COMPLEXITY IN COMPLEX ANALYSIS

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Abstract. We show that the classical kernel and domain functions associated to an n-connected domain in the plane are all given by rational combinations of three or *fewer* holomorphic functions of *one* complex variable. We characterize those domains for which the classical functions are given by rational combinations of only two or fewer functions of one complex variable. Such domains turn out to have the property that their classical domain functions all extend to be meromorphic functions on a compact Riemann surface, and this condition will be shown to be equivalent to the condition that an Ahlfors map and its derivative are algebraically dependent. We also show how many of these results can be generalized to finite Riemann surfaces.

1. Introduction. On a simply connected domain $\Omega \neq \mathbb{C}$ in the plane, the classical Bergman kernel $K(z,w)$ associated to Ω is given by

$$K(z,w)=\frac{f_a'(z)\overline{f_a'(w)}}{\pi(1-f_a(z)\overline{f_a(w)})^2},$$

where $f_a(z)$ is the Riemann mapping function mapping Ω one-to-one onto the unit disc $D_1(0)$ with $f_a(a)=0$ and $f_a'(a)>0$. Thus, the Bergman kernel is a rational combination of just *two* holomorphic functions of *one* complex variable. I have recently proved in [9] that the Bergman kernel and many other objects of potential theory associated to a finitely multiply connected domain are rational combinations of only *three* holomorphic function of *one* complex variable, namely two Ahlfors maps plus the derivative of one Ahlfors map. In [8], I proved that the three functions of z given by $S(z,A_j)$, where $S(z,w)$ is the Szegő kernel and A_j, $j=1,2,3$, are three fixed points in the domain, generate the Szegő kernel, the Bergman kernel, and many other objects of potential theory. In this paper, I shall unify these results and I shall show that the three functions of one variable that generate these classical functions can be taken from a rather long list of functions. I also showed in [10] that there exist certain multiply connected domains in the plane that are particularly simple in the sense that their Bergman and Szegő kernels are generated by only *two* functions of one complex variable. In this paper, I shall present all these results in a single improved framework and I will thereby be able to characterize those domains whose kernel functions are particularly simple. These domains are characterized by a condition on the Ahlfors map which turns out to be equivalent to the condition that the Bergman kernel and other objects of

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potential theory associated to the domain extend to a compact Riemann surface as single valued meromorphic functions. At the heart of these results is a relationship between the Szegő kernel of a domain and proper holomorphic maps of the domain to the unit disc. Because we can prove similar relationships on any finite Riemann surface, we will see that many of these results can be generalized to this setting.

2. Statement of main results. Because the results of this paper are easiest to state and prove for domains in the plane and because all of the main technical advances can be understood in this setting, we shall devote the first part of the paper to planar domains. In the last section of the paper, we sketch the rather routine argument to generalize the results to the case of finite Riemann surfaces.

We shall be able to prove our main results for finitely connected domains in the plane such that no boundary component is a point. For the moment, however, assume that Ω is a bounded n-connected domain in the plane bounded by n non-intersecting C^∞ smooth simple closed curves, γ_j, $j = 1, ...n$. For a point a in Ω, let f_a denote the Ahlfors map associated to the pair (Ω, a). This map is an n-to-one (counting multiplicities) proper holomorphic mapping of Ω onto the unit disc. Furthermore, $f_a(a) = 0$, and f_a is the unique holomorphic function mapping Ω into the unit disc maximizing the quantity $|f_a'(a)|$ with $f_a'(a) > 0$. It is well known that f_a extends C^∞ smoothly up to the boundary $b\Omega$ of Ω and that $|f_a| = 1$ on $b\Omega$.

Let $K(z, w)$ denote the Bergman kernel associated to Ω and let ω_j denote the harmonic measure function which is harmonic on Ω with boundary values of one on the boundary component γ_j and zero on the other boundary components. Let $F_j'(z)$ denote the holomorphic function given by $(1/2)(\partial/\partial z)\omega_j(z)$. (The prime is traditional; F_j' is not the derivative of a holomorphic function on Ω.) Let $G(z, a)$ denote the classical Green's function associated to Ω.

The Szegő kernel associated to Ω is the kernel for the orthogonal projection of $L^2(b\Omega)$ (with respect to arc length measure) onto the subspace consisting of L^2 boundary values of holomorphic functions on Ω, i.e., the Hardy space $H^2(b\Omega)$.

When Ω is merely an n-connected domain in the plane such that no boundary component is a point, we define the Szegő kernel associated to Ω as follows. There exists a biholomorphic mapping Φ mapping Ω one-to-one onto a bounded domain Ω^a in the plane with real analytic boundary. The standard construction yields a domain Ω^a that is a bounded n-connected domain with C^∞ smooth real analytic boundary whose boundary consists of n non-intersecting simple closed real analytic curves. The function Φ' has a single valued holomorphic square root on Ω (see [4, page 43]). Let superscript a's indicate that a kernel function is associated to Ω^a. Kernels without superscripts are associated to Ω. The Szegő kernel associated to Ω is *defined* via

$$S(z, w) = \sqrt{\Phi'(z)}\ S^a(\Phi(z), \Phi(w))\overline{\sqrt{\Phi'(w)}}.$$

We shall also define the Garabedian kernel associated to Ω via the natural transformation formula,

$$L(z, w) = \sqrt{\Phi'(z)}\ L^a(\Phi(z), \Phi(w))\sqrt{\Phi'(w)}.$$

(See [4, page 24] or §3 of this paper for the definition of the Garabedian kernel in a smoothly bounded domain.) Various other transformation formulas hold for the objects of potential theory mentioned above. For example, the Bergman kernels transform via

$$K(z,w)=\Phi'(z)K^a(\Phi(z),\Phi(w))\overline{\Phi'(w)},$$

and the Green's functions satisfy

$$G(z,w)=G^a(\Phi(z),\Phi(w)),$$

and the functions associated to harmonic measure satisfy

$$\omega_j(z)=\omega_j^a(\Phi(z)) \quad \text{and} \quad F_j'(z)=\Phi'(z)F_j^{a\prime}(\Phi(z))$$

(provided, of course, that we stipulate that the boundary components have been numbered so that Φ maps the j-th boundary component of Ω to the j-th boundary component of Ω^a). The Ahlfors map f_b associated to a point $b\in\Omega$ is the holomorphic function mapping Ω into $D_1(0)$ with $|f_b'(b)|$ maximal and $f_b'(b)>0$. It is easy to see that the Ahlfors map satisfies

$$f_b(z)=\lambda f^a_{\Phi(b)}(\Phi(z))$$

for some unimodular constant λ and it follows that $f_b(z)$ is a proper holomorphic mapping of Ω onto $D_1(0)$. The double of Ω may also be defined in a standard way by using Φ and the double of Ω^a.

The work in this paper is motivated by the following result from [9].

Theorem 2.1. *Suppose Ω is an n-connected domain in the plane such that no boundary component is a point. There exist points a and b in Ω and complex rational functions R and Q of four complex variables such that the Bergman kernel can be expressed in terms of the two Ahlfors maps f_a and f_b via*

$$K(z,w)=f_a'(z)\overline{f_a'(w)}R(f_a(z),f_b(z),\overline{f_a(w)},\overline{f_b(w)})$$

and the Szegő kernel can be expressed via

$$S(z,w)^2=f_a'(z)\overline{f_a'(w)}Q(f_a(z),f_b(z),\overline{f_a(w)},\overline{f_b(w)}).$$

Alternatively, the Szegő kernel can be expressed via

$$S(z,w)=S(z,a)S(a,w)Q_2(f_a(z),f_b(z),\overline{f_a(w)},\overline{f_b(w)})$$

where Q_2 is rational. Furthermore, the functions F_j' are given by

$$F_j'(z)=f_a'(z)P(f_a(z),f_b(z))$$

where P is a rational function of two complex variables. Also, every proper holomorphic mapping of Ω onto the unit disc is a rational combination of f_a and f_b.

There are similar formulas for the complex derivative of the Green's function $(\partial/\partial z)G(w,z)$ and for the Poisson kernel given in [9].

These results show that the analytic objects of potential theory and conformal mapping in a multiply connected domain are all given as rational combinations of only *three* holomorphic functions of one complex variable. In [10], I showed that, when the kernel functions are algebraic, these analytic objects are given by rational combinations of only *two* holomorphic functions of one complex variable. In the present paper, we characterize this property in the following theorem. We shall say that a kernel function such as the Bergman kernel $K(z,w)$ is given as a rational combination of two functions of one complex variable on Ω if there exist holomorphic functions G_1 and G_2 on Ω such that $K(z,w)$ can be written as a rational combination of $G_1(z)$, $G_2(z)$, $\overline{G_1(w)}$, and $\overline{G_2(w)}$.

Theorem 2.2. *Suppose Ω is an n-connected domain in the plane with $n > 1$ such that no boundary component is a point. The following conditions are equivalent.*

(1) *The Bergman kernel $K(z, w)$ is given as a rational combination of only two functions of one complex variable on Ω.*
(2) *The Szegő kernel $S(z, w)$ is given as a rational combination of only two functions of one complex variable on Ω.*
(3) *The Bergman kernel, Szegő kernel, the classical functions $F_j'(z)$ associated to Ω, and all proper holomorphic mappings of Ω onto the unit disc are all given as rational combinations of the same two functions of one complex variable on Ω.*
(4) *The domain Ω can be realized as a subdomain of a compact Riemann surface $\mathcal{R}$ such that $S(z, w)$ and $K(z, w)$ extend to $\mathcal{R} \times \mathcal{R}$ as single valued meromorphic functions, and as such, can be expressed as rational combinations of any two functions on Ω which extend to $\mathcal{R}$ to form a primitive pair for $\mathcal{R}$. Also, every proper holomorphic mapping of Ω onto the unit disc as well as the functions $F_k'(z)$, $k = 1, \ldots, n-1$, extend to be single valued meromorphic functions on $\mathcal{R}$. Furthermore, the complement of Ω in $\mathcal{R}$ is connected.*
(5) *There exists a single proper holomorphic mapping of Ω onto the unit disc which satisfies an identity of the form*

$$P(f'(z), f(z)) = 0$$

on Ω for some complex polynomial P, i.e., f and f' are algebraically dependent.
(6) *For every proper holomorphic mapping f of Ω onto the unit disc, f and f' are algebraically dependent.*

A novel feature of the proof of Theorem 2.2 is the use of the Zariski-Castelnuovo Theorem from Algebra.

We remark that it is always possible to choose the two functions that form the primitive pair in condition (4) to be holomorphic on Ω (see Farkas and Kra [15]).

It shall fall out from the proof of Theorem 2.2 that if the Bergman or Szegő kernel associated to a finitely connected domain with no pointlike boundary components is generated by a *single* function of one variable, then the domain must be simply connected. In fact, if the domain under study is simply connected, the proof of Theorem 2.2 becomes considerably easier and a similar result can be established without using Lüroth's Theorem in place of the Zariski-Castelnuovo Theorem.

Theorem 2.2a. *Suppose Ω is a simply connected domain in the plane not equal to $\mathbb{C}$. The following conditions are equivalent.*

(1) *The Bergman or Szegő kernels are given as rational combinations of only one function of one complex variable on Ω.*
(2) *The domain Ω can be realized as a subdomain of a compact Riemann surface $\mathcal{R}$ such that $S(z, w)$ and $K(z, w)$ extend to $\mathcal{R} \times \mathcal{R}$ as single valued meromorphic functions.*
(3) *A Riemann mapping of Ω onto the unit disc and its derivative are algebraically dependent.*

Many frequently encountered domains satisfy the polynomial condition (5) in Theorem 2.2 for some proper map. For example, this condition holds for generalized quadrature domains of the type studied by Aharonov and Shapiro (see [1, page 64] and [18]) and it holds for domains that have algebraic kernel functions (see [7,10]). This last class of domains is the same as the set of domains given by a connected component Ω of a set of the form $\{z : |f(z)| < 1\}$ where f is an algebraic function without singularities in Ω, i.e., Ω is a domain such there exists an algebraic proper holomorphic mapping from it onto the unit disc (see [7]).

