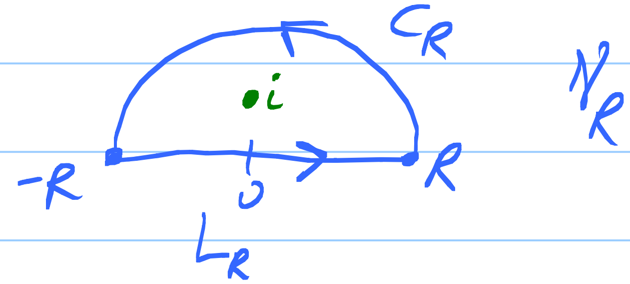
MA 530 Exam 2 Solutions

1. Let $I = \int_0^\infty \frac{\cos x}{x^2+1}\,dx$. $\frac{\cos x}{x^2+1}$ is even and $\frac{\sin x}{x^2+1}$ is odd. (I is absolutely integrable by comparison with $\int_0^\infty \frac{1}{x^2+1}\,dx = \frac{\pi}{2}$.) Let $f(z) = \frac{e^{iz}}{z^2+1}$.

Note that $\int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+1}\,dz = 2I + 0i$. Integrate $f(z)$ around



Notice that $|e^{iz}| = |e^{i(x+iy)}| = e^{-y} \le 1$ on γ_R.

So $$\left|\int_{C_R} \frac{e^{iz}}{z^2+1}\,dz\right| \le \left(\operatorname{Max}_{C_R} \left|\frac{e^{iz}}{z^2+1}\right|\right)\pi R \le \frac{\pi R}{R^2-1}$$

$\to 0$ as $R \to \infty$.

$$\int_{L_R} \frac{e^{iz}}{z^2+1}\,dz = \int_{-R}^{R} \frac{\cos x}{x^2+1}\,dx + i\underbrace{\int_{-R}^{R} \frac{\sin x}{x^2+1}\,dx}_{0}$$

$\to 2I$ as $R \to \infty$.

Finally, the Residue Theorem yields

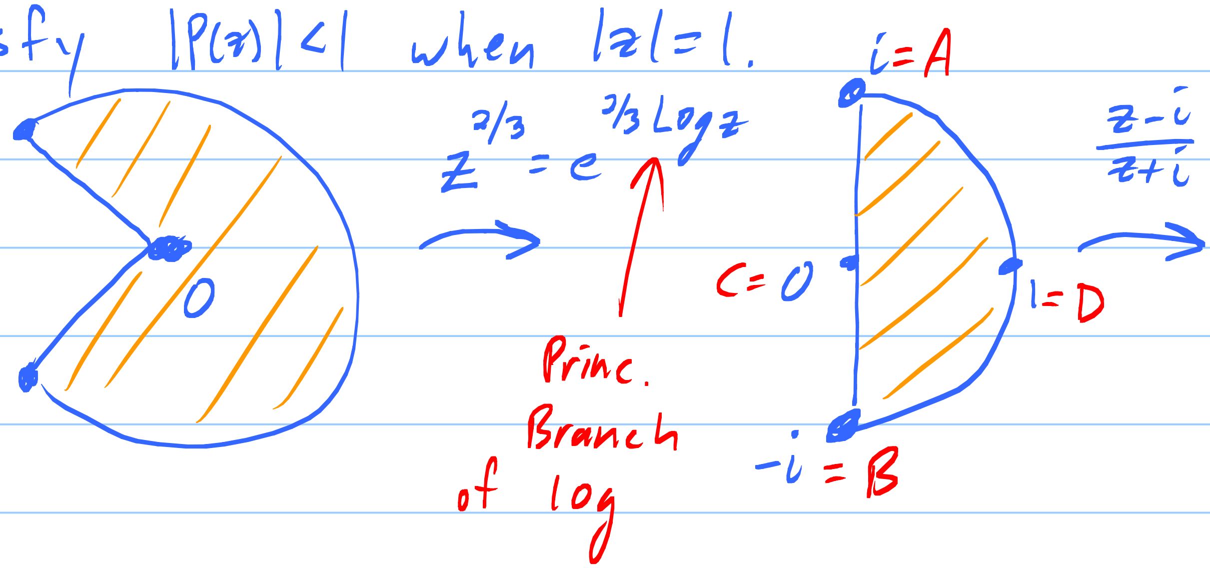
$$\left(\int_{C_R} + \int_{L_R}\right) f(z)\,dz = 2\pi i \operatorname{Res}_i \frac{e^{iz}}{z^2+1}$$

