THE ADJOINT OF A COMPOSITION OPERATOR VIA ITS ACTION ON THE SZEGŐ KERNEL

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ABSTRACT. The adjoint of the classic composition operator on the Hardy space of the unit disc determined by a holomorphic self map of the unit disc is well known to send the Szegő kernel function associated to a point in the unit disc to the Szegő kernel associated to the image of that point under the self map. The purpose of this paper is to show that a constructive proof that holomorphic functions that extend past the boundary can be well approximated by complex linear combinations of the Szegő kernel function gives an explicit formula for the adjoint of a composition operator that yields a new way of looking at these objects and provides inspiration for new ways of thinking about operators that act on linear spans of the Szegő kernel. Composition operators associated to multivalued self mappings will arise naturally, and out of necessity. A parallel set of ideas will be applied to composition operators on the Bergman space.

In honor of Dima's 60-th!

1. Introduction

The classic composition operator C_φ associated to a holomorphic self map φ of the unit disc $D_1(0)$ is defined via

$$(C_\varphi h)(z) = h(\varphi(z)).$$

It is well known to be a bounded operator on the Hardy space of the unit disc. The adjoint of C_φ satisfies

$$\langle C_\varphi h, g\rangle = \langle h, C_\varphi^* g\rangle \tag{1.1}$$

for all h and g in the Hardy space, where the inner product is the standard one on the boundary associated to L^2 of the boundary with respect to arc length measure ds. We refer the reader to [11] for the basic facts about composition operators.

The Szegő kernel associated to the unit disc is

$$S(z,w) = \frac{1}{(2\pi)(1 - z\bar{w})}.$$

Let S_a be defined via $S_a(z) = S(z,a)$. It is a well known fact in the theory of composition operators that $C_\varphi^* S_a = S_{\varphi(a)}$. Indeed, to see this, let $g = S_a$

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in equation (1.1) for a point a in the unit disc and use the fact that pairing a function in the Hardy space with S_a yields point evaluation for the function at a. Hence,

$$h(\varphi(a)) = \langle h, C_\varphi^* S_a \rangle$$

for all h in the Hardy space. Since the Szegő kernel is characterized by the point evaluation property, we must conclude that $C_\varphi^* S_a = S_{\varphi(a)}$.

We now set some notation that we will use throughout this paper. Suppose that Ω is a bounded domain in the plane with smooth real analytic boundary and let $b\Omega$ denote its boundary. We let $H^2(b\Omega)$ denote the Hardy space (with respect to boundary arc length measure) and $H^2(\Omega)$ denote the Bergman space (with respect to Lebesgue area measure). Let $A^{-\infty}(\Omega)$ denote the space of holomorphic functions on Ω that grow at worst as a constant times an inverse power of the distance to the boundary near the boundary, and let $A^\infty(\Omega)$ denote the space of holomorphic functions in $C^\infty(\overline{\Omega})$.

Let S_a denote the Szegő kernel $S(z, a)$ viewed as a function of $z \in \Omega$ for fixed $a \in \Omega$. Let Σ denote the complex linear span of S_a as a ranges over points in the unit disc. It was proved in [4] that a holomorphic function in $C^\infty(\overline{\Omega})$ can be approximated in that space by functions in Σ (also see Chapter 8 in [5] for a proof). The proof there uses Stokes' theorem and spreads the sum over points ranging over the whole interior of the domain. It also uses the fact that the Szegő projection preserves $C^\infty(\overline{\Omega})$ and is therefore more of an existence proof than a constructive proof. One of the purposes of this paper is to write out a constructive simplified proof of this fact in the real analytic boundary setting that will easily allow us to apply the adjoint of a composition operator to the result. The construction will use a contour integral, and so the points in the sum will fall on curves. In this way, we will obtain a concrete expression for the adjoint. The construction will also reveal the relevance of double quadrature domains, domains that satisfy a quadrature identity on holomorphic functions both with respect to area measure and boundary arc length measure, to these considerations.

I would like to thank Carl Cowen, Eva Gallardo-Gutiérrez, and their students for introducing me to this subject and for teaching me so much about it over the years.

2. A constructive proof of the density of Σ and consequences

We begin this section by considering the problem of approximating a function on the unit disc by a linear combination of the Szegő kernel function S_{a_k} as the points a_k range over a finite set of points in the disc. Let Ω denote the unit disc $D_1(0)$ and assume that $h(z)$ is a holomorphic function on Ω that extends holomorphically to a larger disc $D_R(0)$. The construction we are about to describe is related to the fact that the unit disc is a quadrature domain both with respect to area measure and boundary arc length measure. Being an area quadrature domain means that the Schwarz function for the unit disc $S(z)$ extends meromorphically to the disc and being an arc length quadrature domain means that $S'(z)$ is the square of a function that extends meromorphically to the disc. These results are due to Aharonov an Shapiro [1] and Shapiro and Ullemar [21] in the simply connected case and Gustafsson [14, 15] in the multiply. (See [20] and [13] for more about quadrature domains and the history of the subject, and see [8] for a summary of results on double quadrature domains.)

