Math 530

Homework 8

1. Suppose that f is a non-vanishing analytic function on the complex plane minus the origin. Let γ denote the curve given by $z(t) = e^{it}$ where $0 \le t \le 2\pi$. Suppose that

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz$$

is divisible by 3. Prove that f has an analytic cube root on $\mathbb{C} - \{0\}$.

2. Suppose that $\{a_k\}_{k=1}^N$ is a finite sequence of distinct complex numbers and that f is analytic on $\mathbb{C} - \{a_k : k = 1, 2, \ldots, N\}$. Prove that there exist constants c_j, $j = 1, 2, \ldots, N$, such that

$$f(z) - \sum_{k=1}^{N} \frac{c_k}{z - a_k}$$

has an analytic antiderivative on $\mathbb{C} - \{a_k : k = 1, 2, \ldots, N\}$.

3. How many zeroes does the polynomial

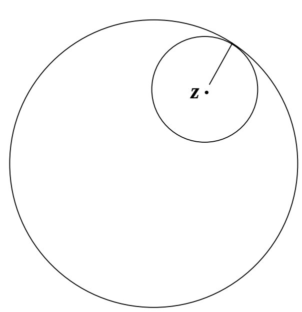
$$z^{1998} + z + 2001$$

have in the first quadrant? Explain your answer.

4. Suppose that u is continuous on $\overline{D_1(0)}$ and that

$$(*) \qquad u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + (1 - |z|)e^{i\theta})\, d\theta$$

for each $z \in D_1(0)$. (This equality means that u is only known to satisfy the averaging property on circles like the one pictured below.) Prove that u is harmonic in $D_1(0)$.



Note: $(*)$ means that $u(z)$ is equal to the average of u over the internally tangent circle centered at z.