THE CAUCHY INTEGRAL FORMULA, QUADRATURE DOMAINS, AND RIEMANN MAPPING THEOREMS

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Abstract. It is well known that a domain in the plane is a quadrature domain with respect to area measure if and only if the function z extends meromorphically to the double, and it is a quadrature domain with respect to boundary arc length measure if and only if the complex unit tangent vector function $T(z)$ extends meromorphically to the double. By applying the Cauchy integral formula to $\bar{z}$, we will shed light on the density of area quadrature domains among smooth domains with real analytic boundary. By extending $\bar{z}$ and $T(z)$ and applying the Cauchy integral formula to the Szegő kernel, we will obtain conformal mappings to nearby arc length quadrature domains and even domains that are like the unit disc in that they are simultaneously area and arc length quadrature domains. These "double quadrature domains" can be thought of as analogues of the unit disc in the multiply connected setting and the mappings so obtained as generalized Riemann mappings. The main theorems of this paper are not new, but the methods used in their proofs are new and more constructive than previous methods. The new computational methods give rise to numerical methods for computing generalized Riemann maps to nearby quadrature domains.

1. Introduction

The Riemann mapping theorem states that any simply connected domain in the plane that is not equal to the whole plane is biholomorphic to the unit disc, which is a well known quadrature domain with respect to both area measure and boundary arc length measure, i.e., a *double quadrature domain.* This classic theorem was generalized in [6] to state that any finitely connected domain Ω in the plane, $\Omega \neq \mathbb{C}$, is biholomorphic to a double quadrature domain. Furthermore, if the domain is a bounded domain bounded by finitely many nonintersecting Jordan curves, the mapping function can be taken to be arbitrarily close to the identity in the uniform topology up to the boundary. If the boundary curves are C^∞ smooth, the mapping can be taken to be arbitrarily close to the identity in $C^\infty(\overline{\Omega})$. If the boundary curves are further assumed to be smooth real analytic curves, the mapping extends holomorphically past the boundary and can be taken to be close to the identity on a neighborhood of the closure of the domain. The proofs of these theorems given in [6] are rather long and use ideas from Riemann surface theory as well as regularity properties of the Bergman and Szegő

2010 *Mathematics Subject Classification.* 30C20; 30C40; 31A35.
Key words and phrases. Bergman kernel, Szegő kernel, double quadrature domains.

projections. The double quadrature domains that arise in the proofs are *one point* double quadrature domains like the unit disc. (The quadrature identities can and will involve derivatives if the target domain is not a disc). In this paper, we relate these problems to the Cauchy integral formula in such a way that the proofs can be greatly shortened and simplified to the point where they could be included in a textbook on the subject. The constructive methods in our proofs give rise to potential numerical methods for finding the conformally equivalent quadrature domains. See [8] for recent advances in the study of quadrature domains and why they are useful.

It is interesting to note that a bounded domain is an area quadrature domain if and only if the boundary values of $\bar{z}$ are equal to the values of a meromorphic function on the double restricted to the boundary, and it is an arc length quadrature domain if and only the complex unit tangent vector function $T(z)$ is equal to the values of a meromorphic function on the double restricted to the boundary. We deduce many of our results by applying the Cauchy integral formula to holomorphic extensions of $\bar{z}$ and $T(z)$ from the boundary. It will be surprising to see how many different ways there are to obtain nearby quadrature domains of various kinds, yielding several different computational methods.

2. The Cauchy integral and the Schwarz function

In this paper, we will always suppose the Ω is a bounded domain in $\mathbb{C}$ bounded by finitely many nonintersecting smooth real analytic curves.

For a function u in $C^\infty(\overline{\Omega})$, the Cauchy integral formula (or Pompeiu's formula) is

$$u(z) = \frac{1}{2\pi i}\int_{b\Omega} \frac{u(w)}{w-z}\, dw + \frac{1}{2\pi i}\iint_{\Omega} \frac{\frac{\partial u}{\partial \bar{w}}(w)}{w-z}\, dw \wedge d\bar{w}. \tag{2.1}$$

Taking $u(z) = \bar{z}$ in the formula yields

$$\bar{z} = \frac{1}{2\pi i}\int_{b\Omega} \frac{\bar{w}}{w-z}\, dw + \frac{1}{2\pi i}\iint_{\Omega} \frac{1}{w-z}\, dw \wedge d\bar{w}. \tag{2.2}$$

Define $Q(z)$ to be the function given by the boundary integral and $\lambda(z)$ to be the function given by the double integral. Note that $Q(z)$ is a holomorphic function on Ω that extends holomorphically past the boundary (since $\bar{w}$ is real analytic in a neighborhood of the boundary, see [4, p. 49]). Hence, it follows that $\lambda(z)$, being equal to $\bar{z}-Q(z)$ is a complex valued harmonic function on Ω that extends harmonically to an open set containing $\overline{\Omega}$. We will show that, on the boundary, $\lambda(z)$ can be well approximated in C^∞ by a holomorphic rational function, and we will use this to show that the domain can be well approximated by an area quadrature domain.