On the unit disc, the Schwarz function is $S(z) = 1/z$. (The Schwarz function notation $S(z)$ is not to be confused with the notations $S(z,w) = S_w(z)$ that we reserve for the Szegő kernel.) Note that $z = 1/\bar{z}$ on the boundary of the unit disc and the element of arc length ds satisfies $ds = z\,d\bar{z} = (1/\bar{z})d\bar{z}$. As before, let $\langle \cdot, \cdot \rangle$ denote the L^2 inner product on the unit circle with respect to arc length measure. The facts just mentioned and the reproducing property of the Szegő kernel allow us to write

$$h(w) = \langle h, S_w \rangle = \int_{b\Omega} h(z)\,\overline{S_w(z)}\ ds = \int_{b\Omega} h(1/\bar{z})\,\overline{S_w(z)}\,\frac{1}{\bar{z}}\ d\bar{z},$$

and since the integrand is antiholomorphic on $\{z : (1/R) < |z| \leq 1\}$, Cauchy's theorem yields that this integral is equal to the integral of the same function over a smaller circle of radius r where $(1/R) < r < 1$. These considerations yield the following lemma.

Lemma 2.1. *Suppose that h is holomorphic on a disc $D_R(0)$ where $R > 1$. Then h can be expanded in terms of the Szegő kernel via*

$$h(w) = \int_{C_r(0)} h(1/\bar{z})\,\overline{S_w(z)}\,\frac{1}{\bar{z}}\ d\bar{z},$$

where $C_r(0)$ is a circle of radius r such that $1/R < r < 1$. It follows by approximating this integral by a finite Riemann sum that h can be uniformly approximated on any disc $D_\rho(0)$ with $1 < \rho < R$ by a finite linear combination of the functions $\overline{S_w(a_k)} = S(w, a_k)$ where the points a_k range over a circle of radius r with $(1/R) < r < (1/\rho) < 1$.

We can now apply the adjoint of a composition operator to the the approximation of h by a linear combination of Szegő kernels to see that $C_\varphi^* h$ is approximated by a linear combination of the functions $S(w, \varphi(a_k))$. Next, we may let this Riemann sum tend to its original integral and note that uniform convergence implies convergence in the Hardy space, which in turn implies uniform convergence on compact subsets, to see that

$$\left(C_\varphi^* h\right)(w) = \int_{C_r(0)} h(1/\bar{z})\,\overline{S_w(\varphi(z))}\,\frac{1}{\bar{z}}\ d\bar{z}, \tag{2.1}$$

where $C_r(0)$ denotes the circle of radius r about zero. This formula shows that $C_\varphi^* h$ also extends holomorphically past the boundary of the unit disc, a fact that is well-known to the experts. This formula illustrates the utility of characterizing the adjoint of a composition operator by its action on the Szegő kernel and we will use similar ideas when we generalize the operators in what follows. However, the last formula should not be considered as something new in and of itself in the case of the disc. It is more of a new way to view the adjoint by fixating on its action on the Szegő kernel. Indeed, we can let r tend back to one to obtain Cowen and Gallardo's formula [12],

$$\left(C_\varphi^* h\right)(w) = \frac{1}{2\pi}\int_{z\in b\Omega} h(z)\,\frac{1}{1-w\,\overline{\overline{\varphi(z)}}}\,ds$$

for $w \in \Omega$, using the H^∞ boundary values of φ in the integral. We have derived Cowen and Gallardo's formula in case h extends holomorphically past the boundary, but we can take a sequence of such functions converging in $L^2(b\Omega)$ to a function in the Hardy space, noting that convergence in $H^2(b\Omega)$ implies uniform convergence on compact subsets, to see that the formula holds for h in the Hardy space, as Cowen and Gallardo demonstrate. (The techniques of Cowen and Gallardo were further refined in [10] and [16].)

We remark here that we have approximated a function that extends holomorphically past the boundary by a linear combination of Szegő kernel functions that approximate it uniformly in a neighborhood of the boundary. If we wanted to approximate a function h in A^∞ or H^2 or a holomorphic function that merely extended continuously up to the boundary in the respective topologies of those spaces, we would first replace $h(z)$ by $h(rz)$ with $r < 1$ (which approximates h in each space) and apply our result to $h(rz)$ to get an approximation.

Another common operator studied by researchers in composition operators is the weighted composition operator given by

$$(W_{\psi,\varphi}h)(z) = \psi(z)h(\varphi(z)),$$

where φ is a holomorphic map of the unit disc into itself and ψ is a bounded holomorphic function. The adjoint of this operator satisfies

$$W_{\psi,\varphi}^* S_a = \overline{\psi(a)} S_{\varphi(a)}.$$

Assuming that h extends holomorphically past the boundary, we may proceed as above in the unweighted case to obtain the formula

$$(W_{\psi,\varphi}^* h)(w) = \int_{C_r(0)} h(1/\bar{z})\,\overline{\psi(z)}\,\overline{S_w(\varphi(z))}\,\frac{1}{\bar{z}}\,d\bar{z},$$

and we may let r tend to one and relax the extension assumptions about the functions involved in this computation. As in the unweighted case, the formula leads back to Cowen and Gallardo's formula for the adjoint of the weighted composition operator.

3. The Szegő span and adjoints of composition operators

Define the m-th a-bar derivative $S_a^m(z)$ of the Szegő kernel in the second variable a via

$$S_a^m(z) = \frac{\partial^m}{\partial \bar{w}^m} S(z,w)\big|_{w=a}$$

and let $S_a^0 = S_a$. The *Szegő span* is the complex linear span of the functions S_a^m as a ranges over points in the domain and m ranges over all nonnegative integers. It is an easy exercise to see that the Szegő span of the unit disc is the space of all complex rational functions without poles in the closed unit disc. We now claim that the classical composition operators studied above are adjoints of continuous linear operators on the Hardy space of the unit disc that preserve the Szegő span, i.e., that preserve the space of complex rational functions without poles in the closed disc. Indeed, formula (1.1) can be differentiated with respect to $\bar{a}$ to see that

$$C_\varphi^* S_a^1 = \overline{\varphi'(a)} S_{\varphi(a)}^1.$$

This computation can be repeated to see that C_φ^* maps the Szegő span into itself. Thus C_φ is the adjoint of a continuous linear operator that preserves the Szegő span. Similar reasoning shows that the adjoints of the weighted composition operators also preserve the Szegő span.

