Math 530

Homework 2

1. Suppose that $\varphi(z)$ is a continuous function on the trace of a path γ. Prove that the function

$$f(z) = \int_\gamma \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta$$

is analytic on $\mathbb{C} - \operatorname{tr}\gamma$.

2. Suppose that a_n is a sequence of non-zero complex numbers. Show that if

$$R = \lim_{n\to\infty} |a_n|/|a_{n+1}|$$

exists, then R is equal to the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$. Find an example of a sequence of non-zero terms a_n such that this limit fails to exist, and yet $\limsup\limits_{n\to\infty} |a_n|^{1/n}$ is equal to one, and hence the associated power series has radius of convergence equal to one.

3. If $U = \sum_{n=0}^{\infty} u_n$ and $V = \sum_{n=0}^{\infty} v_n$ are given by the sum of absolutely convergent series, show that $UV = \sum_{n=0}^{\infty} p_n$ where $p_n = \sum_{k=0}^{n} u_k v_{n-k}$ and that this sum converges absolutely.

4. Use the result of the previous problem and the binomial theorem to give a proof of the formula

$$e(z+w) = e(z)\cdot e(w)$$

where $e(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

5. Suppose that an analytic function is written in polar form

$$f(re^{i\theta}) = U(r,\theta) + iV(r,\theta).$$

Derive the polar form of the Cauchy-Riemann equations,

$$rU_r = V_\theta \qquad \text{and} \qquad U_\theta = -rV_r.$$

Prove that if U and V are continuously differentiable and satisfy the polar Cauchy-Riemann equations on some polar rectangle, then $f(re^{i\theta}) = U(r,\theta) + iV(r,\theta)$ defines an analytic function there. Use this result to verify that the function $\operatorname{Log}(re^{i\theta}) = \operatorname{Ln} r + i\theta$ is analytic on $\{re^{i\theta} : r > 0, -\pi < \theta < \pi\}$.

6. We know that if γ is a curve in the complex plane parameterized by a function $z(t)$ which is a continuously differentiable function from the interval $[a,b]$ into $\mathbb{C}$ and f is analytic on an open set containing $\operatorname{tr}(\gamma)$, then $w(t) = f(z(t))$ is differentiable on $[a,b]$ and $w'(t) = f'(z(t))z'(t)$. Use this result to prove that $\int_\gamma z^n\, dz = 0$ for any *closed* path γ and integer $n \neq -1$, assuming that $\operatorname{tr}\gamma$ does not contain the origin if $n < 0$.