MEASURABLE FUNCTIONS

1. ONE-LINERS

Problem 1. Suppose \( \{ f > \lambda \} \) is measurable for each rational number \( \lambda \). Is \( f \) measurable?

Problem 2. Let \((X, \mathcal{F}, \mu)\) be a measure space. Suppose \( f \) is a measurable real-valued function defined on \( \mathbb{R} \), and put \( g(x) = 0 \) if \( f(x) \) is rational, and \( g(x) = 1 \) if \( f(x) \) is irrational. Is \( g \) measurable?

Problem 3. Let \((X, \mathcal{F}, \mu)\) be a measure space. Suppose \( f \) is measurable and \( B \in \mathcal{B}_1 \) is a Borel subset of \( \mathbb{R} \). Does it then follow that \( f^{-1}(B) \in \mathcal{F} \)?

Problem 4. Suppose \( f \) is a measurable real-valued function defined on \( \mathbb{R} \), and let \( \phi \) be a real-valued Borel-measurable function defined on \( \mathbb{R} \). Show that the composition \( \phi \circ f \) is measurable.

Problem 5. Suppose \( f \) is measurable and show that for each reals \( r, s > 0 \), the truncations \( f_{r,s} \) are measurable:

\[
 f_{r,s}(x) = \begin{cases} 
 r & \text{if } f(x) > r, \\
 f(x) & \text{if } -s \leq f(x) \leq r, \\
 -s & \text{if } f(x) < -s.
\end{cases}
\]

Problem 6. Let \( \{ f_n \} \) be a sequence of measurable functions defined on \( \mathbb{R} \), and let \( A = \{ x \in \mathbb{R} : \lim_n f_n(x) \text{ exists} \} \). Show that \( A \) is measurable.

Problem 7. Let \( m(X) < \infty \), let \( \lambda_n > 0, n \in \mathbb{N} \), and suppose \( \{ f_n \} \) is a sequence of extended real-valued measurable functions defined on \( X \) that satisfies \( \sum_{n=1}^\infty m\{ |f_n| > \lambda_n \} < \infty \). Prove that \( \limsup_n |f_n|/\lambda_n \leq 1 \) a.e.

Problem 8 (NEW!). Suppose that \( f \) is Lebesgue measurable and \( \phi \) is real-valued continuous and has the following property: For any null set \( N \), \( \phi^{-1}(N) \in \mathcal{L} \). Show that \( f \circ \phi \) is Lebesgue measurable.

2. ADVANCED PROBLEMS

Problem 9. Show that if \( f \) is an everywhere finite real-valued measurable function, then it is the uniform limit of a sequence of elementary
functions\(^1\) Also, unless \(f\) is bounded, it is not the uniform limit of a sequence of simple functions.

**Problem 10.** Suppose \((X, \mathcal{F}, \mu)\) is a \(\sigma\)-finite measure space and let \(f\) be a measurable function defined on \(X\). Show that the function \(\mu\{|f| > \lambda\}; \lambda > 0\), is non-increasing and right-continuous. Furthermore, if \(f, f_1\) and \(f_2\) are non-negative and measurable, and \(\eta_1, \eta_2\) are non-negative real numbers so that \(f \leq \eta_1 f_1 + \eta_2 f_2\) \(\mu\)-a.e., then for any \(\lambda > 0\),

\[
\mu\{f > (\eta_1 + \eta_2)\lambda\} \leq \mu\{f_1 > \lambda\} + \mu\{f_2 > \lambda\}.
\]

**Problem 11.** Show that if \(\{f_n\}\) is a sequence of measurable functions such that \(\{|f_n|\}\) is nondecreasing and \(f = \lim_n f_n\), then \(\lim_n \mu\{|f_n| > \lambda\} = \mu\{|f| > \lambda\}\) for any \(\lambda > 0\).

**Problem 12.** Let \(f\) be a real-valued function defined on \([0, 1]\) such that \(f'\) exists for all \(x \in (0, 1)\). Prove that \(f'\) is measurable.

**Problem 13.** Suppose \(\{A_n\}\) is a sequence of Lebesgue-measurable subsets of \(\mathbb{R}\), and let \(f_n = \chi_{A_n}, n \in \mathbb{N}\). Find a necessary and sufficient condition for the sequence \(\{f_n\}\) to:

(i) converge a.e.