It seems like a rather interesting problem to determine all the continuous linear operators on the Hardy space that preserve the Szegő span. Among such operators are the adjoints of generalized composition operators of the following type studied by Cowen and Gallardo and their students.

Example 1. Suppose that φ is the multivalued inverse of a finite Blaschke product. (A simple example of such a thing is the multivalued N-th root function.) This implies that φ is a finite-to-finite multivalued holomorphic mapping with algebraic singularities at perhaps finitely many points. Such maps have a mapping degree N. At all but finitely many points in a (finite) set V, we may list local holomorphic maps $\{\varphi_k\}_{k=1}^N$ that represent φ and that map into the unit disc. Each function element can be analytically continued around $\Omega - V$ to come back to another function element in the list and all the elements can be reached in this way from a single element. The points of V are algebraic singularities for the multivalued functions so obtained. (Also, each $|\varphi_k(z)|$ tends to one as $|z|$ tends to one, making φ an *irreducible proper holomorphic self correspondence* of the disc.)

The operator $C_\varphi : h \mapsto \sum_{k=1}^N h \circ \varphi_k$ is a generalized composition operator. The sum is a well defined holomorphic function on $\Omega - V$ and the points in V are clearly removable singularities for the sum. At points in V, continuity shows that the sum is equal to a fixed linear combination of values of h. For example, when φ is given by the multivalued N-th root function, $(C_\varphi h)(z)$ is equal to the sum of $h(\zeta)$ as the ζ rangle over the N-th roots of z if $z \neq 0$, and $(C_\varphi h)(0) = N\, h(0)$.

For points a in $\Omega - V$, the adjoint satisfies $C_\varphi^* S_a = \sum_{k=1}^N S_{\varphi_k(a)}$. For points a in V, $C_\varphi^* S_a$ must also be equal to such a sum by continuity in a and the continuous nature of algebraic functions at finite valued singular points. In fact, the sum could be realized as a sum of N terms with some repeated terms corresponding to points having a local multiplicity. In case φ is the N-th root function, $C_\varphi^* S_0 = NS_0$. Similar reasoning to that above shows that the sum extends to the disc in the a variable to be antiholomorphic in a. By differentiating with respect to $\bar{a}$ in the a variable, it can be seen that C_φ^* maps the Szegő span into the Szegő span. It is also special in that it maps S_a into Σ for each a.

Example 2. A similar interesting example of a weighted composition operator of a generalized type is described via the multivalued inverse of the function z^N as follows. Let $\varphi_k(z)$, $k = 1, \ldots, N$, denote the N branches of the N-th root on $\Omega - \{0\}$, and let $F(w) = w^N$. The generalized composition operator λ given by

$$\lambda : h \mapsto \sum_{k=1}^{N} \varphi_k'(h \circ \varphi_k) \tag{3.1}$$

is a very interesting operator. Notice the pole-like algebraic singularities that occur in the weight functions φ_k'. Even so, it is not hard to see that zero is a removable singularity for the sum and the value at zero is a constant times the $(N-1)$-st derivative of h at zero. Indeed, because $|\varphi_k'|^2$ is the Jacobian determinant for the local maps viewed as mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$, the operator λ takes $L^2(\Omega)$ into itself. Since an isolated singularity of a holomorphic function that is locally in L^2 is removable, we may use the change of variables formula and the averaging property of analytic functions to see that π times the value of $\sum_{k=1}^N \varphi_k'(h \circ \varphi_k)$ at zero is equal to

$$\langle \sum_{k=1}^{N} \varphi_k'(h \circ \varphi_k), 1 \rangle_\Omega = \langle h, F'(1 \circ F) \rangle_\Omega = \langle h, N z^{N-1} \rangle_\Omega = \pi \frac{h^{N-1}(0)}{(N-1)!},$$

where $\langle u, v \rangle_\Omega$ denotes the L^2 inner product with respect to area measure on Ω. The last equality follows from the orthogonality of the monomials on the unit disc and Taylor's formula. (See Chap. 16 of [5] for a complete and careful explanation of this line of reasoning.) It is straightforward to show that λ preserves the Hardy space because the branches are C^∞ smooth up to the boundary. The adjoint of λ satisfies

$$\lambda^* S_a = \sum_{k=1}^{N} \overline{\varphi_k'(a)} S_{\varphi_k(a)}$$

at points $a \neq 0$. At $a = 0$, the adjoint is equal to $1/(N-1)!$ times the $(N-1)$-st derivative of $S(z, w)$ in $\bar{w}$ evaluated at $w = 0$, i.e.,

$$\lambda^* S_0 = \frac{1}{(N-1)!} S_0^{(N-1)}.$$

Note that λ^* maps the functions S_a into Σ for $a \neq 0$, but to a higher order function in the Szegő span at $a = 0$. The formula for $\lambda^* S_a$ can also be differentiated in the a variable to see that λ^* is an operator on the Hardy space that preserves the Szegő span and, therefore, λ is the adjoint of an operator on the Hardy space that preserves the Szegő span.