Theorem 2.1 is actually a special case of a more general theorem that we prove here. Let $\widehat{\Omega}$ denote the double of Ω and let $R(z)$ denote the antiholomorphic involution on $\widehat{\Omega}$ which fixes the boundary of Ω. Let $\widetilde{\Omega} = R(\Omega)$ denote the reflection of Ω in $\widehat{\Omega}$ across the boundary. The double of Ω is a compact Riemann surface and hence the field of meromorphic functions on $\widehat{\Omega}$ is generated by a pair of meromorphic functions (G_1, G_2) on $\widehat{\Omega}$ known as a primitive pair (see Farkas and Kra [15]). The construction of primitive pairs given in [15] shows that we may assume that G_1 and G_2 are holomorphic on $\Omega \subset \widehat{\Omega}$.

We now define a class $\mathcal{A}$ of meromorphic functions on Ω. Recall that $G(z, a)$ denotes the classical Green's function associated to Ω.

The class $\mathcal{A}$ consists of

(1) the functions $F_j'(z)$, $j = 1, \ldots, n$,
(2) functions of z of the form $\frac{\partial}{\partial z} G(z, a)$ for fixed a in Ω,
(3) functions of z of the form $D_a \frac{\partial}{\partial z} G(z, a)$ where D_a denotes a differential operator of the form $\frac{\partial^n}{\partial a^n}$ or $\frac{\partial^n}{\partial \bar{a}^n}$, and a is a fixed point in Ω,
(4) functions of z of the form $S(z, a_1)S(z, a_2)$ where a_1 and a_2 are fixed points in Ω,
(5) and linear combinations of functions above.

If Ω has C^∞ smooth boundary, we allow the points a in (2) and (3) of the definition of the class $\mathcal{A}$ to be in the larger set $\overline{\Omega}$.

Theorem 2.1 will be generalized as follows.

Theorem 2.3. *Suppose that Ω is a finitely connected domain in the plane such that no boundary component is a point. Let G_1 and G_2 denote any two meromorphic functions on Ω that extend to the double of Ω to form a primitive pair, and let $A(z)$ denote any function from the class $\mathcal{A}$ other than the zero function. The Bergman kernel associated to Ω can be expressed as*

$$K(z, w) = A(z)\overline{A(w)}R_1(G_1(z), G_2(z), \overline{G_1(w)}, \overline{G_2(w)})$$

where R_1 is a complex rational function of four complex variables. Similarly, the Szegő kernel can be expressed as

$$S(z, w)^2 = A(z)\overline{A(w)}R_2(G_1(z), G_2(z), \overline{G_1(w)}, \overline{G_2(w)})$$

where R_2 is rational, and the functions F_j' can be expressed

$$F_j'(z) = A(z)R_3(G_1(z), G_2(z))$$

where R_3 is rational. Furthermore, every proper holomorphic mapping of Ω onto the unit disc is a rational combination of G_1 and G_2.

The proof of Theorem 2.3 hinges on the fact proved in §6 that, when Ω has real analytic boundary, functions in the class $\mathcal{A}$ can be seen to be equal to meromorphic functions H on $\overline{\Omega}$ that satisfy an identity of the form

$$H(z)T(z) = \overline{J(z)T(z)}$$

for $z \in b\Omega$ where $J(z)$ is another meromorphic function on $\overline{\Omega}$ and $T(z)$ represents the complex unit tangent vector at z pointing in the direction of the standard orientation of $b\Omega$. We prove in §6 that the class $\mathcal{A}$ is the largest set of functions with this property.

It is remarkable how many different common functions fall into the class $\mathcal{A}$. For example, the Bergman kernel is in $\mathcal{A}$ because it is related to the classical Green's function via ([14, page 62], see also [4, page 131])

$$K(z,w) = -\frac{2}{\pi}\frac{\partial^2 G(z,w)}{\partial z \partial \bar{w}}.$$

Another kernel function on $\Omega \times \Omega$ that we shall need is given by

$$\Lambda(z,w) = -\frac{2}{\pi}\frac{\partial^2 G(z,w)}{\partial z \partial w}.$$

(In the literature, this function is sometimes written as $L(z,w)$ with anywhere between zero and three tildes and/or hats over the top. We have chosen the symbol Λ here to avoid confusion with our notation for the Garabedian kernel above.) Note that $\Lambda(z,w)$ is in $\mathcal{A}$ as a function of z for each fixed w in Ω. Furthermore, given a proper holomorphic map f from Ω onto the unit disc, the quotient f'/f is in $\mathcal{A}$ by virtue of the fact that $\ln|f|$ is a linear combination of Green's functions. Since $\bar{f} = 1/f$ on $b\Omega$, we shall be able to see that $f'(z)T(z) = -\overline{T(z)f'(z)/f(z)^2}$ for $z \in b\Omega$ on a smooth domain, and we will be able to deduce that f' itself is in $\mathcal{A}$ from Lemma 6.3 below.

It is also remarkable the variety of functions that can appear as members of a primitive pair. The argument in [9, page 332] reveals that for any proper holomorphic map f_1 from Ω onto the unit disc, there exists a second proper map $f_2 : \Omega \to D_1(0)$ such that (f_1, f_2) forms a primitive pair for the double of Ω. (In fact, the second map can be taken to be an Ahlfors map.) When f_1'/f_1 is taken to be equal to $A(z)$ in Theorem 2.3 and G_1 and G_2 are taken to be this primitive pair, the Bergman kernel $K(z,w)$ is expressed as $f_1'(z)\overline{f_1'(w)}$ times a rational function of $f_1(z)$, $f_2(z)$, $\overline{f_1(w)}$ and $\overline{f_2(w)}$. This result makes it very easy to prove the result given in [7] that the Bergman kernel associated to a finitely connected domain is an algebraic function if and only if there exists a proper holomorphic map from the domain onto the unit disc which is algebraic. Indeed, if f_1 is algebraic, then f_1' is algebraic, and since f_1 and f_2 extend to the double to form a primitive pair, they must be algebraically dependent (see Farkas and Kra [15, page 248]). Hence, f_2 is an algebraic function of f_1 and this shows that $K(z,w)$ is algebraic. The reverse implication is even easier (see [7,11,12]). Similarly, the statement about the Szegő kernel in Theorem 2.3 yields an easy proof of the result given in [7] that the Szegő kernel associated to a finitely connected domain is an algebraic function if and only if there exists a proper holomorphic map from the domain onto the unit disc which is algebraic.

It is shown in [8] that, for almost any three points a_1, a_2, and b in Ω, the two quotients $K(z,a_1)/K(z,b)$ and $K(z,a_2)/K(z,b)$ extend to the double of Ω and form a primitive pair. When these two functions are used as G_1 and G_2 and when $K(z,b)$ is used as the function $A(z)$ from the class $\mathcal{A}$, we find that $K(z,w)$ is a rational combination of $K(z,a_1)$, $K(z,a_2)$, and $K(z,b)$ and conjugates of $K(w,a_1)$, $K(w,a_2)$, and $K(w,b)$ (as was also shown in [8]). It now follows from Theorem 2.3

that all the functions in the class $\mathcal{A}$ are rational combinations of $K(z,a_1)$, $K(z,a_2)$ and $K(z,b)$. This reinforces the idea that the Bergman kernel contains *everything* there is to know about a domain.

Let $G_z(z,w)$ denote the derivative $(\partial/\partial z)G(z,w)$ of the Green's function. Similar reasoning to that given in [8] shows that, for almost any three points a_1, a_2, and b in Ω, the two quotients $G_z(z,a_1)/G_z(z,b)$ and $G_z(z,a_2)/G_z(z,b)$ form a primitive pair for the double of Ω. Hence, Theorem 2.3 yields that the Bergman kernel is a rational combination of the three functions of one variable given by $G_z(z,a_1)$, $G_z(z,a_2)$, and $G_z(z,b)$. A similar result has been proved for the Szegő kernel in [8]. Let us summarize these results in the statement of the following theorem.

Theorem 2.4. *Suppose that Ω is a finitely connected domain in the plane such that no boundary component is a point. The Bergman kernel $K(z,w)$ associated to Ω can be expressed as a rational combination of the three functions*

(1) $K(z,A_1)$, $K(z,A_2)$, $K(z,A_3)$, *or*
(2) $S(z,A_1)$, $S(z,A_2)$, $S(z,A_3)$, *or*
(3) $G_z(z,A_1)$, $G_z(z,A_2)$, $G_z(z,A_3)$,

where A_1, A_2, and A_3 are three fixed points in Ω.

Recall that $L(z,a)$ denotes the Garabedian kernel associated to Ω. We now define another class of meromorphic functions on Ω relevant to the Szegő kernel. The class $\mathcal{B}$ consists of

(1) functions of z of the form $S(z,a)$ or $L(z,a)$ for fixed points a in Ω,
(2) functions of z of the form $\frac{\partial^m}{\partial \bar{a}^m}S(z,a)$ or $\frac{\partial^m}{\partial a^m}L(z,a)$ for fixed points a in Ω,
(3) and linear combinations of functions above.

If Ω has C^∞ smooth boundary, we allow the points a in (1) and (2) in the definition of the class $\mathcal{B}$ to be in the larger set $\overline{\Omega}$.

The Szegő kernel can be expressed in a manner similar to the Bergman kernel as follows.

Theorem 2.5. *Suppose that Ω is a finitely connected domain in the plane such that no boundary component is a point. Let G_1 and G_2 denote any two meromorphic functions on Ω that extend to the double of Ω to form a primitive pair, and let $B(z)$ denote any function from the class $\mathcal{B}$ other than the zero function. The Szegő kernel associated to Ω can be expressed as*

$$S(z,w) = B(z)\overline{B(w)}R(G_1(z), G_2(z), \overline{G_1(w)}, \overline{G_2(w)})$$

where R is a complex rational function of four complex variables.

The proof of Theorem 2.5 hinges on the fact proved in §6 that, when Ω has real analytic boundary, functions in the class $\mathcal{B}$ can be seen to be equal to meromorphic functions G on $\overline{\Omega}$ that satisfy an identity of the form

$$G(z) = \overline{H(z)T(z)}$$

for $z \in b\Omega$ where $H(z)$ is another meromorphic function on $\overline{\Omega}$ and, as before, $T(z)$ represents the complex unit tangent vector at z pointing in the direction of the standard orientation of $b\Omega$. We prove in §6 that the class $\mathcal{B}$ is the largest set of functions with this property.

It is shown in [8] that, for almost any three points a_1, a_2, and b in Ω, the two quotients $S(z,a_1)/S(z,b)$ and $S(z,a_2)/S(z,b)$ form a primitive pair for the double of Ω. When these two functions are used as G_1 and G_2 and when $S(z,b)$ is used as $B(z)$ in Theorem 2.5, we find that $S(z,w)$ is a rational combination of $S(z,a_1)$, $S(z,a_2)$ and $S(z,b)$, and the conjugates of $S(w,a_1)$, $S(w,a_2)$ and $S(w,b)$, as stated in Theorem 2.4. It now also follows from Theorem 2.5 that all the functions in the class $\mathcal{B}$ are rational combinations of $S(z,a_1)$, $S(z,a_2)$ and $S(z,b)$. It can be shown that the class $\mathcal{A}$ is equal to the complex linear span of the set of products of two functions in the class $\mathcal{B}$. Hence, it is reasonable to say that the Szegő kernel contains *even more* information about a domain than the Bergman kernel does.

It is reasonable to wonder why on earth one might want to take complicated linear combinations for $A(z)$ or $B(z)$ in Theorems 2.3 and 2.5 instead of a single simple function from the classes. However, if Ω is a quadrature domain in the sense that

$$\iint_\Omega h(z)\, dA = \sum_{j=1}^N h(w_j)$$

for finitely many fixed points w_j in Ω and all holomorphic functions h in the Bergman space, then

$$\sum_{j=1}^N K(z,w_j) \equiv 1$$

and we may take $A(z) \equiv 1$ in Theorem 2.3. Hence, if Ω is a quadrature domain in this sense, then the Bergman kernel is a rational combination of any two functions that extend to the double of Ω to form a primitive pair. Similarly, if Ω is a quadrature domain in the sense that

$$\int_{b\Omega} h(z)\, ds = \sum_{j=1}^N h(w_j)$$

for finitely many fixed points w_j in Ω and all holomorphic functions h in the Hardy space, then

$$\sum_{j=1}^N S(z,w_j) \equiv 1$$

and we may take $B(z) \equiv 1$ in Theorem 2.5. Hence, if Ω is a quadrature domain in this sense, then the Szegő *and* Bergman kernels are rational combinations of any two functions that extend to the double of Ω to form a primitive pair.