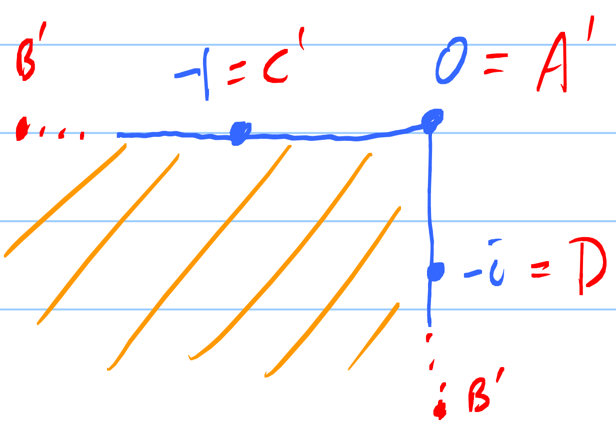
$$0 + 2I = 2\pi i \cdot \frac{e^{i(i)}}{2i} = \pi e^{-1} \text{ as } R \to \infty.$$

So $I = \dfrac{\pi}{2e}$.

2. Use Rouché's Thm. Let $f(z) = z^n$ and $g(z) = -a_{n-1}z^{n-1} - \cdots - a_1 z - a_0$. If $|P(z)| < 1$ when $|z| = 1$, then $|f(z) - g(z)| = |P(z)| < 1 = |z^n| = |f(z)|$ when $|z| = 1$. Hence f and g have the same number of zeroes in $D_1(0)$. f has n zeroes. But g is a polyomial of degree $n-1$ or less. Hence g is either $\equiv 0$ or g has $n-1$ zeroes or less. This is a contradiction. So P cannot satisfy $|P(z)| < 1$ when $|z| = 1$.

3.



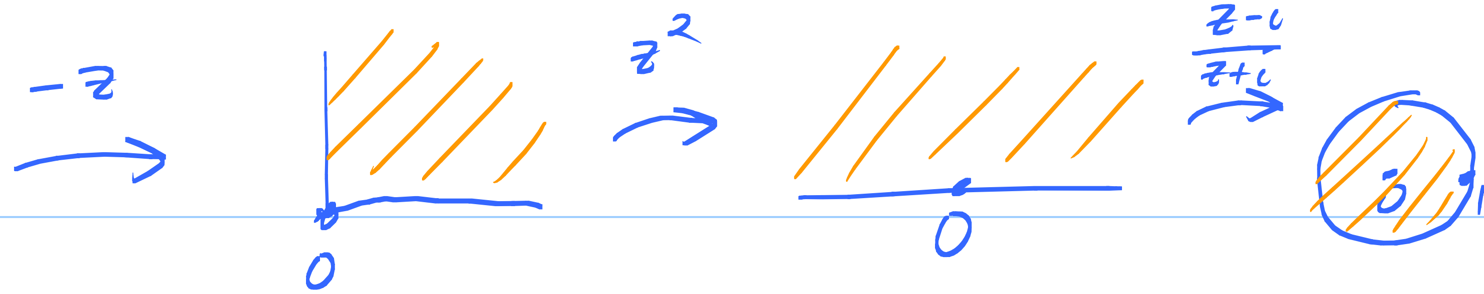


$$\frac{1-i}{1+i} = \frac{-2i}{2} = -i = D'$$

$$\frac{0-i}{0+i} = -1 = C'$$

$$0 = A'$$

$$\infty = B'$$



4. a) If Z_u is empty, then u is either always positive or always negative. Let v be a harmonic conjugate for u. Then $f = u + iv$ is entire and maps $\mathbb{C}$ into either the left or the right half plane, and hence misses an open set. Hence f must be constant. It follows that u must be a non-zero constant.

b) If Z_u is contained in $D_R(0)$, then u must be always positive or always negative on $\{z : |z| > R\}$. If $z_0 \in Z_u$, then

$$0 = u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{u(z_0 + 2Re^{it})}_{\substack{\text{always } >0 \\ \text{or} \\ \text{always } <0}}\, dt. \quad \lightning$$

Hence, Z_u must be bounded.