In the next section, we will see that multivaluedness arises naturally in the description of adjoints of operators that preserve the Szegő span and that we should not be surprised to see pole-like algebraic singularities in the weight functions that appear.

We remark that generalized composition operators like the two above can be defined using any irreducible *proper holomorphic self correspondence of the unit disc* φ to itself. See [2] or [6] for details about proper holomorphic correspondences, and see [22] for where the study of holomorphic correspondences originated. See also Cowen and Gallardo [12] for more examples and more types of generalized composition operators.

In order to study operators that preserve the Szegő span, it will be necessary to allow higher order composition operators of the form

$$h \mapsto \frac{d^m}{dz^m}(h \circ \varphi)$$

and the corresponding weighted versions multiplied by a holomorphic function ψ in $A^{-\infty}(\Omega)$. (We will also be forced to allow φ and ψ to be multivalued later.) Such operators might not be bounded operators on the Hardy space (for φ that map near the boundary of the disc, for example), hence we will also have to leave the Hardy space. We will now show that the natural spaces on which to study such operators and their adjoints are $A^{-\infty}(\Omega)$ and $A^{\infty}(\Omega)$.

First, we show that the higher order composition operators are continuous operators on the space $A^{-\infty}(\Omega)$ of functions that grow at worst as an inverse power of the distance to the boundary near the boundary. To see this, assume that φ is a holomorphic self map of the unit disc. Let $d(z) = 1 - |z|$ denote the distance to the boundary function. If $\varphi(0) = 0$, the Schwarz lemma implies that

$$d(\varphi(z)) \geq d(z).$$

If $\varphi(0) \neq 0$, we can compose with a Möbius transformation to make zero a fixed point and note that a Möbius transformation distorts the boundary distance via a small constant, large constant inequality. Hence, there is a positive constant c such that

$$d(\varphi(z)) \geq c\, d(z).$$

This inequality implies that $d(\varphi(z))^{-1}$ is bounded by a constant times $d(z)^{-1}$. Hence, $h \mapsto h \circ \varphi$ is a continuous operator from $A^{-\infty}(\Omega)$ to itself. Since multiplication by a function in $A^{-\infty}(\Omega)$ and differentiation are also continuous operators from $A^{-\infty}(\Omega)$ to itself, we conclude that the more general composition operators are too.

Notice that the Szegő span is a space of functions that extend C^{∞} smoothly to the boundary and we are studying operators that preserve the span. Hence, it will not be surprising that we will expect our adjoints to preserve $A^{\infty}(\Omega)$. We now make these vague urges precise.

It is well known that the L^2 inner product on the boundary extends from $H^2(b\Omega) \times H^2(b\Omega)$ to $A^{-\infty}(\Omega) \times A^{\infty}(\Omega)$ and exhibits these two spaces as being mutually dual (see [17, 18, 24, 23]). We claim that this fact together with the fact that composition operators are continuous operators on $A^{-\infty}(\Omega)$ give an instant proof that C_φ^* maps $A^{\infty}(\Omega)$ into itself (a fact that is well-known to the experts). Indeed, if h is in $A^{\infty}(\Omega)$ and φ is a holomorphic self map of the unit disc, then

$$g \mapsto \langle C_\varphi g, h \rangle$$

is a continuous linear functional on $A^{-\infty}(\Omega)$. Hence, there is an H in $A^{\infty}(\Omega)$ such that

$$\langle C_\varphi g, h\rangle = \langle g, H\rangle.$$

Since $H^2(b\Omega)$ is contained in $A^{-\infty}(\Omega)$, and since the pairing agrees with the L^2 pairing when g (and $C_\varphi g$) have L^2 boundary values, we must conclude that $H = C_\varphi^* h$ and that C_φ^* therefore maps $A^{\infty}(\Omega)$ into itself. Hence, the adjoint in this more general sense is an extension of the classic adjoint and it makes good sense to use the same notation for the more general adjoint. The same reasoning can be used to show that the adjoint of a classic weighted composition operator preserves $A^{\infty}(\Omega)$, and in fact, so do the adjoints of our higher order composition operators when we view the operators as operators on $A^{-\infty}(\Omega)$ and their adjoints as operators on $A^{\infty}(\Omega)$. In this context, the higher order adjoints can also be shown to preserve the Szegő span.

We now turn to studying operators that preserve $A^{\infty}(\Omega)$ and the Szegő span.

4. Operators that preserve the Szegő span

We continue to let Ω denote the unit disc. Suppose that Λ is a continuous linear operator from $A^{\infty}(\Omega)$ to itself that preserves the Szegő span. This implies that $(\Lambda S_a)(z)$ is a rational function of z without poles in the closed unit disc for each point a in the unit disc. Let

$$H(z, a) = (\Lambda S_a)(z).$$

Note that H is holomorphic and rational in z for fixed a and antiholomorphic in a (as can be seen by differentiating under the operator). Also, because Λ is a continuous operator on $A^{\infty}(\Omega)$ and the C^s norm of S_a grows at worst as a power of the distance of a to the boundary, it follows that $H(z, a)$ is the conjugate of a function in $A^{-\infty}(\Omega)$ as a function of a for each z in the unit disc.

We will now prove that $H(z, a)$ is a quotient of polynomials in z with coefficients that are conjugates of holomorphic functions of a in $A^{-\infty}(\Omega)$. The argument is very similar to one used in [7, p. 1366], which is based on an old argument found in Bochner and Martin [9] to show that a function that is rational in each variable separately is rational.