Most of the results of this paper depend on a special formula for the Szegő kernel. It is a standard construction to produce the Szegő projection and kernel with respect to a weight function on the boundary of Ω when the boundary of Ω is sufficiently smooth. Suppose that Ω is a domain in the plane bounded by finitely many C^∞ smooth curves. (It will be clear from the proofs given later that this smoothness condition could be greatly relaxed, but we will be able to reap enough consequences in the smooth case that we will not bother trying to generalize the result here.) Given a positive C^∞ weight function φ on $b\Omega$, let $S(z,w)$ denote the Szegő kernel function defined on $\Omega \times b\Omega$ which reproduces holomorphic functions on Ω with respect to the weight φ in the sense that

$$h(z) = \int_{w \in b\Omega} S(z,w) h(w)\, \varphi(w)\, ds$$

for points z in Ω and holomorphic functions h on Ω that are in the Hardy space associated to Ω. We shall prove that the weighted Szegő kernel $S(z,w)$ is a rational combination of finitely many holomorphic functions of one complex variable on Ω in the following precise sense.

Let $S_0(z,w)$ denote $S(z,w)$, and let $S_{\bar{n}}(z,w)$ denote $(\partial^n/\partial \bar{w}^n) S(z,w)$.

Theorem 2.6. *Suppose that Ω is a domain in the plane bounded by finitely many C^∞ smooth curves. Suppose that $f : \Omega \to D_1(0)$ is a proper holomorphic mapping that has zeroes at $a_1, \ldots, a_N$ with multiplicities $M(1), \ldots, M(N)$, respectively. Given a positive C^∞ weight function φ on $b\Omega$, the weighted Szegő kernel $S(z,w)$ with respect to φ satisfies*

$$S(z,w) = \frac{1}{1 - f(z)\overline{f(w)}} \left(\sum_{i,j=1}^{N} \sum_{n=0}^{M(i)} \sum_{m=0}^{M(j)} c_{ijnm} S_{\bar{n}}(z, a_i)\, \overline{S_{\bar{m}}(w, a_j)} \right)$$

for some constants c_{ijnm}.

The constants c_{ijnm} will be described more fully in the proof of this theorem given in §8.

When the weight φ is taken to be the Poisson kernel associated to a point a in Ω, we will be able to say more about the associated Szegő kernel and how it is related to the objects of potential theory, see §10. This viewpoint will also allow us to find interesting generalizations to the case where Ω is a finite Riemann surface, see §11.

3. Preliminaries. For the moment, suppose that Ω is a bounded n-connected domain in the plane with C^∞ smooth boundary, i.e., a domain whose boundary $b\Omega$ is given by finitely many non-intersecting C^∞ simple closed curves. Let γ_j, $j = 1, \ldots, n$, denote the n non-intersecting C^∞ simple closed curves which define the boundary of Ω, and suppose that γ_j is parameterized in the standard sense by $z_j(t)$. Let $T(z)$ be the C^∞ function defined on $b\Omega$ such that $T(z)$ is the complex number representing the unit tangent vector at $z \in b\Omega$ pointing in the direction of the standard orientation. This complex unit tangent vector function is characterized by the equation $T(z_j(t)) = z_j'(t)/|z_j'(t)|$.

Let $\widehat{\Omega}$ denote the double of Ω and let $R(z)$ denote the antiholomorphic involution on $\widehat{\Omega}$ which fixes the boundary of Ω. Let $\widetilde{\Omega} = R(\Omega)$ denote the reflection of Ω in $\widehat{\Omega}$ across the boundary. We shall frequently use the following fact. If g and h are meromorphic functions on Ω which extend continuously to the boundary such that

$$g(z) = \overline{h(z)} \qquad \text{for } z \in b\Omega,$$

then g extends to the double of Ω as a meromorphic function. Indeed, the function $\overline{h(R(z))}$ gives the holomorphic extension of g to $\widetilde{\Omega}$. For example, since a proper holomorphic map f from Ω to the unit disc extends smoothly up to the boundary and has modulus one there, it follows that

$$f(z) = 1/\overline{f(z)} \qquad \text{for } z \in b\Omega,$$

and, hence, that f extends to be meromorphic on the double of Ω.

We now take a moment to recite some standard facts that we shall assume the reader knows. Let $A^\infty(\Omega)$ denote the space of holomorphic functions on Ω that are in $C^\infty(\overline{\Omega})$. Let $L^2(b\Omega)$ denote the space of complex valued functions on $b\Omega$ that are square integrable with respect to arc length measure ds. The Hardy space of functions in $L^2(b\Omega)$ that are the L^2 boundary values of holomorphic functions on Ω shall be written $H^2(b\Omega)$. This space is equal to the closure in $L^2(b\Omega)$ of $A^\infty(\Omega)$ (see [4] for a proof of this elementary fact).

Let $S(z,a)$ denote the classical Szegő kernel associated to the classical Szegő projection P, which is the orthogonal projection of $L^2(b\Omega)$ onto $H^2(b\Omega)$. For each fixed point $a \in \Omega$, $S(z,a)$ extends to the boundary as a function of z to be a function in $A^\infty(\Omega)$. Furthermore, $S(z,a)$ has exactly $(n-1)$ zeroes in Ω (counting multiplicities) and does not vanish at any points z in the boundary of Ω.

The classical Garabedian kernel $L(z,a)$ is a kernel related to the Szegő kernel via the identity

$$\frac{1}{i}L(z,a)T(z) = S(a,z) \qquad \text{for } z \in b\Omega \text{ and } a \in \Omega. \tag{3.1}$$

For fixed $a \in \Omega$, the kernel $L(z,a)$ is a holomorphic function of z on $\Omega - \{a\}$ with a simple pole at a with residue $1/(2\pi)$. Furthermore, as a function of z, $L(z,a)$ extends to the boundary and is in the space $C^\infty(\overline{\Omega} - \{a\})$. Also, $L(z,a)$ is non-zero for all (z,a) in $\overline{\Omega} \times \Omega$ with $z \neq a$.

The kernel $S(z,w)$ is holomorphic in z and antiholomorphic in w on $\Omega \times \Omega$, and $L(z,w)$ is holomorphic in both variables for $z, w \in \Omega$, $z \neq w$. We note here that $S(z,z)$ is real and positive for each $z \in \Omega$, and that $S(z,w) = \overline{S(w,z)}$ and $L(z,w) = -L(w,z)$. Also, the Szegő kernel reproduces holomorphic functions in the sense that

$$h(a) = \langle h, S(\cdot, a)\rangle$$

for all $h \in H^2(b\Omega)$ and $a \in \Omega$, where the inner product is taken in $L^2(b\Omega)$.

Given a point $a \in \Omega$, the Ahlfors map f_a associated to the pair (Ω, a) is related to the Szegő kernel and Garabedian kernel via

$$f_a(z) = \frac{S(z,a)}{L(z,a)}. \tag{3.2}$$

Note that $f_a'(a) = 2\pi S(a,a) \neq 0$. Because f_a is n-to-one, f_a has n zeroes. The simple pole of $L(z,a)$ at a accounts for the simple zero of f_a at a. The other $n-1$ zeroes of f_a are given by $(n-1)$ zeroes of $S(z,a)$ in $\Omega - \{a\}$. Let $a_1, a_2, \dots, a_{n-1}$ denote these $n-1$ zeroes (counted with multiplicity). It was proved in [5] (see also [4, page 105]) that, if a is close to one of the boundary curves, the zeroes $a_1, \dots, a_{n-1}$ become distinct simple zeroes. It follows from this result that, for all but at most finitely many points $a \in \Omega$, $S(z,a)$ has $n-1$ distinct simple zeroes in Ω as a function of z.

The Bergman kernel and the kernel $\Lambda(z,w)$ defined in §2 satisfy an identity analogous to (3.1):

$$\Lambda(w,z)T(z) = -K(w,z)\overline{T(z)} \qquad \text{for } w \in \Omega \text{ and } z \in b\Omega \tag{3.3}$$

(see [4, page 135]). We remark that it follows from well known properties of the Green's function that $\Lambda(z,w)$ is holomorphic in z and w and is in $C^\infty(\overline{\Omega}\times\overline{\Omega}-\{(z,z): z\in\overline{\Omega}\})$. If $a\in\Omega$, then $\Lambda(z,a)$ has a double pole at $z=a$ as a function of z and $\Lambda(z,a)=\Lambda(a,z)$ (see [4, page 134]). If Ω has real analytic boundary, then the kernels $K(z,w)$, $\Lambda(z,w)$, $S(z,w)$, and $L(z,w)$, extend meromorphically to $\overline{\Omega}\times\overline{\Omega}$ (see [4, page 103, 132–136]). The derivative of the Green's function also satisfies an identity similar to (3.3):

$$\frac{\partial G}{\partial z}(z,w)T(z) = -\overline{\frac{\partial G}{\partial z}(z,w)T(z)}, \tag{3.4}$$

for $w\in\Omega$ and $z\in b\Omega$ (see [4, page 134]). We shall also need the identity,

$$S(z,a_1)S(z,a_1)T(z) = -\overline{L(z,a_1)L(z,a_2)T(z)} \tag{3.5}$$

for a_1 and a_2 in Ω and $z\in b\Omega$, which follows from (3.1).

The transformation formulas for the Szegő and Garabedian kernels given in §2 and the biholomorphic map to a domain with real analytic boundary allow us to certify that the Ahlfors map is given by formula (3.2) even in case the domain under study is merely a finitely connected domain such that no boundary component is a point.

4. Proofs of Theorems 2.2 and 2.2a. We first assume that Ω is n-connected with $n>1$. Suppose that the Szegő kernel $S(z,w)$ associated to Ω is a rational combination of only two holomorphic functions G_1 and G_2 on Ω. To be precise, suppose that $S(z,w)$ is equal to a rational combination of $G_1(z)$, $G_2(z)$, $\overline{G_1(w)}$, and $\overline{G_2(w)}$. It is shown in [8] that the Bergman kernel $K(z,w)$ associated to Ω and all the other functions mentioned in Theorem 2.2 are rational combinations of functions of z of the form $S(z,A)$ for three fixed points A in Ω. Hence $K(z,w)$ is also a rational combination of the two holomorphic functions G_1 and G_2.

Let f_1 and f_2 denote two Ahlfors maps associated to Ω that generate the field of meromorphic functions on the double of Ω (see [9]). Let $\mathbb{C}(f_1,f_2)$ denote the field of functions generated by the two Ahlfors maps (which can be identified with the field of meromorphic functions on the double in the obvious manner).

It is shown in [7] that any proper holomorphic map from Ω onto the unit disc can be expressed as a rational combination of finitely many functions of z of the form $K(z,w)$ and $(\partial/\partial\bar{w})^m K(z,w)$ for w in $f^{-1}(0)$ (see also [11,12]). It follows that f_1 and f_2 are in the field $\mathbb{C}(G_1,G_2)$ of functions on Ω generated by G_1 and G_2. Hence, $\mathbb{C}(f_1,f_2)\subset\mathbb{C}(G_1,G_2)$.

We now claim that the functions G_1 and G_2 must be algebraically dependent, i.e., that there exists a complex polynomial $P(z,w)$ of two complex variables such that $P(G_1(z),G_2(z))\equiv 0$. Indeed if there did not exist such a polynomial, then we could apply the Zariski-Castelnuovo Theorem as follows to deduce that f_1 and f_2 must be algebraically independent, which cannot be the case for two functions that extend meromorphically to the double of Ω. (The Zariski-Castelnuovo Theorem states that an intermediate field between $\mathbb{C}(x,y)$ and $\mathbb{C}$ must be of the form $\mathbb{C}(u,v)$ or $\mathbb{C}(u)$ where u and v are algebraically independent.) Since

$$\mathbb{C}(G_1,G_2)\supset\mathbb{C}(f_1,f_2)\supset\mathbb{C},$$

the algebraic independence of G_1 and G_2 would imply that $\mathbb{C}(f_1, f_2) = \mathbb{C}(u, v)$ where u and v are algebraically independent elements of the field, or $\mathbb{C}(f_1, f_2) = \mathbb{C}(u)$ where u is a single element of the field. The first case is clearly impossible because any two elements of the field of meromorphic functions on the double are algebraically dependent (see Farkas and Kra [15, p. 248]). The second case implies that the double of Ω is the Riemann sphere, which only happens if Ω is simply connected.

