

Math 530
Practice problems

1. Suppose that f is analytic in a neighborhood of the closed unit disc $\{z : |z| \leq 1\}$. If $|f(z)| < 1$ when $|z| = 1$, prove that f must have at least one *fixed point* in the unit disc, i.e., that there must be at least one point z with $|z| < 1$ such that $f(z) = z$.
2. How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the unit disk?
3. How many roots of the equation $z^4 - 6z + 3 = 0$ fall in the annulus $\{z : 1 < |z| < 2\}$?
4. Suppose that u is a (real valued) harmonic function on a domain. Show that u cannot have any isolated zeroes.
5. If u is harmonic on a domain Ω , what can you say about the subset of Ω where the gradient of u vanishes?
6. Find an analytic function that maps $\{z : 0 < \operatorname{Re} z < 1\}$ one-to-one onto the unit disk. Use the inverse of this map to show that there exists a bounded harmonic function on the unit disk whose harmonic conjugates are unbounded on the unit disk. (Remark: A similar idea can be used to show that there is a harmonic function which extends continuously to the closed disk that does not have a bounded harmonic conjugate on the disk.)
7. Let $C_1(0)$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and let f be a function that is analytic on $D_r(0)$ for some $r > 1$. Prove that if $f(C_1(0)) \subset C_1(0) \setminus \{1\}$, then f is a constant function.
8. Find a one-to-one conformal mapping of the first quadrant onto the unit disc.