We assume that Λ is not the zero operator. Thus, $H(z, a)$ cannot be identically zero and there is a nonempty open set $U \times V$ in $\Omega \times \Omega$ where H is nonvanishing. Because $H(z, a)$ is rational in z for each a in V, there are nonnegative integers $M(a)$ and $N(a)$, and coefficients $A_k(a)$ and $B_k(a)$ such that

$$H(z, a)\left(\sum_{k=0}^{N(a)} A_k(a) z^k\right) + \sum_{k=0}^{M(a)} B_k(a) z^k = 0. \tag{4.1}$$

By insisting that the polynomials in (4.1) have no common factors, we uniquely specify $M(a)$ and $N(a)$. Because H is nonvanishing, we may divide equation

(4.1) by a nonzero constant (which depends on a) so that we may assume that

$$\sum_{k=0}^{N(a)} |A_k(a)|^2 + \sum_{k=0}^{M(a)} |B_k(a)|^2 = 1. \tag{4.2}$$

The set $\mathcal{O}_{M,N}$ of points a in V where $M(a) \leq M$ and $N(a) \leq N$ is closed in V, and since $V = \cup \mathcal{O}_{M,N}$, the Baire category theorem implies that there is a nonempty open subset of V on which $M(a)$ and $N(a)$ are uniformly bounded. Call this possibly smaller set V now. By allowing some coefficients to be zero, we may assume that (4.1) and (4.2) hold with N and M in place of $N(a)$ and $M(a)$ for all a in V. Let $q = N + M + 2$. Writing out (4.1) for q points $z_1, z_2, \ldots, z_q$ in U yields a linear system with a nonzero solution by virtue of (4.2). Hence the determinant of the system is zero, i.e.,

$$\det \begin{bmatrix} H(z_1,a) & H(z_1,a)z_1 & \ldots & H(z_1,a)z_1^N & 1 & z_1 & \ldots z_1^M \\ H(z_2,a) & H(z_2,a)z_2 & \ldots & H(z_2,a)z_2^N & 1 & z_2 & \ldots z_2^M \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ H(z_q,a) & H(z_q,a)z_q & \ldots & H(z_q,a)z_q^N & 1 & z_q & \ldots z_q^M \end{bmatrix} \equiv 0$$

We now assume that $z_1, \ldots, z_{q-1}$ are distinct points in V and we replace z_q by a variable z in V. If we expand the determinant along the bottom row, we obtain an equation like (4.1) where the coefficients $A_k(a)$ and $B_k(a)$ are antiholomorphic functions of a that are conjugates of functions in $A^{-\infty}(\Omega)$. This equation extends to hold for all points z and a in the unit disc. If at least one of the coefficients $B_k(a)$ is not identically zero, our claim follows. If all the B_k are zero, then all the principal minors corresponding to elements in the bottom row are zero. We consider each of the principal minors separately and let the z_k variable in the bottom row of each to be a genuine variable z again. We repeat this process on each until we get a nonvanishing minor. Since $H(z,a)$ is nonvanishing, there are plenty of 1×1 minors of the matrix that are nonzero. Let m be the smallest positive integer such that the determinant of every $m \times m$ submatrix of our matrix is identically zero, but there is an $(m-1) \times (m-1)$ submatrix whose determinant is not identically zero. Let $\mathbb{M}$ denote the $m \times m$ submatrix (which must contain a column with the functions H in it because the Vandermonde determinant is not zero). We can let the last z_k variable in the bottom row of $\mathbb{M}$ to be z again and repeat the above argument to conclude that $H(z,a)$ is a function of the promised form. This completes the proof.

Now that we have determined the form of $H(z,a)$, we turn to expressing our operator Λ in terms of H. Assume that h is a holomorphic on a neighborhood of the closed unit disc. We may express h as a limit of linear combination of the functions S_a via an integral, and then let the sum converge to the integral under the operator as we did via equation (2.1) in §2 to obtain

$$(\Lambda h)(z) = \int_{C_r(0)} H(z,w)h(1/\bar{w})\,\frac{1}{\bar{w}}\,d\bar{w}.$$

We may now let $r \to 1$ as we did in §2 to obtain

$$(\Lambda h)(z) = \int_{w \in b\Omega} h(w) H(z,w)\, ds,$$

where the boundary integral is understood in terms of the extension of the L^2 pairings to $A^{-\infty}(\Omega) \times A^{\infty}(\Omega)$ because the conjugate of $H(z,w)$ is in $A^{-\infty}(\Omega)$ as a function of w for each z in Ω. Finally, we may take a sequence of h_j that extend holomorphically past the boundary converging in $A^{\infty}(\Omega)$ to h to show that integral formula for Λ extends to hold for h in $A^{\infty}(\Omega)$.