Since the boundary is real analytic, there exists a *Schwarz function* $S(z)$ that is holomorphic on a neighborhood of the boundary and satisfies $\bar{z} = S(z)$ on the boundary (see Davis [7], Shapiro [11], or Aharonov and Shapiro [1]). To make this paper self-contained, we will briefly show how to produce $S(z)$. Since the boundary is real analytic, there is a parametrization $\zeta(t)$ of the boundary that is real analytic in the real variable t, i.e., $\zeta(t)$ is equal to a power series in t. Locally, we may let t wander off the real line into a disc centered at a point on the real axis of the complex plane. We obtain a holomorphic function $\zeta(\tau)$ of τ that agrees with $\zeta(t)$ when $\tau = t$ is real. Since $\zeta'(t)$ is nonvanishing, $\zeta(\tau)$ has a local holomorphic inverse $\zeta^{-1}(z)$. The Schwarz function is given locally near the boundary by $\overline{\zeta(\,\overline{\zeta^{-1}(z)}\,)}$.

Suppose now that $S(z)$ is holomorphic on the set of points that fall within a distance $\delta > 0$ of the boundary. Let χ be a function in $C^\infty(\mathbb{C})$ that is equal to one on a neighborhood of the boundary of Ω and that has compact support inside the set of points within a distance of δ of the boundary of Ω. We may think of $\chi(z)S(z)$ as a function in $C_0^\infty(\mathbb{C})$ that is holomorphic on a neighborhood of the boundary of Ω or as a function restricted to Ω that is holomorphic near the boundary. Since $S(z) = \bar{z}$ on the boundary, note that if we let $u(z) = \chi(z)S(z)$, then $u(z) = \bar{z}$ on the boundary and $\Psi := (\partial/\partial\bar{z})(\chi S) = S(z)(\partial\chi/\partial\bar{z})$ is in $C_0^\infty(\Omega)$.

We now apply the Cauchy integral formula to the function $u(z) = \chi(z)S(z)$ and, since Ψ has compact support, let z go to the boundary to obtain

$$\bar{z} = Q(z) + \frac{1}{2\pi i}\iint_\Omega \frac{\Psi(w)}{w - z}\, dw \wedge d\bar{w}. \tag{2.3}$$

for $z \in b\Omega$. Note that the double integral is equal to $\lambda(z)$ on the boundary. Because Ψ is smooth and has compact support in Ω, we may approximate the integral by a (finite) Riemann sum when z is in the boundary to obtain a rational function $R(z)$ with simple poles in Ω that is as uniformly close to $\lambda(z)$ on a neighborhood of the boundary (and consequently C^∞ close on $b\Omega$). As we continue, we will assume that this approximation is as close as we need to meet various conditions.

Assume now that, in addition to having a smooth real analytic boundary, Ω is also simply connected. We may solve the Dirichlet problem with boundary data $\lambda(z)-R(z)$ to obtain a small harmonic function v given as $v = h(z)+\overline{H(z)}$ where $h(z)$ and $H(z)$ are holomorphic functions on Ω that extend holomorphically to an open set containing $\overline{\Omega}$. The functions h and H can be made as close to zero on an open set containing $\overline{\Omega}$ as desired by choosing $R(z)$ to be close enough to $\lambda(z)$. We will give explicit formulas for h and H at the end of this section that will make this point very clear. The formulas will also yield a method for computing h and H. However, for the time being, to see that h and H are small, note that standard elliptic theory yields a solution to the Dirichlet problem v that is harmonic and close to zero on a neighborhood of the closure of Ω. We may decompose v as $h + \overline{H}$ where h and H are holomorphic since Ω is simply connected. Now, since $h' = \partial v/\partial z$ and $H' = \partial\overline{v}/\partial z$, which are both close to zero, we may replace h and H by appropriate antiderivatives that are close to zero on a neighborhood of the closure of Ω.

We may now write

$$\bar{z} - \overline{H(z)} = Q(z) + R(z) + h(z) \tag{2.4}$$

for z in the boundary. We can make $z - H(z)$ as close to the identity as desired by improving the approximation by $R(z)$. Let $\mathcal{S}$ be equal to the meromorphic function on the right hand side of the identity and let $f(z) = z - H(z)$. We may assume that $f(z)$ is a biholomorphic mapping that is as close to the identity on a neighborhood of the closure of Ω as desired. We now claim that f maps Ω to an area quadrature domain. Indeed, let $F = f^{-1}$ and note that $z = f(F(z))$ on $f(\Omega)$. For z in the boundary of $f(\Omega)$, we may write

$$\bar{z} = \mathcal{S}(F(z)),$$

and we see that the meromorphic function on the right is the Schwarz function for $f(\Omega)$ and that it extends meromorphically to $f(\Omega)$. This shows that $f(\Omega)$ is an area quadrature domain. (Alternatively, equation (2.4) shows that f extends to the double of Ω as a meromorphic function and Gustafsson's theorem [9] yields that $f(\Omega)$ is a quadrature domain.) Note that the points in the quadrature identity for the domain $f(\Omega)$ are the images under f of the points used in the Riemann sum approximation to the integral. Since the resulting poles are simple poles, the quadrature identity does not involve any derivatives.

