Math 530
Homework 6

1. Use the zero counting formula

\[ N = \frac{1}{2\pi i} \int_{C_r} \frac{f'(z)}{f(z)} \, dz \]

to give yet another proof of the Fundamental Theorem of Algebra. (This proof will show that a polynomial of degree \( N \) has \( N \) roots, counted with multiplicity.)

2. Prove that an isolated singularity of \( f(z) \) is removable as soon as either \( \text{Re} \ f(z) \) or \( \text{Im} \ f(z) \) is bounded above or below near the singularity.

3. Suppose that \( f_n \) is a sequence of analytic functions on a domain \( \Omega \) containing \( \{ z : |z| \leq 1 \} \) and suppose that \( f_n \) is uniformly Cauchy on the set \( \{ z : |z| = 1 \} \).
   Show that \( f_n \) converges uniformly on \( \{ z : |z| < 1 \} \) to a function \( f \) which is analytic there.

4. Show that an isolated singularity of \( f(z) \) cannot be a pole of \( \exp f(z) \).

5. Prove that if \( h \) is an analytic branch of \( f^{1/n} \), then \( h'/h = f'/(nf) \).

6. Derive the formula

\[ \frac{1}{2\pi} \int_{0}^{2\pi} \cos^{2n} \theta \, d\theta = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \]

by integrating

\[ \frac{1}{z} \left( z + \frac{1}{z} \right)^{2n} \]

around the unit circle.