It is interesting to consider operators that preserve the Szegő span in familiar ways. For example, consider an operator Λ that maps S_a to a constant times the Szegő kernel based at some point in Ω for each a in Ω. If we write

$$\Lambda S_w = c_w S_{b_w},$$

then we may define functions $\psi(w) = \overline{c_w}$ and $\varphi(w) = b_w$. Hence,

$$H(z,w) = \frac{\overline{\psi(w)}}{2\pi(1 - z\,\overline{\varphi(w)})}.$$

Setting $z = 0$ shows that ψ is holomorphic and in $A^{-\infty}(\Omega)$. If we differentiate with respect to z repeatedly and set $z = 0$, we see that $\psi\varphi^k$ is holomorphic for each k. If follows that φ is holomorphic. Also, φ maps into the unit disc, so the adjoint of Λ must be a classic weighted composition operator with weight ψ in $A^{-\infty}(\Omega)$. In order for this composition operator to be bounded in the Hardy space, we would need to place further conditions on ψ. Since the norm in $L^2(b\Omega)$ of S_a is the square root of $S(a,a) = (1 - |a|^2)^{-1}/(2\pi)$, we see from the identity $\Lambda S_a = \overline{\psi(a)} S_{\varphi(a)}$ that the the boundary behavior of ψ is controlled by the $L^2(b\Omega)$ operator norm of Λ and the boundary behavior of φ. If φ is a Möbius transformation (or a finite Blaschke product), then we can deduce that ψ must be bounded on the unit disc to make Λ (and therefore the adjoint of Λ) bounded on the Hardy space. If φ maps the unit disc into a compact subset of the unit disc, ψ could merely be a function in the Hardy space and the composition operator would be bounded on the Hardy space.

A more interesting case to consider is an operator Λ that maps the functions S_a into Σ in the manner that the adjoint of the operator of Example 2 of §3 does. If one applies partial fractions to $H(z,w)$ (which is rational in z), one must obtain

$$H(z,w) = \sum_{k=1}^{N} \frac{\overline{\psi_k(w)}}{1 - z\,\overline{\varphi_k(w)}}. \tag{4.3}$$

As Example 1 and 2 show, it is reasonable to demand that the values of the $\varphi_k(w)$ be distinct, except on a discrete set of points were multiplicity leads to coalescence. Also, Example 2 shows that there can be a discrete set where higher powers of the terms occur corresponding to higher order terms in the Szegő span. Note that in Example 2, we leave the space Σ on a discrete set, so we shall allow for this possibility here. We will now show that the functions ψ_k and

φ_k can be seen to be (possibly) multivalued holomorphic functions of a special type. The functions $1/\overline{\varphi_k(w)}$ are the roots of a polynomial in z with coefficients that are functions of w that are conjugates of functions in $A^{-\infty}(\Omega)$. Hence they are N-valued holomorphic functions with perhaps algebraic singularities at isolated points in the unit disc. (Rudin's paper [19] shows how to prove such things.) Since the φ_k must map into the unit disc, they have no pole-like algebraic singularities. Hence there is a discrete set of points E in the unit disc such that the φ_k are holomorphic germs on $\Omega - E$ that can be continued at will on $\Omega - E$. The germs so obtained map into the unit disc. Keeping Example 2 in mind, it is reasonable to stipulate that, by perhaps enlarging the discrete set E, we may assume the values of $\varphi_k(w)$ are N distinct complex numbers for w in $\Omega - E$. We now wish to do a calculation that shows why we should not expect the functions ψ_k to be independent from the φ_k. Pick a point w_0 in $\Omega - E$ and one of the functions φ_j such that φ_j is holomorphic and nonvanishing on a neighborhood of w_0 in $\Omega - E$. Since $\varphi_j(w_0)$ is distinct from $\varphi_k(w_0)$ for $k \neq j$ and the point $W_0 = 1/\overline{\varphi_j(w_0)}$ is a point outside the closed unit disc, we may find a small circle C_ϵ about W_0 such that none of the other points $1/\overline{\varphi_k(w_0)}$, $k \neq j$, fall inside C_ϵ. The residue theorem yields that $-i$ times the conjugate of $\frac{\psi_j(w)}{\varphi_j(w)}$ is equal to

$$\int_{C_\epsilon} H(z,w)\, dz$$

for w near w_0. The function of w given by the integral is antiholomorphic in w near w_0. This shows that, locally, the ψ_k might be holomorphic germs that depend on the φ_k as we follow paths in $\Omega - E$, and that singularities and multivaluedness of the φ_k might give rise to the same behavior in the ψ_j. We now consider a more sophisticated way to understand this association.

As we did in the simpler case, we may differentiate (4.3) with respect to z and then set $z = 0$ to obtain a system,

$$\sum_{k=1}^{N} \psi_k \varphi_k^m = H_m \tag{4.4}$$

for $m = 0, \ldots, N-1$, where the H_m are holomorphic functions in $A^{-\infty}(\Omega)$. We may use Cramer's rule to solve for the ψ_k to obtain

$$\psi_j = \frac{\det V_j}{\det V},$$

where V is the Vandermonde-type matrix,

$$V = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \varphi_1 & \varphi_2 & \cdots & \varphi_N \\ \varphi_1^2 & \varphi_2^2 & \cdots & \varphi_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{N-1} & \varphi_2^{N-1} & \ldots & \varphi_N^{N-1} \end{bmatrix},$$

and V_j is the matrix obtained from V by replacing the j-th column with the column $(H_0, \ldots, H_{N-1})^T$. We next expand the determinants in the numerator and denominator along the j-column and divide the top and bottom by the determinant of the $(N-1)\times(N-1)$ Vandermonde-type matrix obtained from the φ_k with $k\neq j$ to obtain

$$\psi_j=\frac{\sum_{k=1}^N g_k H_k}{\sum_{k=1}^N g_k\varphi_j^k},$$

where the functions g_k are symmetric functions of the φ_k with $k\neq j$ that are holomorphic on $\Omega-E$ and have removable singularities at the points in E. Consequently, each ψ_j is a well defined holomorphic germ on $\Omega-E$ that can be continued at will around $\Omega-E$. Since the φ_j have at worst algebraic singularities at points in E, the singularities of ψ_j at points in E are at worst algebraic singularities. That algebraic pole-like singularities can occur was seen in Example 2 (3.1) of §3.