Suppose that f is a proper holomorphic mapping of Ω onto the unit disc. It is proved in [7] that both $f(z)$ and $f'(z)$ can be expressed as a rational combinations of finitely many functions of z of the form $K(z, w)$ and $(\partial/\partial\bar{w})^m K(z, w)$ for values of w in the finite set $f^{-1}(0)$ (see also [11,12]). Hence both f and f' are rational combinations of G_1 and G_2. Since G_1 and G_2 satisfy a polynomial identity of the form

$$P(G_1(z), G_2(z)) \equiv 0$$

on Ω, we may state that G_2 is an algebraic function of G_1. Hence, both f and f' are algebraic functions of G_1, and hence f' must be an algebraic function of f, and it follows that f and f' must be algebraically dependent.

We now consider the simply connected case. Suppose that the Szegő kernel associated to Ω is a rational combination of only one holomorphic function G. Since the Bergman kernel is a constant times the square of the Szegő kernel, it follows that it is also a rational combination of G. Let f denote a Riemann map associated to Ω mapping Ω one-to-one onto the unit disc. Note that f extends to the double of Ω and that it generates the field of meromorphic functions on the double of Ω (see [15]).

The same proof given in [7] for proper maps shows that $f(z)$ and $f'(z)$ are both rational combinations of functions of z of the form $K(z, w)$ and $(\partial/\partial\bar{w})K(z, w)$ and $w = f^{-1}(0)$. Hence both f and f' are rational combinations of G. Hence f' must be an algebraic function of f, and it follows that f and f' must be algebraically dependent.

To finish the proofs of Theorems 2.2 and 2.2a, we now turn to the business of constructing a special Riemann surface attached to the domain in case a proper map onto the unit disc and its derivative are algebraically dependent.

5. Construction of a Riemann surface. We assume at first that Ω is a finitely connected domain in the plane with C^∞ smooth boundary. Suppose also that there exits a proper holomorphic mapping f from Ω onto the unit disc with the property that f and f' are algebraically dependent, i.e., there exists a polynomial $P(z, w)$ such that $P(f'(z), f(z)) \equiv 0$ on Ω. We now construct the Riemann surface mentioned in Theorems 2.2 and 2.2a.

Let $S_{\bar{m}}(z, a)$ denote $(\partial^m/\partial\bar{a}^m)S(z, a)$ and let $L_m(z, a)$ denote $(\partial^m/\partial a^m)L(z, a)$. Notice that we may differentiate the conjugate of (3.1) m-times with respect to $\bar{a}$ to obtain

$$S_{\bar{m}}(z, a) = i\overline{L_m(z, a)T(z)}$$

for $z \in b\Omega$ and $a \in \Omega$. When the square of this identity is combined with the identity

$$\frac{f'(z)}{f(z)}T(z) = -\overline{f'(z)T(z)/f(z)} \quad \text{for } z \in b\Omega$$

(which is obtained by differentiating $\log|f(z(t))| = 0$ with respect to t when $z(t)$ is a parameterization of a boundary curve), we see that

$$\frac{S_{\bar{m}}(z,a)^2 f(z)}{f'(z)}$$

is equal to the conjugate of

$$\frac{L_m(z,a)^2 f(z)}{f'(z)}$$

for $z \in b\Omega$ and any point a in Ω. This identity reveals that $S_{\bar{m}}(z,a)^2 f(z)/f'(z)$ extends to the double of Ω as a meromorphic function of z. Since $f(z)$ extends to the double of Ω, it follows that $S_{\bar{m}}(z,a)^2/f'(z)$ extends to the double of Ω as a meromorphic function. The algebraic dependence of f and f' means that they satisfy a polynomial equation $P(f'(z), f(z)) \equiv 0$ on Ω. Since $f(z)$ extends to the double of Ω as a meromorphic function, this polynomial identity reveals how to extend $f'(z)$ to the double of Ω as a finitely valued function with at most finitely many algebraic singularities. Furthermore, since $S_{\bar{m}}(z,a)^2/f'(z)$ extends to the double of Ω as a meromorphic function, we may state that $S_{\bar{m}}(z,a)$ extends to the double of Ω as a finitely valued function of z with at most finitely many algebraic singularities for each fixed point a in Ω.

Theorem 2.6 states that the Szegő kernel is given by a rational combination of f and finitely many functions of the form $S_{\bar{m}}(z,a_j)$. Let us call the functions in the list of finitely many functions of the form $S_{\bar{m}}(z,a_j)$ just mentioned *core functions*. We may view the core functions (from the viewpoint of Weierstrass) as being finitely valued multivalued functions that can be analytically continued to the double of Ω. There is a finite set of points E in double of Ω at which one or more of the function elements associated to the core functions has an algebraic singularity. Choose a point A_0 in $\Omega - E$ to act as a base point. We construct $\mathcal{R}$ by performing analytic continuation of each of the core functions *simultaneously*, starting at A_0 and moving all around $\widehat{\Omega}$, paying special attention to the points in E. Away from E, the lifting of germs along curves to a Riemann surface over the double of Ω is routine and obvious. When we analytically continue up to a point p in E, it may happen that none of the germs of the function elements of the core functions become singular at p. In this case, we lift and analytically continue through p without incident. If, on the other hand, at least one of the elements is singular at p, then we construct a local coordinate system at the point $\tilde{p}$ above p as follows. Consider the function elements of the core functions that are obtained as we analytically continue them up to p along a curve. Each of these elements can be viewed as a function element of a Puiseux expansion at p in a local coordinate ζ where p corresponds to the origin. Hence, for each core function $C(z)$ there is a positive integer λ such that the substitution $z = p + (\zeta)^\lambda$ makes $C(\zeta)$ analytic and continuable in ζ through $\zeta = 0$. (Note that the number λ is equal to one if $C(z)$ does not have a singularity at p.) Let m be equal to the least common multiple of all the numbers λ associated to the core functions. We can now define a local uniformizing variable that is suitable for each of the function elements in the obvious manner: $z = p + (\zeta)^m$. This coordinate function allows us to lift all the core functions so as to be defined and single valued on a disc centered at $\zeta = 0$ and we use it to define a local chart near $\tilde{p}$.

It is clear that we can identify Ω as a subdomain of the Riemann surface $\mathcal{R}$ by virtue of the fact that the core functions all have *preferred* germs in Ω given by the values they have via the definition of the kernel functions on the domain Ω. A point $\tilde{p} \in \mathcal{R}$ over $p \in \Omega$ shall be identified as the point p in $\Omega \subset \mathcal{R}$ if the germs of all the core functions are equal to their preferred germs at p. Note that there will be other sheets of $\mathcal{R}$ above Ω if it happens that one or more of the core functions continue back to $p \in \Omega$ and are not equal to their preferred germs over Ω. The Riemann surface $\mathcal{R}$ is clearly compact because of the finite valuedness of the core functions on the compact $\widehat{\Omega}$. Note that any meromorphic function on $\widehat{\Omega}$ can be defined as a meromorphic function on $\mathcal{R}$. Hence, $f(z)$ can be extended to $\mathcal{R}$ as a meromorphic function. Since the core functions and f extend to $\mathcal{R}$, it follows that the Szegő kernel extends to be meromorphic on $\mathcal{R} \times \mathcal{R}$. Since the Bergman kernel is a rational combination of three functions of z of the form $S(z, A)$, it follows that the Bergman kernel extends to be meromorphic on $\mathcal{R} \times \mathcal{R}$. Similarly, so do the functions F_j'.

To see that the complement of Ω in $\mathcal{R}$ is connected, we can follow exactly the same pole counting procedure given in [10] where a similar Riemann surface is attached to a domain with algebraic kernel functions. This procedure yields that each boundary component of Ω is attached to the Riemann surface exactly once and that the complement of Ω in $\mathcal{R}$ is connected.

Let G_1 and G_2 denote a primitive pair for $\mathcal{R}$. Theorem 2.6 states that the Szegő kernel is a rational combination of $f(z)$ and the core functions. Since $f(z)$ and the core functions extend to be meromorphic functions on $\mathcal{R}$, we may now state that the Szegő kernel is a rational combination of G_1 and G_2. Since the Bergman kernel associated to Ω and all the other functions mentioned in Theorem 2.2 are rational combinations of functions of z of the form $S(z, A)$ for three fixed points A in Ω, they too are rational combinations of G_1 and G_2. The proof of Theorem 2.2 is complete in case Ω has smooth boundary.

If Ω does not have smooth boundary, we shall use the identity

$$P(f(z), f'(z)) \equiv 0, \tag{5.1}$$

which is assumed to hold for a proper holomorphic mapping $f(z)$ of Ω onto the unit disc, to show that the boundary of Ω must be given by piecewise C^∞ smooth real analytic curves. Let $F(w)$ denote a local inverse defined on a small open subset of $D_1(0)$ to the proper holomorphic map $f(z)$. If we replace z by $F(w)$ in (5.1) we obtain $P(w, 1/F'(w)) \equiv 0$, and this shows that $F'(w)$ is an algebraic function. Hence F can be analytically continued past the boundary of the unit disc except at possibly finitely many points where F' has an algebraic singularity. Since the continuation of f maps the boundary of the unit disc into the boundary of Ω, it is easy to deduce that the boundary of Ω is given by piecewise real analytic curves. (Only minor complications are introduced if the point at infinity is in the boundary of Ω and these are bypassed by standard arguments.) Now the same construction of $\mathcal{R}$ we used above can be carried out and the pole counting argument of [10] given in the case of piecewise real analytic boundary applies to yield that the complement of Ω in $\mathcal{R}$ is connected. The rest of the proof is the same.

6. The proofs of Theorems 2.3 and 2.5. We shall see momentarily that we will be able to reduce our problem to the case where Ω has C^∞ smooth real analytic boundary. The following lemmas will allow us to see that the two classes $\mathcal{A}$ and $\mathcal{B}$

are natural and that they are the largest classes of functions that can appear in the statements of Theorems 2.3 and 2.5.

Lemma 6.1. *Suppose that Ω is a finitely connected domain in the plane with C^∞ smooth real analytic boundary. On such a domain, the class $\mathcal{B}$ is equal to the set of meromorphic functions G on $\overline{\Omega}$ that satisfy an identity of the form*

$$G(z) = \overline{H(z)T(z)} \tag{6.1}$$

for $z \in b\Omega$, where $H(z)$ is another meromorphic function on $\overline{\Omega}$.

When we deal with the class $\mathcal{A}$ we shall need the following lemma.

Lemma 6.2. *Suppose that Ω is a finitely connected domain in the plane with C^∞ smooth real analytic boundary. On such a domain, the class $\mathcal{A}$ is equal to the set of meromorphic functions G on $\overline{\Omega}$ that satisfy an identity of the form*

$$G(z)T(z) = \overline{H(z)T(z)} \tag{6.2}$$

for $z \in b\Omega$, where $H(z)$ is another meromorphic function on $\overline{\Omega}$.

These two lemmas have the important consequence that if g_1/g_2 is a quotient of two functions in the class $\mathcal{A}$ (or two functions in the class $\mathcal{B}$) where $g_2 \not\equiv 0$, then g_1/g_2 extends to the double of Ω as a meromorphic function. Indeed, this fact follows directly from the identity of the form $g_1(z)/g_2(z) = \overline{h_1(z)/h_2(z)}$ on $b\Omega$ that would be satisfied by the quotient of two such functions.

Proof of Lemma 6.1. Notice that (3.1) shows that $L(z,a)$ and $S(z,a)$ satisfy the similar identities:

$$S(z,a) = \overline{-iL(z,a)T(z)}$$
$$L(z,a) = \overline{-iS(z,a)T(z)}.$$

Furthermore, these identities can be differentiated with respect to a or $\bar{a}$.

$$S_{\bar{m}}(z,a) = \overline{-iL_m(z,a)T(z)}$$
$$L_m(z,a) = \overline{-iS_{\bar{m}}(z,a)T(z)}.$$

Hence, functions in the class $\mathcal{B}$ satisfy the condition of formula (6.1).

