f, if any, have even order. Prove that f has a meromorphic square root on Ω.