5. Composition operators on the Bergman space

We close by sketching a course of action to treat composition operators on the Bergman space in a highly parallel manner to what we have done in the Hardy space setting. Once again, we let Ω denote the unit disc and φ a holomorphic self map of the unit disc. The composition operator C_φ is defined the same way as in the Hardy space setting, but the adjoint is defined by means of the $L^2(\Omega)$ inner product,

$$\langle u,v\rangle=\iint_\Omega u\,\bar v\ dA,$$

where dA denotes area measure, and the composition operator and its adjoint act on the Bergman space. Let

$$K(z,w)=\frac{1}{\pi(1-z\bar w)^2}$$

denote the Bergman kernel of the unit disc and write $K_a(z)=K(z,a)$ as we did for the Szegő kernel. The adjoint satisfies the identity, $C_\varphi^*K_a=K_{\varphi(a)}$. If h is a holomorphic function that extends holomorphically past the boundary of the unit disc, we wish to approximate h by a finite linear combination of the Bergman kernel functions K_{a_j} at points a_j in Ω by simplifying the approximation given in [3] (see also [5, Chap. 15]) in the case of the unit disc. We can easily write down a function ψ such that ψ is real analytic on a neighborhood of the unit circle, zero on the circle, and $\partial\psi/\partial z$ is equal to h on a neighborhood of the circle. Indeed, if H is a holomorphic antiderivative for h, then ψ can be defined via

$$\psi(z)=H(z)-\chi(z)H(1/\bar z)$$

where χ is a C^∞ cut off function that is one near the unit circle and zero on a large compact subset of the unit disc. The cut off function is chosen so that $H(1/\bar z)$ is antiholomorphic on a neighborhood of the support of χ in the unit disc.

The complex Green's identity reveals that, if g is holomorphic on a neighborhood of the closure of the unit disc, then

$$\iint_\Omega \frac{\partial \psi}{\partial z}\ \bar{g}\ dz \wedge d\bar{z} = \iint_\Omega \frac{\partial}{\partial z}(\psi\, \bar{g})\ dz \wedge d\bar{z} = \int_{b\Omega} \psi\, \bar{g} d\bar{z},$$

and this last integral is zero because ψ vanishes on the boundary. Hence, $\partial\psi/\partial z$ is orthogonal to holomorphic functions that extend past the boundary, which are dense in the Bergman space. Hence, $\partial\psi/dz$ is orthogonal to the Bergman space and h is equal to the Bergman projection of $h - (\partial\varphi/\partial z)$. Note that $\Phi := h - (\partial\varphi/\partial z)$ is a compactly supported function in $C_0^\infty(\Omega)$, and, in fact, $\Phi = H(1/\bar{z})(\partial\chi/\partial z)$. We may now approximate h as a sum of Bergman kernels as follows.

$$\begin{aligned} h(z) &= \iint_\Omega K(z,w) h(w)\, dA = \iint_\Omega K(z,w) \left(h(w) - \frac{\partial \psi}{\partial w} \right) dA \\ &= \iint_\Omega K(z,w)\Phi(w)\, dA = \iint_\Omega K(z,w) H(1/\bar{w}) \frac{\partial \chi}{\partial w}\, dA, \end{aligned}$$

and we may approximate this integral by a Riemann sum $\sum c_j K_{a_j}$ where the points in the Riemann sum run over the compact set containing the support of Φ. The approximation yields a sum that is holomorphic on a neighborhood of the closure that can be made uniformly close to h on a bigger disc $D_\rho(0)$ for some $\rho > 1$.

We next apply C_φ^* to the Riemann sum and take the limit of Riemann sums converging back to the original integral to obtain the formula

$$(C_\varphi^* h)(z) = \lim \sum c_j K_{\varphi(a_j)} = \iint_\Omega K(z, \varphi(w))\, H(1/\bar{w}) \frac{\partial \chi}{\partial w}\, dA. \tag{5.1}$$

This formula can be used as the starting point to prove a parallel line of results for the adjoints of composition operators on the Bergman space via their action on the Bergman kernel. For example, it can be read off that the adjoint preserves the space of holomorphic functions that extend holomorphically past the boundary in the Bergman space setting, a well known fact.

Similar arguments and estimates to those used to analyze the adjoint in the Hardy space setting can be used in the Bergman space setting. Indeed, the L^2 inner product with respect to area also extends to $A^{-\infty}(\Omega) \times A^\infty(\Omega)$ and exhibits these two spaces as being mutually dual (see Korenblum [17, 18] or see chapter 30 of [5] for an alternate proof). As in the Hardy space setting, it can be seen that the adjoint in the Bergman space setting maps $A^\infty(\Omega)$ to itself, another known fact.

It is important to mention that the complex Green's formula can be undone in (5.1) to obtain Cowen and Gallardo's formula,

$$(C_\varphi^* h)(z) = \iint_\Omega K(z, \varphi(w))\, h(w)\, dA.$$

We have shown that this formula is valid when $z \in \Omega$ and h extends past the boundary. Density and convergence theorems can be used to show that the formula is valid in more generality, as shown by Cowen and Gallardo.