Now suppose that G and H are meromorphic functions on $\overline{\Omega}$ that satisfy

$$G(z) = \overline{H(z)T(z)}$$

for $z \in b\Omega$. When a is in Ω, we note that $L(z,a)$ has a single simple pole at a as a function of z and $S(z,a)$ has no singularities in $\overline{\Omega}$, and $L_m(z,a)$ has a single pole of order $m+1$ at a and $S_{\bar{m}}(z,a)$ has no singularities in $\overline{\Omega}$. Hence, we may subtract linear combinations of the identities in the paragraph above from (6.1) designed to remove the singularities of G and H in Ω from both sides of the equation. In the end, we obtain an identity of the form

$$G_1(z) = \overline{H_1(z)T(z)}$$

where G_1 and H_1 are holomorphic on Ω with possibly finitely many poles on $b\Omega$. When a is in the boundary, the function $S(z,a)$ has a single simple pole at a in $\overline{\Omega}$ and $S_{\bar{m}}(z,a)$ has a single pole of order $m+1$ at a in $\overline{\Omega}$. Hence, we can subtract linear combinations of $S(z,a)$ and $S_{\bar{m}}(z,a)$ from G_1 to eliminate the poles of G_1 in $\overline{\Omega}$ and subtract the corresponding linear combinations of $-iL(z,a)$ and $-iL_m(z,a)$ from H_1 to obtain an identity of the form

$$g(z) = \overline{h(z)T(z)}$$

where g is holomorphic on $\overline{\Omega}$ and h is holomorphic on Ω with possibly finitely many poles on $b\Omega$. However, the identity $g(z) = \overline{h(z)T(z)}$ on $b\Omega$ shows that h does not tend to infinity at any point on the boundary, so h actually has no poles on $\overline{\Omega}$. Hence, both g and h are holomorphic on $\overline{\Omega}$. Now we can conclude from the fact that $\overline{h(z)T(z)}$ is orthogonal to holomorphic functions in $L^2(b\Omega)$ that g and h must both be zero. The proof of our claim is complete.

Proof of Lemma 6.2. The three identities

$$\begin{aligned} F_j'(z)T(z) &= -\overline{F_j'(z)T(z)} \\ G_z(z,a)T(z) &= -\overline{G_z(z,a)T(z)} \\ S(z,a_1)S(z,a_2)T(z) &= -\overline{L(z,a_1)L(z,a_2)T(z)} \end{aligned}$$

for $z \in b\Omega$ (see [4, pages 80, 134], (3.3), (3.4), and (3.5)) reveal that functions in the class $\mathcal{A}$ satisfy the condition of formula (6.2).

Now suppose that G and H are meromorphic functions on $\overline{\Omega}$ that satisfy

$$G(z)T(z) = \overline{H(z)T(z)}$$

for $z \in b\Omega$. Note that the function $G_z(z,a)$ has a simple pole at a even when a is in $b\Omega$. Differentiate the identity $G_z(z,a)T(z) = -\overline{G_z(z,a)T(z)}$ ($z \in b\Omega$) m times with respect to a. The function $(\partial^m/\partial a^m)G_z(z,a)$ on the left hand side of the formula is a meromorphic function of z on $\overline{\Omega}$ with a single pole of order $m+1$ at $z=a$. This is true even if $a \in b\Omega$. When $a \in \Omega$, the function $(\partial^m/\partial \bar{a}^m)G_z(z,a)$ on the right hand side of the formula is a holomorphic function of z on $\overline{\Omega}$ with no singularity at $z=a$. If we differentiate the same formula m times with respect to $\bar{a}$, then the function $(\partial^m/\partial \bar{a}^m)G_z(z,a)$ on the left hand side of the formula is a holomorphic function of z on $\overline{\Omega}$ with no singularity at $z=a$ when $a \in \Omega$ and the function $(\partial^m/\partial a^m)G_z(z,a)$ on the right hand side of the formula is a meromorphic function of z on $\overline{\Omega}$ with a single pole of order $m+1$ at $z=a$. Hence, it is possible to subtract linear combinations of functions in the class $\mathcal{A}$ from the identity $GT = \overline{HT}$ to remove all the poles of $G(z)$ in $\overline{\Omega}$ and all the poles of H of order two or more in Ω to obtain an identity of the form

$$g(z)T(z) = \overline{h(z)T(z)}$$

where g is holomorphic on $\overline{\Omega}$ and h is meromorphic on $\overline{\Omega}$ with only finitely many simple poles in Ω and finitely many other poles on $b\Omega$. However, none of the poles of h can actually occur on the boundary because the identity $gT = \overline{hT}$ shows that h cannot blow up there. Let $\{b_j\}_{j=1}^N$ denote the simple poles of h in Ω and choose a point a in Ω distinct from these points. The functions $S(z,b_j)S(z,a)$ are holomorphic on $\overline{\Omega}$ and satisfy the identity

$$S(z,b_j)S(z,a)T(z) = -\overline{L(z,b_j)L(z,a)T(z)}$$

for $z \in b\Omega$. Since $L(z,b_j)L(z,a)T(z))$ has a simple pole at b_j, we may subtract a linear combination of these identities from $g(z)T(z) = \overline{h(z)T(z)}$ to eliminate the poles of h at the points b_j. We obtain

$$g_1(z)T(z) = \overline{h_1(z)T(z)}$$

where g_1 is holomorphic on $\overline{\Omega}$ and h_1 is meromorphic on $\overline{\Omega}$ with possibly a single simple pole at a. However, if $g_1(z)T(z) = \overline{h_1(z)T(z)}$ is integrated around the boundary with respect to arc length measure, the left hand side is zero by Cauchy's Theorem and the right hand side is equal to $2\pi i$ times the residue of h_1 at a. So, in fact, h_1 has no poles in $\overline{\Omega}$. It is shown in [4, page 80] that holomorphic functions g_1 and h_1 that satisfy $g_1(z)T(z) = \overline{h_1(z)T(z)}$ on the boundary must be linear combinations of the functions F_j', and this shows that G is in the class $\mathcal{A}$ and the proof of our claim is complete.

Recall that the classes $\mathcal{A}$ and $\mathcal{B}$ were defined differently for domains with smooth boundary than for domains without. We now let $\mathcal{A}$ and $\mathcal{B}$ denote the classes defined by restricting the points a in the definitions of the classes $\mathcal{A}$ and $\mathcal{B}$ to be in Ω. If Ω has smooth boundary, then we let $\mathcal{A}^+$ and $\mathcal{B}^+$ denote the classes obtained by letting the points a in the definitions also fall on the boundary.

Lemma 6.3. *Suppose that $\Phi : \Omega \to \Omega_a$ is a biholomorphic mapping between a domain Ω and a finitely connected domain Ω_a with C^∞ smooth real analytic boundary such that no boundary component is a point. The transformation*

$$h \mapsto \Phi'(z)h(\Phi(z))$$

is a one-to-one linear map of the class $\mathcal{A}$ associated to Ω_a onto the class $\mathcal{A}$ associated to Ω. If $b\Omega$ is C^∞ smooth, then this transformation is also a one-to-one linear map of the class $\mathcal{A}^+$ associated to Ω_a onto the class $\mathcal{A}^+$ associated to Ω. The transformation

$$h \mapsto \sqrt{\Phi'(z)}h(\Phi(z))$$

is a one-to-one linear map of the class $\mathcal{B}$ associated to Ω_a onto the class $\mathcal{B}$ associated to Ω. If $b\Omega$ is C^∞ smooth, then this transformation is also a one-to-one linear map of the class $\mathcal{B}^+$ associated to Ω_a onto the class $\mathcal{B}^+$ associated to Ω.

The proof of Lemma 6.3 is straightforward and uses only the well known transformation properties of the functions that generate the two classes. We omit the proof.

Proof of Theorem 2.5. Assume for the moment that Ω has smooth real analytic boundary. Fix a point a in Ω so that the zeroes $a_1, \dots, a_{n-1}$ of $S(z,a)$ are distinct simple zeroes. I proved in [4, Theorem 3.1] that the Szegő kernel can be expressed via

$$S(z,w) = \frac{1}{1 - f_a(z)\overline{f_a(w)}} \left(c_0 S(z,a)\overline{S(w,a)} + \sum_{i,j=1}^{n-1} c_{ij} S(z,a_i)\,\overline{S(w,a_j)} \right) \tag{6.3}$$

where $f_a(z)$ denotes the Ahlfors map associated to (Ω, a), $c_0 = 1/S(a,a)$, and the coefficients c_{ij} are given as the coefficients of the inverse matrix to the invertible matrix $[S(a_j, a_k)]$. This shows that

$$S(z,w)(1 - f_a(z)\overline{f_a(w)})$$

is a linear combination of functions of the form $g(z)\overline{h(w)}$ where g and h are in the class $\mathcal{B}$. Let $B(z)$ be any non-zero function in the class $\mathcal{B}$. The remark after the statement of Lemma 6.2 asserts that the quotient of any two functions in the class $\mathcal{B}$ extends to the double of Ω as a meromorphic function. It follows that

$$\frac{S(z,w)(1 - f_a(z)\overline{f_a(w)})}{B(z)\overline{B(w)}} \tag{6.4}$$

is equal to a linear combination of functions of the form $g(z)/B(z)$ times the conjugate of $h(w)/B(w)$, and because these quotients are meromorphic on the double, they can be expressed as rational functions of the primitive pair. The Ahlfors map itself extends to the double as a meromorphic function, and hence the claim about the Szegő kernel is proved.

In case Ω does not have smooth real analytic boundary, we use the well known fact that there exists a biholomorphic map Φ from Ω onto a bounded domain Ω_a with smooth real analytic boundary. The transformation formulas for the Szegő kernel, the Ahlfors maps, and the functions in the class $\mathcal{B}$ reveal that formula (6.4) transforms by simple composition with Φ, i.e., the terms involving Φ' cancel. Similarly, the linear combination of terms of the form $g(z)/B(z)$ times the conjugate of $h(w)/B(w)$ where g and h are in the class $\mathcal{B}$ enjoy the same property, and these extend to the double as meromorphic functions. The proof is complete.

Proof of Theorem 2.3. Assume for the moment that Ω has smooth real analytic boundary. The Bergman kernel $K(z,w)$ is related to the Szegő kernel via the identity

$$K(z,w) = 4\pi S(z,w)^2 + \sum_{i,j=1}^{n-1} A_{ij} F_i'(z)\overline{F_j'(w)}, \tag{6.5}$$

Notice that formula (6.3) and Lemma 6.2 yield that $S(z,w)^2$ is a linear combination of functions in the class $\mathcal{A}$ divided by $(1 - f_a(z)\overline{f_a(w)})^2$. Hence, if we divide (6.5) by $A(z)\overline{A(w)}$ where $A(z)$ is a non-zero function in the class $\mathcal{A}$, then we see that

$$\frac{K(z,w)}{A(z)\overline{A(w)}}$$

is a linear combination of functions of the form $g(z)/A(z)$ times conjugates of $h(w)/A(w)$ times either constants or $(1 - f_a(z)\overline{f_a(w)})^{-2}$, where g and h are in the class $\mathcal{A}$. Since all these functions of z and w extend to the double of Ω as meromorphic functions, Theorem 2.3 follows. Now the same argument given in the last paragraph of the proof of Theorem 2.5 shows that we can drop the hypothesis that the boundary of Ω be smooth and real analytic. The proof is complete.

7. The Szegő kernel with weights. Suppose that Ω is a bounded n-connected domain in the plane with C^∞ smooth boundary, i.e., a domain whose boundary $b\Omega$ is given by finitely many non-intersecting C^∞ simple closed curves. Recall that $L^2(b\Omega)$ denotes the space of complex valued functions on $b\Omega$ that are square integrable with respect to arc length measure ds. Given a positive real valued C^∞ function φ on the boundary of Ω, let $L^2_\varphi(b\Omega)$ denote the space of complex valued functions on $b\Omega$ that are square integrable with respect to $\varphi(s)ds$. The Hardy space of functions in $L^2(b\Omega)$ that are the L^2 boundary values of holomorphic functions on Ω shall be written $H^2(b\Omega)$. This space is equal to the closure in $L^2(b\Omega)$ of $A^\infty(\Omega)$ (see [4]) and can be identified in a natural way with the subspace of $L^2_\varphi(b\Omega)$ equal to the closure of $A^\infty(\Omega)$ in that space. Thus, we need not define $H^2_\varphi(b\Omega)$ separately.

The inner products associated to $L^2(b\Omega)$ and $L^2_\varphi(b\Omega)$ shall be written

$$\langle u, v\rangle = \int_{b\Omega} u\, \bar{v}\, ds, \quad \text{and} \quad \langle u, v\rangle_\varphi = \int_{b\Omega} u(s)\, \overline{v(s)}\, \varphi(s) ds,$$

respectively. We let $S(z,a)$ denote the classical Szegő kernel associated to the classical Szegő projection P, which is the orthogonal projection of $L^2(b\Omega)$ onto $H^2(b\Omega)$, and we let $\sigma(z,w)$ denote the kernel associated to the orthogonal projection P_φ of the weighted space $L^2_\varphi(b\Omega)$ onto $H^2(b\Omega)$. The arguments presented in [4] showing that P maps $C^\infty(b\Omega)$ into itself can easily be modified to show that P_φ also maps $C^\infty(b\Omega)$ into itself. Also, if the boundary curves of Ω are C^∞ smooth *real analytic* curves and φ is real analytic on $b\Omega$, then P_φ maps real analytic functions on $b\Omega$ into the space of holomorphic functions on Ω that extend to be holomorphic on a neighborhood of $\overline{\Omega}$ (see [4, page 41]).

