

Math 530
Homework 4

1. For what values of z is the series $\sum_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n$ convergent? Same question for

$$\sum_{n=0}^{\infty} \frac{z^n}{1+z^{2n}}.$$

2. If f is analytic on the unit disc and $|f(z)| \leq 1/(1-|z|)$, find the best estimate of $|f^{(n)}(0)|$ that the Cauchy Estimates will yield.
3. Show that the successive derivatives of an analytic function at a point a can never satisfy $|f^{(n)}(a)| > n!n^n$. Formulate a sharper theorem of the same kind.
4. Prove that

$$\cos(\theta + \psi) = \cos \theta \cos \psi - \sin \theta \sin \psi$$

without mentioning trigonometry or angles.

5. Suppose that f is analytic on a disk $D_\epsilon(0)$ and satisfies the differential equation $f'' = f$. Prove that f is given by $A \cosh z + B \sinh z$, where A and B are constants.
6. If $f(z) = \sum a_n z^n$, what is $\sum n^3 a_n z^n$?
7. Prove that an entire function f such that $\operatorname{Re} f(z) > 0$ for all z must be constant.
8. Prove that there is no analytic function f on the unit disk such that $f(1/n) = 2^{-n}$ for $n = 2, 3, 4, \dots$
9. Give a proof of the Fundamental Theorem of Algebra based on the Maximum Principle.