RIEMANN-STIELTJES INTEGRATION. BOUNDED VARIATION

Problem 1. Assume $\{f_n\}$ is a sequence of real-valued nondecreasing functions defined on $I = [a, b]$, and suppose $f(x) = \lim_n f_n(x)$ exists for all $x \in I$. Is f necessarily nondecreasing?

Problem 2. Assume f is a bounded real-valued function defined on $I = [a, b]$ and let $\mathcal{F} = \{g : g \text{ defined on } I, g \text{ non-increasing and } g(x) \geq f(x) \text{ for } x \in I\}$. Show that $f^\star(x) = \sup\{f(y) : x \leq y \leq b\}$, for $x \in I$, belongs to $\mathcal{F}$, and in fact it is the smallest element there. Moreover, if f is continuous at x, so is $f^\star$.

Problem 3. Show that a monotone function $f\colon [a, b] \to \mathbb{R}$ has, at most, countably many discontinuities, and that all are of the first kind. Conversely, if D is an at most countable subset of $[a, b]$, construct a monotone function $f\colon [a, b] \to \mathbb{R}$ such that $D = \{x \in [a, b] : f \text{ is discontinuous at } x\}$.

Problem 4. A real-valued function f defined on $I = [a, b]$ is said to be Lipschitz there if there is a constant c such that $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in I$. Show that if f is Lipschitz on I, it is BV there.

Problem 5. Let f, g be BV on $I = [a, b]$. Show that f, g are bounded on I, and that for any real number η, $f + \eta g$ is BV on I and $V(f + \eta g; a, b) \leq V(f; a, b) + |\eta| V(g; a, b)$.

Problem 6 (Fall'01)**.** Let $f, g \in BV$ on $I = [a, b]$. Show that $fg \in BV(I)$, and that if $|g(x)| \geq \varepsilon > 0$ for $x \in I$, then also $f/g \in BV(I)$. Estimate $V(fg; a, b)$ and $V(f/g; a, b)$ in terms of $V(f; a, b)$, $V(g; a, b)$ and ε.

Problem 7. Let f, g be real-valued functions defined on $I = [a, b]$, and suppose that f and g differ at finitely many values. show that $f \in BV(I)$ if and only if $g \in BV(I)$, and that $V(f; a, b) = V(g; a, b)$.

Problem 8. Characterize those real numbers α, β for which $f(x) = x^\alpha \sin(x^{-\beta})$, $x \neq 0$, $f(0) = 0$ is BV on $[0, 1]$. Verify that the choice $\alpha = 2$, $\beta = 3/2$ gives an example of a function which is BV on I, differentiable there, and yet f' is unbounded.

Problem 9 (Fall'03)**.** Let $\{q_1, q_2, \dots\}$ be an enumeration of the set of rational numbers q with $0 < q < 1$. Define $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2^{-n} & \text{if } x = q_n, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f has bounded variation.

Problem 10 (Fall'03)**.** Give an example of a function $f\colon [0,1] \to \mathbb{R}$ such that $f = 0$ almost everywhere and f does not have bounded variation, and justify your answer.

Problem 11 (Spring'04)**.** Find all the functions $f\colon [0,1] \to \mathbb{R}$ with bounded variation satisfying

$$f(x) + \left(T_0^x f\right)^{1/2} = 1, \text{ for all } x \in [0,1],$$

and

$$\int_0^1 f(x)\, dx = 1/3.$$

Problem 12 (Fall'05)**.** Suppose f is of bounded variation on $[0,1]$. Prove that so is e^f.