Any function v in the subspace $H^2(b\Omega)^\perp$ of $L^2(b\Omega)$ (which is the orthogonal complement of $H^2(b\Omega)$ in $L^2(b\Omega)$) can be written

$$v = \overline{HT},$$

for a unique H in $H^2(b\Omega)$ (see [4, p. 13]). Consequently, every function u in $L^2(b\Omega)$ has an orthogonal decomposition of the form

$$u = h + \overline{HT},$$

where $h \in H^2(b\Omega)$ and $H \in H^2(b\Omega)$. We can easily deduce from this fact that every function u in $L^2_\varphi(b\Omega)$ has an orthogonal decomposition of the form

$$u = h + \varphi^{-1}\overline{HT},$$

where $h \in H^2(b\Omega)$ and $H \in H^2(b\Omega)$. Indeed, let $h = P_\varphi u$. Then $u-h$ is orthogonal to $H^2(b\Omega)$ with respect to the weighted inner product. Hence $(u-h)\varphi$ is orthogonal to $H^2(b\Omega)$ with respect to the standard inner product on $L^2(b\Omega)$. Hence $(u-h)\varphi = \overline{HT}$ for a unique $H \in H^2(b\Omega)$ and our claim is proved. This orthogonal decomposition allows us to define a *weighted Garabedian kernel* as follows. The Cauchy integral formula reveals that the *weighted Cauchy kernel* $C_a(z)$, which is given as the conjugate of

$$\frac{1}{2\pi i}\frac{\varphi(z)^{-1}T(z)}{z-a},$$

reproduces holomorphic functions with respect to the weighted inner product in the sense that

$$h(a) = \langle h, C_a \rangle_\varphi.$$

Hence, it follows that

$$\sigma(z, a) = P_\varphi C_a$$

and the orthogonal decomposition for C_a is

$$C_a(z) = \sigma(z, a) + \varphi^{-1}\overline{H_a T},$$

where H_a is in $H^2(b\Omega)$. By analogy with the definition of the classical Garabedian kernel (see [4, p. 24]), we define the weighted Garabedian kernel $\lambda(z, a)$ to be given by

$$\lambda(z, a) = \frac{1}{2\pi}\frac{1}{z-a} - iH_a(z).$$

Notice that $\sigma(z, a)$ and $\lambda(z, a)$ satisfy the identity

$$\overline{\sigma(z, a)} = \frac{1}{i\varphi(z)}\lambda(z, a)T(z) \tag{7.1}$$

for $z \in b\Omega$ and $a \in \Omega$. Since $\sigma(z, a) = P_\varphi C_a$, it follows that $\sigma(z, a)$ is in $A^\infty(\Omega)$ as a function of z for each fixed $a \in \Omega$ if φ is C^∞ smooth. Furthermore, for a fixed point $a \in \Omega$, $\lambda(z, a)$ is holomorphic function of z on $\Omega - \{a\}$ with a simple pole at a with residue $1/(2\pi)$ and $\lambda(z, a)$ extends C^∞ smoothly to $b\Omega$ if φ is C^∞ smooth. In case the boundary of Ω is real analytic and φ is real analytic on $b\Omega$, both $\sigma(z, a)$ and $\lambda(z, a)$ extend holomorphically past the boundary in z for each fixed a in Ω. We record here for future use the derivative

$$\frac{\partial^n}{\partial a^n}\sigma(a, z) = \frac{1}{i\varphi(z)}\frac{\partial^n}{\partial a^n}\lambda(z, a)T(z) \tag{7.2}$$

of (7.1) with respect to a.

The weighted Szegő kernel reproduces holomorphic functions with respect to the weighted inner product in the sense that

$$h(a) = \langle h, \sigma(\cdot, a)\rangle_\varphi$$

for $h \in H^2(b\Omega)$. This last identity may be differentiated with respect to a to yield that

$$h^{(n)}(a) = \langle h, (\partial^n/\partial \bar{a}^n)\sigma(\cdot, a)\rangle_\varphi$$

for $h \in H^2(b\Omega)$. The weighted Szegő kernel satisfies $\sigma(z, a) = \overline{\sigma(a, z)}$.

The conjugate of formula (7.1) is

$$\frac{i}{\varphi(z)}\overline{\lambda(z, a)} = \sigma(z, a)T(z)$$

If we multiply this by (7.1) and divide out the $1/\varphi$ factors, we obtain

$$\sigma(z, a)\lambda(z, a)T(z) = -\overline{\lambda(z, a)\sigma(z, a)T(z)} \quad \text{for } z \in b\Omega$$

and $a \in \Omega$. Hence, it follows from Lemma 6.2 that $\sigma(z,a)\lambda(z,a)$ is in the class $\mathcal{A}$ and hence, even though the weight function may be quite arbitrary, it is possible to relate these kernels to standard objects of potential theory. For example, similar arguments to those used in [8] can be used to show that the Green's function can be expressed via

$$\frac{\partial}{\partial z}G(z,w) = \pi\frac{\sigma(z,w)\lambda(z,w)}{\sigma(w,w)} + \sum_{j=1}^{n-1} c_j(w)F_j'(z)$$

where the functions $c_j(w)$ can be easily determined as in §5 of [8] to be given by a linear combination of $n-1$ explicit harmonic functions and finitely many other functions that are rational combinations of the basic holomorphic functions that comprise σ and λ.

8. An orthonormal basis for the weighted Hardy space. We suppose, as we did in §7, that Ω is a bounded n-connected domain in the plane with C^∞ smooth boundary and that φ is a real valued positive C^∞ weight function on $b\Omega$. Suppose that $f : \Omega \to D_1(0)$ is a proper holomorphic mapping of Ω into the unit disc. Such a mapping extends C^∞ smoothly to the boundary and is a finite branched covering map of some finite order M. Let $a_1, \ldots, a_N$ denote the zeroes of f in Ω and let $M(k)$ denote the multiplicity of the zero of f at a_k. Of course, $M = \sum_{k=1}^N M(k)$. Notice that, because $|f(z)| = 1$ for z in $b\Omega$, it follows that $f(z) = 1/\overline{f(z)}$ for z in $b\Omega$.

Let $\sigma_0(z,w)$ denote $\sigma(z,w)$, let $\sigma_{\bar n}(z,w)$ denote $(\partial^n/\partial\bar w^n)\sigma(z,w)$, and let

$$\sigma_{m\bar n}(z,w) := \frac{\partial^{m+n}}{\partial z^m \partial \bar w^n}\sigma(z,w).$$

We shall now prove that the set of functions

$$h_{inp}(z) = \sigma_{\bar n}(z,a_i)f(z)^p,$$

where $1 \le i \le N$, $0 \le n \le M(i)$, and $p \ge 0$, forms a basis for the Hardy space $H^2(b\Omega)$ and that

$$\langle h_{inp}, h_{jmq}\rangle_\varphi = \begin{cases} 0, & \text{if } p \neq q, \\ \sigma_{m\bar n}(a_j,a_i), & \text{if } p = q. \end{cases} \tag{8.1}$$

First, we must show that the set of functions above spans a dense subset of $H^2(b\Omega)$. Indeed, suppose that $g \in H^2(b\Omega)$ is orthogonal to the span. Notice that the reproducing property of the weighted Szegő kernel yields that

$$0 = \langle g, \sigma_{\bar n}(\cdot, a_j)\rangle_\varphi = g^{(n)}(a_j)$$

for $0 \le n \le M(j)$, and therefore g vanishes at $a_1, \ldots, a_N$ to the same order that f does. Suppose we have shown that g vanishes to order m times the order that f vanishes at each a_j, $j = 1, \ldots, N$. It follows that g/f^m has removable singularities at each a_j and so it can be viewed as an element of $H^2(b\Omega)$. We shall now show that

g/f^m must vanish to the same order at each a_j that f does. Since $1/f(z) = \overline{f(z)}$ when $z \in b\Omega$, we may write

$$0 = \langle g, \sigma_{\bar{n}}(\cdot, a_j) f^m \rangle_\varphi = \langle g/f^m, \sigma_{\bar{n}}(\cdot, a_j) \rangle_\varphi$$

and this last quantity is equal to the n-th derivative of g/f^m at a_j. Since this is zero for $0 \le n \le M(j)$, we conclude that g/f^m vanishes to the same order that f does at each a_j. Hence, g vanishes to order $m+1$ times the order that f vanishes at each a_j, $j = 1, \dots, N$. By induction, we may conclude that g vanishes to infinite order at each a_j and hence, $g \equiv 0$. This proves the density.

We now turn to the proof of (8.1). We may suppose that $p \ge q$. The fact that $\overline{f} = 1/f$ on $b\Omega$ yields that

$$\langle h_{inp}, h_{jmq} \rangle_\varphi = \langle \sigma_{\bar{n}}(z, a_i) f(z)^{p-q}, \sigma_{\bar{m}}(z, a_j) \rangle_\varphi .$$

The reproducing property of the weighted Szegő kernel yields that this last last inner product is equal to

$$\frac{\partial^m}{\partial z^m} \left[\sigma_{\bar{n}}(z, a_i) f(z)^{p-q} \right]$$

evaluated at $z = a_j$. Since the multiplicity of the zero of f at a_j is greater than or equal to m, this quantity is zero if $p > q$. If $p = q$, then the $f(z)$ term is not present and the proof of identity (8.1) complete.

It is now easy to see that the functions h_{inp} are linearly independent. Indeed, identity (8.1) reveals that we need only check that, for fixed p, the functions h_{inp}, $i = 1, \dots, N$, $n = 0, \dots, M(i)$, are linearly independent, and this is true because a relation of the form

$$\sum_{i=1}^{N} \sum_{n=0}^{M(i)} C_{in} \sigma_{\bar{n}}(z, a_i) \equiv 0$$

implies, via the reproducing property of the weighted Szegő kernel, that every function g in the Hardy space satisfies

$$\sum_{i=1}^{N} \sum_{n=0}^{M(i)} \overline{C_{in}} g^{(n)}(a_i) = 0,$$

and it is easy to construct polynomials g that violate such a condition.

We next orthonormalize the sequence $\{h_{inp}\}$ via the Gram-Schmidt procedure. Formula (8.1) shows that we need only orthonormalize the functions h_{inp}, $i = 1, \dots, N$, $n = 0, \dots, M(i)$ for each fixed p. If we are careful to perform the Gram-Schmidt procedure in exactly the same order with respect to the indices i and n at each level p, we obtain an orthonormal set $\{H_{inp}\}$ which is related to our original set via a formula,

$$H_{inp}(z) = \sum_{j=1}^{N} \sum_{m=0}^{M(j)} b_{ijnm} h_{jmp},$$

where, because $|f| = 1$ on $b\Omega$, *the coefficients* b_{ijnm} *do not depend on* p. This last fact is critical in what follows.

The weighted Szegő kernel can be written in terms of our orthonormal basis via

$$\sigma(z,w)=\sum_{p=0}^{\infty}\sum_{i=1}^{N}\sum_{n=1}^{M(i)} H_{inp}(z)\,\overline{H_{inp}(w)}.$$

The geometric sum

$$\sum_{p=0}^{\infty} f(z)^p\,\overline{f(w)^p}=\frac{1}{1-f(z)\overline{f(w)}}$$

can be factored from the expression for $\sigma(z,w)$ to yield a formula like the one in the following theorem. (Note that by taking the weight function to be identically one, we obtain a proof of Theorem 2.6.)