The examples of composition operators based on multivalued mappings given in §3 are also bounded operators on the Bergman space. The Bergman span can be defined in a way completely analogous to the Szegő span, and the adjoints of the examples of §3 can be seen to preserve the Bergman span. On the unit disc, the Bergman span is the space of all rational functions with at worst residue free poles outside the closure of the unit disc. The same reasoning used in §3 and §4 can be used to show that an operator Λ on $A^\infty(\Omega)$ that preserves the Bergman span is given by

$$(\Lambda h)(z) = \iint_\Omega H(z,w)\, h(w)\, dA,$$

where $H(z,w) = (\Lambda K_w)(z)$ is a quotient of polynomials in z with coefficients that are conjugates of holomorphic functions in $A^{-\infty}(\Omega)$. The function $H(z,w)$ is rational in z with at worst residue free poles outside the closure of the unit disc, is antiholomorphic in w, and the conjugate of $H(z,w)$ is in $A^{-\infty}(\Omega)$ as a function of w for each z in the disc.

6. Composition operators between double quadrature domains

We have seen the relevance of the extendibility of the Schwarz function and its derivative in our deliberations. For this reason, it would be interesting to consider composition operators associated to mappings between double quadrature domains. On a double quadrature domain, the Szegő span contains, not only the complex polynomials, but the set of all complex rational functions with poles outside the closure of the domain. Also, functions in the Szegő span are algebraic functions that are rational in z and $\bar{z}$ when restricted to the boundary. Hence, adjoints of composition operators that preserve the Szegő span would map rational functions with poles outside the closure of one domain to algebraic functions on the other that are rational functions of z and $\bar{z}$ on the boundary. Consequently, the results of §4 seem likely to generalize to this setting. Since smooth finitely connected domains are biholomorphic to nearby double quadrature domains, such domains might take the place of the unit disc in the simply connected case.

References

[1] Aharonov, D. and H. S. Shapiro, *Domains on which analytic functions satisfy quadrature identities*, Journal D'Analyse Mathématique **30** (1976), 39–73.

[2] Bell, S. and E. Bedford, *Boundary Behavior of Proper Holomorphic Correspondences.* Math. Ann. **272** (1985) 505–518.

[3] Bell, S., *Non-vanishing of the Bergman kernel at boundary points of certain domains in* $\mathbb{C}^n$, Math. Ann. **244** (1979), 69–74.

[4] ———, *Unique continuation theorems for the* $\bar\partial$*-operator and applications*, J. of Geometric Analysis **3** (1993), 195–224.

[5] ———, *The Cauchy transform, potential theory, and conformal mapping*, 2nd Edition, CRC Press, Boca Raton, 2016.

[6] ———, *Algebraic mappings of circular domains in C^n*, Proceedings of the special year in several complex variables at the Mittag-Leffler Institute, 1987-88, Princeton University Press, Mathematical Notes, vol. **38** (1993), 126–135.

[7] ———, *Complexity of the classical kernel functions of potential theory*, Indiana University Mathematics Journal **44** (1995), 1337–1369.

[8] Bell, S., Gustafsson, B., and Sylvan, Z., *Szegő coordinates, quadrature domains, and double quadrature domains*, Computational Methods and Function Theory **11** (2011), No. 1, 25–44.

[9] Bochner, S. and W. Martin, *Several complex variables*, Princeton Univ. Press, Princeton, 1948.

[10] Bourdon, P. and J. Shapiro, *Adjoints of rationally induced composition operators*, J. Func. Anal. **255** (2008), 1995–2012.

[11] Cowen, C. and B. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press, Boca Raton, 1995.

[12] Cowen, C. and E. Gallardo, *A new class of operators and a description of adjoints of composition operators*, J. Func. Anal. **238** (2006), 447–462.

[13] Ebenfelt, B. Gustafsson, D. Khavinson, and M. Putinar, *Quadrature domains and their applications*, Operator Theory: Advances and Applications **156**, Birkhäuser, Basel, 2005.

[14] Gustafsson, B., Quadrature domains and the Schottky double, *Acta Applicandae Math.* **1** (1983), 209–240.

[15] Gustafsson, B., Applications of half-order differentials on Riemann surfaces to quadrature identities for arc-length, *Journal D'Analyse Math.* **49** (1987), 54–89.

[16] Hammond, C., J. Moorhouse, and M. Robbins, *Adjoints of composition operators with rational symbol*, J. Math. Anal. Appl. **341** (2008), 626–639.

[17] Korenblum, B. *An extension of the Nevanlinna theory*, Acta Math. **135** (1975), 187–219.

[18] ———, *A Beurling-type theorem*, Acta Math. **138** (1976), 265–293.

[19] Rudin, W., *Proper holomorphic mappings between circular domains*, Indiana Math. J. **31** (1982), 701–720.

[20] Shapiro, H. S., The Schwarz function and its generalization to higher dimensions, Univ. of Arkansas Lecture Notes in the Mathematical Sciences, Wiley, New York, 1992.

[21] Shapiro, H. and C. Ullemar, *Conformal mappings satisfying certain extremal properties and associated quadrature identities*, Research Report TRITA-MAT-1986-6, Royal Inst. of Technology, 40 pp., 1981.

[22] Stein, K., *Topics on holomorphic correspondences*, Rocky Mt. J. Math. **2** (1964), 443–463.

[23] Straube, E., *Harmonic and Analytic Functions Admitting a Distribution Boundary Value*, Ann. Scuola Norm. Sup. Pisa **11**, Nr. 4 (1984-1985), 559–591.

[24] Taylor, B. A., and D. L. Williams *Ideals in rings of analytic functions with smooth boundary values*, Canad. J. Math. **22** (1970), 1266–1283.

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