(ii) converge uniformly.

**Problem 14.** Show that if \(f\) is a real-valued measurable function, then there exists a sequence \(\{\lambda_n\}\) of real numbers and a sequence \(\{A_n\}\) of measurable sets such that \(f = \sum_{n=1}^{\infty} \lambda_n \chi_{A_n}\) a.e.

**Problem 15.** By means of an example, show that the conclusion of Egorov’s Theorem does not necessarily hold if the \(f_n\)'s are not Lebesgue measurable.

**Problem 16.** Let \(\{f_m\}\) be a sequence of Lebesgue-measurable functions defined on \(I = [a, b]\) and suppose that \(\lim_n f_n = f\) exists a.e. on \(I\). If \(f \neq 0\) a.e. and \(f_n \neq 0\) a.e. on \(I\), prove that given \(\varepsilon > 0\), there exists \(c > 0\) and a sequence \(\{E_n\}\) of measurable subsets of \(I\) such that \(|f_n(x)| \geq c, x \in E_n\) and \(|I \setminus E_n| \leq \varepsilon\) for \(n \in \mathbb{N}\).

3. Qual problems

**Problem 17 (August’99).** Suppose that \(f\) is Lebesgue measurable and \(0 \leq f(x) \leq 1\) on \([0, 1]\), and let \(\lambda(y) = m\{x \in [0, 1] : f(x) > y\}, y \geq 0\), where \(m\) denotes Lebesgue measure. Show that \(\lambda\) is non-increasing and continuous from the right on \([0, \infty)\).

\(^1\)An elementary function is one which assumes at most countably many values.
Problem 18 (January’00). Suppose \( f_n \) are measurable functions on some measure space \((\Omega, \mathcal{A}, \mu)\), \( n \in \mathbb{N} \), and
\[
\sum_{n=1}^{\infty} \mu\{x \in \Omega : |f_n(x)| > 1/n\} < \infty.
\]
Prove that \( f_n \to 0 \) a.e.

Problem 19 (August’00). Let \((X, \mathcal{F}, \mu)\) be a measure space and suppose \( \{f_n\} \) is a sequence of measurable functions with the property that for all \( n \geq 1 \),
\[
\mu\{x \in X : |f_n(x)| \geq \lambda\} \leq Ce^{-\lambda^2/n}
\]
for all \( \lambda > 0 \). (Here \( C \) is a constant independent of \( n \).) Prove that
\[
\limsup_n \frac{f_{2n}}{\sqrt{2^n \log \left(\log(2^n)\right)}} \leq 1, \text{ a.e.}
\]

Problem 20 (August’00). Answer the following questions:
(i) Let \((X, \mathcal{F}, \mu)\) be a finite measure space. Let \( \{f_n\} \) be a sequence of measurable functions. Prove that \( f_n \to f \) in measure if and only if every subsequence \( \{f_{n_k}\} \) contains a further sequence \( \{f_{n_{kj}}\} \) that converges almost everywhere to \( f \).
(ii) Let \((X, \mathcal{F}, \mu)\) be a finite measure space. Let \( F: \mathbb{R} \to \mathbb{R} \) be continuous and \( f_n \to f \) in measure. Prove that \( F(f_n) \to F(f) \) in measure.

Problem 21 (August’03). Let \( f \) be a bounded Lebesgue measurable function on \( \mathbb{R} \). Put
\[
g(x) = \sup \{a \in \mathbb{R} : |\{y : y \in (x, x + 1) \text{ and } f(y) > a\}| > 0\},
\]
where \(|\cdot|\) is the Lebesgue measure. Prove \( \liminf_{x \to 0} g(x) \geq g(0) \).

Problem 22 (January’99, August’06). Let \( f_n: \mathbb{R} \to (-\infty, \infty], n \in \mathbb{N}, \) be a sequence of Lebesgue measurable functions.
(i) Prove that \( g(x) = \sup_n f_n(x) \) is Lebesgue measurable.
(ii) Prove that \( h(x) = \limsup_n f_n(x) \) is Lebesgue measurable.