Theorem 8.1. *Suppose that f is a proper holomorphic mapping of Ω onto the unit disc with zeroes at $a_1,\dots,a_N$ with multiplicities $M(1),\dots,M(N)$, respectively. The weighted Szegő kernel $\sigma(z,w)$ satisfies*

$$\sigma(z,w)=\frac{1}{1-f(z)\overline{f(w)}}\left(\sum_{i,j=1}^{N}\sum_{n=0}^{M(i)}\sum_{m=0}^{M(j)} c_{ijnm}\sigma_{\bar n}(z,a_i)\,\overline{\sigma_{\bar m}(w,a_j)}\right). \tag{8.2}$$

The coefficients c_{ijnm} can easily be determined. Suppose $1\le k\le N$ and $0\le q\le M(k)$. Differentiate (8.2) q times with respect to $\bar w$ and set $w=a_k$ (and recall that f has a zero of multiplicity $M(k)$ at a_k) to obtain

$$\sigma_{\bar m}(z,a_k)=\sum_{i,j=1}^{N}\sum_{n=0}^{M(i)}\sum_{m=0}^{M(j)} c_{ijnm}\sigma_{\bar n}(z,a_i)\,\overline{\sigma_{q\bar m}(a_k,a_j)}$$

We saw an identity like this when we showed above that the functions h_{jnp} are linearly independent for each fixed k. The same reasoning we used there yields that such a relation can only be true if the system,

$$\sum_{j=1}^{N}\sum_{m=0}^{M(j)} c_{ijnm}\overline{\sigma_{q\bar m}(a_k,a_j)}=\begin{cases}1, & \text{if } i=k \text{ and } m=q,\\ 0, & \text{if } i\neq k \text{ or } m\neq q,\end{cases}$$

has a unique solution. This gives us a non-degenerate linear system to solve for the coefficients c_{ijnm}.

In the case that the proper holomorphic mapping f in Theorem 8.1 has simple zeroes, the formula for the weighted Szegő kernel becomes easier to write.

Theorem 8.2. *Suppose that f is a proper holomorphic mapping of Ω onto the unit disc with simple zeroes at $a_1,\dots,a_N$. The weighted Szegő kernel $\sigma(z,w)$ satisfies*

$$\sigma(z,w)=\frac{1}{1-f(z)\overline{f(w)}}\sum_{i,j=1}^{N} c_{ij}\sigma(z,a_i)\,\overline{\sigma(w,a_j)} \tag{8.3}$$

where the coefficients c_{ij} are determined by the condition that the matrix formed by the coefficients $[c_{ij}]$ is the inverse to the matrix $[\sigma(a_k,a_j)]$.

We remark that for any proper holomorphic mapping f from Ω onto the unit disc, it is always possible to choose a Möbius transformation φ so that the proper map $\varphi \circ f$ has simple zeroes. Hence the simpler formula in Theorem 8.2 is always at our disposal.

We can use identity (8.3) and (7.1) to derive similar results for the weighted Garabedian kernel. Assume that w is in $b\Omega$ and multiply (8.3) by $i\varphi(w)\overline{T(w)}$. Use (7.1) to obtain

$$\lambda(w,z) = \frac{1}{1 - f(z)\overline{f(w)}} \sum_{i,j=1}^{N} c_{ij}\sigma(z,a_i)\,\lambda(w,a_j).$$

Finally, replace $\overline{f(w)}$ by $1/f(w)$ to obtain

$$\lambda(w,z) = \frac{f(w)}{f(w) - f(z)} \sum_{i,j=1}^{N} c_{ij}\sigma(z,a_i)\,\lambda(w,a_j).$$

Since both sides of this identity are holomorphic in w, the identity extends to hold for all w in Ω minus the finite set of points $\{a_1,\ldots,a_N\}$ and z. The singularities at $\{a_1,\ldots,a_N\}$ are easily seen to be removable because f vanishes at these points.

Theorem 8.3. *Suppose that f is a proper holomorphic mapping of Ω onto the unit disc with simple zeroes at $a_1,\ldots,a_N$. The weighted Garabedian kernel $\lambda(z,w)$ satisfies*

$$\lambda(w,z) = \frac{f(w)}{f(w) - f(z)} \sum_{i,j=1}^{N} c_{ij}\sigma(z,a_i)\,\lambda(w,a_j) \tag{8.4}$$

where the coefficients c_{ij} are determined by the condition that the matrix formed by the coefficients $[c_{ij}]$ is the inverse to the matrix $[\sigma(a_k,a_j)]$.

9. The weighted Szegő kernel and the double of a domain. Suppose that Ω is an n-connected domain in the plane such that no boundary component is a point and assume further that the boundary of Ω consists of n non-intersecting C^∞ smooth closed curves. As before, we let $\widehat{\Omega}$ denote the double of Ω and $R(z)$ denote the antiholomorphic involution on $\widehat{\Omega}$ which fixes the boundary of Ω and let $\widetilde{\Omega} = R(\Omega)$ denote the reflection of Ω in $\widehat{\Omega}$ across the boundary. Recall that if $f : \Omega \to D_1(0)$ is a proper holomorphic mapping of Ω onto the unit disc, then f extends to be a meromorphic function on $\widehat{\Omega}$.

We shall now prove that for fixed points A_1 and A_0 in Ω, functions of z of the form $\sigma(z,A_1)/\sigma(z,A_0)$ extend as meromorphic functions to the double of Ω. Indeed, if we write the conjugate of formula (7.1), first using $a = A_1$ and then $a = A_0$, and divide the two resulting formulas, we see that $\sigma(z,A_1)/\sigma(z,A_0)$ is equal to the complex conjugate of $\lambda(z,A_1)/\lambda(z,A_0)$ when $z \in b\Omega$. The arguments in [13] can easily be adapted to show that functions of z of the form $\sigma(z,a)$ cannot vanish to infinite order at a boundary point for fixed points a in Ω. Hence, the zeroes of $\sigma(z,A_1)$ and $\sigma(z,A_0)$ on $b\Omega$ are isolated and of finite order. Now the usual reflection argument (which maps a one sided neighborhood of a boundary point of Ω onto the lower half disc) in the construction of the double of Ω shows that $\sigma(z,A_1)/\sigma(z,A_0)$ is a meromorphic function on Ω which extends up to $b\Omega$ with at most finitely many pole-like singularities on $b\Omega$, and the conjugate of $\lambda(R(z),A_1)/\lambda(R(z),A_0)$ is a meromorphic function on $\widetilde{\Omega}$ which extends up to $b\Omega$ from the "outside" of Ω and which agrees with $\sigma(z, A_1)/\sigma(z, A_0)$ on $b\Omega$. Hence, $\sigma(z, A_1)/\sigma(z, A_0)$ extends to $\widehat{\Omega}$ as a meromorphic function. Similar reasoning using (7.1) and (7.2) shows that for fixed points A_1 and A_0 in Ω, functions of z of the form $\sigma_{\bar{n}}(z, A_1)/\sigma(z, A_0)$ extend as meromorphic functions to the double of Ω.

If we fix a point b in Ω and if we divide (8.2) by $\sigma(z, b)\sigma(b, w)$, we see that

$$\frac{\sigma(z, w)}{\sigma(z, b)\sigma(b, w)}$$

is a rational combination of holomorphic functions of z that extend meromorphically to the double of Ω and antiholomorphic functions of w that extend antimeromorphically to the double of Ω. Since the field of meromorphic functions on the double of Ω is generated by just two such functions (a primitive pair), we obtain the following result about the complexity of $\sigma(z, w)$.

Theorem 9.1. *Suppose that Ω is an n-connected domain in the plane such that the boundary of Ω consists of n non-intersecting analytic C^∞ smooth closed curves. Let G_1 and G_2 be a primitive pair for the field of meromorphic functions on the double of Ω. Given any point b in Ω, the weighted Szegő kernel is given by*

$$\sigma(z, w) = \sigma(z, b)\sigma(b, w)R(G_1(z), G_2(z), \overline{G_1(w)}, \overline{G_2(w)}),$$

and thus is a rational combination of only three functions of one complex variable on Ω.

We showed in §8 that it is not possible for a function of the form

$$\sigma(z, A_1) - c\sigma(z, A_0)$$

to vanish identically on Ω. (If it did, then every polynomial $p(z)$ would satisfy $p(A_1) - \bar{c}p(A_0) = 0$ by virtue of the reproducing property of the weighted Szegő kernel, and this is absurd.) Hence, if $A_1 \neq A_0$, then $\sigma(z, A_1)/\sigma(z, A_0)$ extends to $\widehat{\Omega}$ as a non-constant meromorphic function of some finite order m on $\widehat{\Omega}$. We record this result here for future use.

Theorem 9.2. *Suppose that Ω is an n-connected domain in the plane such that the boundary of Ω consists of n non-intersecting C^∞ smooth closed curves. If A_1 and A_0 are distinct points in Ω, then $\sigma(z, A_1)/\sigma(z, A_0)$ extends to the double of Ω as a non-constant meromorphic function.*

10. A special weight function for the Szegő kernel. Assume, as we did in §9, that Ω is an n-connected domain in the plane such that the boundary of Ω consists of n non-intersecting C^∞ smooth closed curves. Let $p(a, z)$ denote the classical Poisson kernel associated to Ω which reproduces harmonic functions in the sense that

$$u(a) = \int_{z \in b\Omega} p(a, z)\, u(z)\; ds$$

when u is harmonic in Ω and continuous up to the boundary. Choose a point A_0 in Ω and define a weight function via

$$\varphi(z) = p(A_0, z).$$

The weighted Szegő kernel associated to this weight has the virtue that

$$\sigma(z, A_0) \equiv 1.$$

Formula (7.1) shows that $\lambda(z, A_0)$ is non-vanishing on the boundary of Ω and the argument principle and formula (7.1) show that $\lambda(z, A_0)$ has exactly $n-1$ zeroes in Ω (counted with multiplity) in the z variable. Furthermore, since

$$p(w, z) = \frac{1}{2\pi}\frac{\partial}{\partial n_z}G(w, z) = \frac{i}{\pi}\frac{\partial}{\partial \bar{z}}G(w, z)\overline{T(z)},$$

it follows that

$$p(A_0, z) = \frac{i}{\pi}\frac{\partial}{\partial \bar{z}}G(A_0, z)\overline{T(z)},$$

and (7.1) reveals that

$$\lambda(z, A_0) = \frac{1}{\pi}\frac{\partial}{\partial \bar{z}}G(A_0, z) \tag{10.1}$$

for $z \in b\Omega$. Since these two functions extend meromorphically inside Ω and have exactly the one simple pole at A_0 with exactly the same residue, they are equal for all z in Ω, too.

Because σ can be expressed as the weighted projection of the weighted Cauchy kernel (see §7), $\sigma(z, a) = P_\varphi C_a$, and because P_φ is a continuous linear operator from $C^\infty(b\Omega)$ into $A^\infty(\Omega)$, we may conclude that $\sigma(z, a)$ has no zeroes in the z variable in $\overline{\Omega}$ for all a sufficiently close to A_0. Furthermore, $\lambda(z, a)$ is non-vanishing on $b\Omega$ and has exactly $n-1$ zeroes in Ω when a is close to A_0. Choose a point $A_1 \neq A_0$ which is close enough so that these two conditions hold. Let $G(z)$ denote the extension of

$$\sigma(z, A_1) = \frac{\sigma(z, A_1)}{\sigma(z, A_0)}$$

to the double of Ω given by Theorem 9.2. Notice that

$$G(z) = \overline{\lambda(R(z), A_1)}/\overline{\lambda(R(z), A_0)}$$

on $\widetilde{\Omega}$. Hence, G has no zeroes and no poles in $\overline{\Omega}$ and some positive number $m \leq n-1$ of zeroes in $\widetilde{\Omega}$ and the same number m of poles in $\widetilde{\Omega}$. The order of G on $\widehat{\Omega}$ is m.

Choose a point $w_0 \neq 0$ in $\mathbb{C}$ close enough to the origin that $G^{-1}(w_0)$ consists of m distinct points in $\widehat{\Omega}$ which all fall in $\widetilde{\Omega}$. We may also choose w_0 so that none of these points is one of the $n-1$ zeroes of $\lambda(R(z), A_0)$ in $\widetilde{\Omega}$. Let $z_1, \ldots, z_m$ denote these m points.

We now wish to show that it is possible to choose a point A_2 in Ω so that the meromorphic extensions of $\sigma(z, A_1)$ and

$$\sigma(z, A_2) = \frac{\sigma(z, A_2)}{\sigma(z, A_0)}$$

to $\widehat{\Omega}$ form a primitive pair for $\widehat{\Omega}$ (meaning that they generate the field of meromorphic functions on the double of Ω). For a fixed point A_2 in Ω, let $H(z)$ be defined to be the meromorphic extension of $\sigma(z, A_2)$ to $\widehat{\Omega}$. Think of H as depending on A_2 even though we have suppressed this fact in the notation. To show that $G(z)$ and $H(z)$ form a primitive pair, we need only choose A_2 so that $H(z)$ separates the m points in $G^{-1}(w_0)$ (see [3, page 321-324]).

Suppose that A_2 is not equal to A_1 and suppose that A_2 is close enough to A_0 that $\lambda(R(z_k), A_2) \neq 0$ for each k.

If $z_i \in \widetilde{\Omega}$ and $z_j \in \widetilde{\Omega}$ are points in $G^{-1}(w_0)$ which are not separated by H, then

$$\frac{\lambda(R(z_i), A_2)}{\lambda(R(z_i), A_0)} = \frac{\lambda(R(z_j), A_2)}{\lambda(R(z_j), A_0)},$$

and so

$$\frac{\lambda(R(z_i), A_2)}{\lambda(R(z_j), A_2)} = c$$

where

$$c = \frac{\lambda(R(z_i), A_0)}{\lambda(R(z_j), A_0)}$$

is a non-zero constant. But the set of points w in Ω where

$$\frac{\lambda(R(z_i), w)}{\lambda(R(z_j), w)} = c$$

is a finite subset of Ω because this function of w extends to the double of Ω as a non-constant meromorphic function since

$$\frac{\lambda(R(z_i), w)}{\lambda(R(z_j), w)}$$

is equal to the conjugate of

$$\frac{\sigma(R(z_i), R(w))}{\sigma(R(z_j), R(w))}$$

on $b\Omega$. Hence $w = A_2$ can be chosen to avoid this possibility for each pair of indices $i \neq j$.

Assume that A_1 and A_2 in Ω are two points in Ω such that the extensions of $\sigma(z, A_1)$ and $\sigma(z, A_2)$ to the double of Ω form a primitive pair for the field of meromorphic functions on the double.

Let f denote an Ahlfors mapping of Ω onto the unit disc. We may suppose that the base point of the map has been chosen so that the zeroes of f are all simple zeroes. We know that $f(z)$ extends meromorphically to $\widehat{\Omega}$. It follows that all the functions that appear on the right hand side of formula (8.3) extend to the double of Ω. Hence, $\sigma(z, w)$ is a rational combination of $\sigma(z, A_1)$, $\sigma(z, A_2)$ and the conjugates of $\sigma(w, A_1)$ and $\sigma(w, A_2)$.

We collect these results in the following theorem.

Theorem 10.1. *Suppose that Ω is an n-connected domain in the plane such that the boundary of Ω consists of n non-intersecting C^∞ smooth closed curves. The weighted Szegő kernel with respect to the Poisson weight*

$$\varphi(z) = p(A_0, z)$$

for a point A_0 in Ω is such that $\sigma(z, a)$ extends to the double of Ω as a non-constant meromorphic function for each a in Ω. Furthermore, there exist two points A_1 and A_2 in Ω such that the extensions of $\sigma(z, A_1)$ and $\sigma(z, A_2)$ to the double of Ω form a primitive pair for the field of meromorphic functions on the double. The kernel $\sigma(z, w)$ is a rational function of $\sigma(z, A_1)$, $\sigma(z, A_2)$, $\overline{\sigma(w, A_1)}$ and $\overline{\sigma(w, A_2)}$.

11. Finite Riemann surfaces. Suppose that Ω is a finite Riemann surface with boundary, i.e. suppose that Ω is a Riemann surface with boundary with compact closure of finite genus and finitely many boundary curves. We assume that there is at most one boundary curve and that none of the boundary curves are pointlike. To make the exposition easier, we shall assume that the boundary curves of Ω are C^∞ smooth real analytic curves. It will be clear that this assumption can be greatly reduced in what follows, but we do not concern ourselves with this here. It is a standard construction to produce the Szegő projection and kernel with respect to a weight function on the boundary of Ω (see [16]). Given a measure ω on the boundary which is given by a positive C^∞ weight function with respect to the standard metric on the boundary, let $S(z,w)$ denote the Szegő kernel function defined on $\Omega \times b\Omega$ which reproduces holomorphic functions on Ω with respect to ω in the sense that

$$h(z) = \int_{w\in b\Omega} S(z,w)h(w)\,d\omega$$

for points z in Ω and holomorphic functions h on Ω that are in the L^2 Hardy space associated to Ω relative to the measure ω. Let P denote the Szegő projection, which is the orthogonal projection of the L^2 space on $b\Omega$ with respect to the weight function ω onto the closed subspace of functions in this space which are the boundary values of holomorphic functions on Ω.

Ahlfors proved that Ahlfors maps exist in the more general setting that we are dealing with now (see [2]). Fix a point a in Ω and let f_a denote an Ahlfors map associated to (Ω, a). This map is a holomorphic map of Ω onto the unit disc such that $f_a(a) = 0$ and which maximizes the derivative $f_a'(a)$ in some coordinate chart. Ahlfors proved that this map is a proper holomorphic mapping of Ω onto the unit disc. Since the boundary of Ω is C^∞ smooth, it follows that f_a extends C^∞ smoothly up to the boundary of Ω. Of course, f_a maps the boundary of Ω into the boundary of the unit disc, i.e. $|f(z)| = 1$ when $z \in b\Omega$.

Exactly the same arguments as those given in §§7-8 can now be applied in this more general context to yield that the Szegő kernel $S(z,w)$ is a rational combination of finitely many holomorphic functions of one complex variable on Ω. Indeed, if the zeroes of f_a are simple, then the formula in Theorem 8.2 holds with $S(z,w)$ in place of $\sigma(z,w)$ and f_a in place of f. If the zeroes are not simple, then the formula in Theorem 8.1 holds where it is understood that the derivatives $S_{\bar n}(z,a_j)$ in the second variable are taken with respect to some arbitrary, but fixed, coordinate chart near a_j.

To show that $S(z,w)$ is actually generated by *only three* holomorphic functions of one variable, we must do a little extra work. Let $G(z,w)$ denote the classical Green's function for Ω. For a fixed point b in Ω, let $\partial_z G(z,b)$ denote the meromorphic one-form $(\partial/\partial z)G(z,b)\,dz$. This form reproduces holomorphic functions on Ω in the sense that

$$h(b) = \int_{z\in b\Omega} h(z)\partial_z G(z,b).$$

There is a C^∞ function $\mathcal{G}_b$ on the boundary of Ω, such that

$$\int_{z\in b\Omega} \psi(z)\partial_z G(z,b) = \int_{z\in b\Omega} \psi(z)\overline{\mathcal{G}_b(z)}\,d\omega$$

for all continuous functions ψ on $b\Omega$. It follows that the function $S_b(z) := S(z,b)$ is the Szegő projection of $\mathcal{G}_b$. Since $(S_b(z) - \mathcal{G}_b)\omega$ is orthogonal to holomorphic functions, we may use a theorem of Read (see [16, page 75] and [17]) to assert that there is a meromorphic one-form A_b on Ω with no singularities on $\overline{\Omega}$ such that

$$\int_{z\in b\Omega} \psi(z)\left(S_b(z)-\overline{\mathcal{G}_b(z)}\right)\,d\omega = \int_{z\in b\Omega} \psi(z) * A_b$$

for all continuous functions ψ on $b\Omega$. Since S_b and $\mathcal{G}_b$ are C^∞ smooth on $b\Omega$, it follows that A_b is C^∞ smooth up to $b\Omega$ and hence, there is a C^∞ smooth function $\alpha_b(z)$ on $b\Omega$ such that

$$\int_{z\in b\Omega} \psi(z) * A_b = \int_{z\in b\Omega} \psi(z)\overline{\alpha_b(z)}\,d\omega$$

for all continuous functions ψ on $b\Omega$. Define $\lambda_b(z)$ to be $\mathcal{G}_b(z)+\alpha_b(z)$. For two points a and b in Ω, the quotient $S_b(z)/S_a(z)$ is a meromorphic function on $\overline{\Omega}$. It is easy to verify that the quotient $\lambda_b(z)/\lambda_a(z)$ is the boundary value of an antimeromorphic function on $\overline{\Omega}$ given as the quotient of two meromorphic one-forms. Hence $S_b(z)/S_a(z)$ extends to the double of Ω as a meromorphic function. This and the fact that we may compose by a Möbius transformation so as to be able to assume that our proper holomorphic map to the disc has simple zeroes is all that we need to be able to use the formula in Theorem 8.2 to deduce that

$$S(z,w)=S(z,a)S(a,w)R(G_1(z),G_2(z),\overline{G_1(w)},\overline{G_2(w)})$$

where R is a rational function and G_1 and G_2 form a primitive pair for the double of Ω.

The arguments in §10 carry over to our finite Riemann surface and Theorem 10.1 holds in this more general context.

We conclude by showing how these results can be applied to the Bergman kernel on Ω. The Bergman kernel on Ω is a differential $(1,1)$ form given by

$$K(w,z)dw\wedge d\bar{z} = \frac{\partial^2}{\partial w\partial\bar{z}}G(w,z)dw\wedge d\bar{z}.$$

Let $\alpha = df/f$ where f is a proper holomorphic mapping of Ω onto the unit disc (such as an Ahlfors map). The proof of Theorem 1.2 given in [9] yields that

$$\frac{K(w,z)dw\wedge d\bar{z}}{\alpha(w)\wedge\overline{\alpha(z)}}$$

can be viewed as a *function* on $\Omega\times\Omega$ which extends to $\widehat{\Omega}\times\widehat{\Omega}$ to be meromorphic in w and antimeromorphic in z. It now follows from the generalization of Theorem 10.1 mentioned above that the Bergman kernel associated to Ω is given as

$$K(w,z)dw\wedge d\bar{z} = R(G_1(w),G_2(w),\overline{G_1(z)},\overline{G_2(z)})df(w)\wedge\overline{df(z)}$$

where R is a complex rational function and G_1 and G_2 form a primitive pair for $\widehat{\Omega}$. It is interesting to note that the two functions $G_1(z)$ and $G_2(z)$ can be taken to be $S(z,A_1)$ and $S(z,A_2)$ for suitably chosen points A_1 and A_2 in Ω and that f is also a rational combination of these two functions.

I leave it for the future to relate other objects of potential theory associated to a finite Riemann surface to this weighted Szegő kernel so as to obtain results about complexity in complex analysis and potential theory.

References

1. D. Aharonov and H. S. Shapiro, *Domains on which analytic functions satisfy quadrature identities*, Journal D'Analyse Mathématique **30** (1976), 39–73.
2. L. L. [sic] Ahlfors, *Open Riemann Surfaces and extremal problems on compact subregions*, Comment. Math. Helv. **24** (1950), 100–134.
3. L. V. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton Univ. Press, Princeton, 1960.
4. S. Bell, *The Cauchy transform, potential theory, and conformal mapping*, CRC Press, Boca Raton, 1992.
5. ______, *The Szegő projection and the classical objects of potential theory in the plane*, Duke Math. J. **64** (1991), 1–26.
6. ______, *Complexity of the classical kernel functions of potential theory*, Indiana Univ. Math. J. **44** (1995), 1337–1369.
7. ______, *Finitely generated function fields and complexity in potential theory in the plane*, Duke Math. J. **98** (1999), 187–207.
8. ______, *The fundamental role of the Szegő kernel in potential theory and complex analysis*, J. für die reine und angewandte Mathematik, in press.
9. ______, *Ahlfors maps, the double of a domain, and complexity in potential theory and conformal mapping,*, J. d'Analyse Mathematique **78** (1999), 329–344.
10. ______, *A Riemann surface attached to domains in the plane and complexity in potential theory*, Houston J. Math., in press.
11. ______, *Proper holomorphic maps that must be rational*, Trans. Amer. Math. Soc. **284** (1984), 425–429.
12. ______, *Proper holomorphic correspondences between circular domains*, Math. Annalen **270** (1985), 393–400.
13. ______, *Unique continuation theorems for the $\bar\partial$-operator and applications*, J. of Geometric Analysis **3** (1993), 195–224.
14. S. Bergman, *The kernel function and conformal mapping*, Math. Surveys 5, AMS, Providence, 1950.
15. H. M. Farkas and I. Kra, *Riemann Surfaces*, Springer-Verlag, New York, 1980.
16. Maurice Heins, *Hardy classes on Riemann surfaces*, Springer-Verlag, Lecture Notes in Math., vol. 98, Berlin, 1969.
17. A. H. Read, *A converse of Cauchy's theorem and applications to extremal problems*, Acta Math. **100** (1958), 1–22.
18. H. S. Shapiro, *The Schwarz function and its generalization to higher dimensions*, Univ. of Arkansas Lecture Notes in the Mathematical Sciences, Wiley, New York, 1992.

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