It is interesting to note that we could approximate the rational function $R(z)$ on a neighborhood of the boundary by a rational function $\rho(z)$ that has a single higher order pole at a point a in Ω. The same construction using $\rho(z)$ in place of $R(z)$ would produce an area quadrature domain with a *one point* quadrature identity at the point $f(a)$ involving the value at $f(a)$ and derivatives at $f(a)$. Avci [2] proved that a simply connected one-point arc length quadrature domain must also be an area quadrature domain. (An alternate proof of this fact appears in [6].) Hence, we have shown how to approximate by a double quadrature domain, a stronger form of generalized Riemann mapping theorem.

We now assume that Ω is a finitely multiply connected domain bounded by $n > 1$ smooth real analytic nonintersecting curves. Let γ_n denote the outer boundary curve, and denote the inner boundary curves by γ_j, $j = 1, \dots, n-1$. The argument above in the simply connected domain case breaks down when we try to solve the Dirichlet problem with boundary data $\lambda - R$ because not every harmonic function can be written as a holomorphic plus an antiholomorphic function. To get around this point, we let $b_1, \dots, b_{n-1}$ be points in the interiors of the bounded components of the complement of Ω, one per component, b_j being inside γ_j, $j = 1, \dots, n-1$. A harmonic function u on Ω can be expressed as a sum

$$h + \overline{H}$$

on Ω, where h and H are holomorphic, if and only if the $n-1$ periods of u around the inner boundary curves vanish. Since the periods of $\ln|z-b_j|$ are independent, this means that, given a harmonic function u on Ω, there are uniquely determined constants c_j such that holomorphic functions h and H exist on Ω such that

$$u - \sum_{j=1}^{n-1} c_j \ln|z - b_j| = h + \overline{H}.$$

A real analytic function $\mu(z)$ restricted to a real analytic curve can be holomorphically extended to a neighborhood of the curve via the complexified parametrization that we used to construct the Schwarz function above. Indeed, $\mu(\zeta(t))$ is given by a convergent power series $p(t)$ and $h(z) = p(\zeta^{-1}(z))$ is the holomorphic function that locally extends the function from the curve to the complex plane. By shrinking the neighborhood of the curve, we obtain a unique single valued holomorphic extension.

Let $\sigma_{kj}(z)$ be a holomorphic function defined on a neighborhood of the curve γ_k that is equal to the real analytic function $\ln|z - b_j|$ along γ_k. By adding up C^∞ cut off functions times the σ_{kj} over all k, we may obtain a function $\sigma^j(z)$ in $C^\infty(\overline{\Omega})$ that is holomorphic near the boundary and equal to $\ln|z - b_j|$ on γ_k, $k = 1, \ldots, n$. If we now apply the Cauchy integral formula to σ^j, we obtain that the boundary values of σ^j are given by

$$\ln|z - b_j| = Q_j(z) + \frac{1}{2\pi i} \iint_\Omega \frac{\Psi_j(w)}{w - z}\, dw \wedge d\bar{w},$$

where Q_j is holomorphic on Ω and Ψ_j has compact support. Since the periods of Q_j are zero, we may approximate the area integral by a Riemann sum to get a rational function of z with periods close to the periods of $\ln|z - b_j|$. Hence, since the periods of $\ln|z-b_j|$ are independent, there are $n-1$ rational functions r_j with only simple poles in Ω such that the periods of the r_j are independent.

Now we may repeat the argument we used in the simply connected case. When we get to equation (2.3), we obtain a rational function $R(z)$ such that $\lambda(z) - R(z)$ is as close to zero in $C^\infty(b\Omega)$ as desired. We may now state that there are holomorphic functions $h(z)$ and $H(z)$ that extend holomorphically to an open set containing $\overline{\Omega}$, that are close to zero in $C^\infty(\overline{\Omega})$, and small constants ϵ_j such that

$$\lambda(z) - R(z) = h(z) + \overline{H(z)} + \sum_{j=1}^{n-1} \epsilon_j r_j(z).$$

Finally, we may complete the argument by noting that

$$\bar{z} - \overline{H(z)} = Q(z) + R(z) + h(z) + \sum_{j=1}^{n-1} \epsilon_j r_j(z)$$

on the boundary. As before, the holomorphic mapping $f(z) = z - H(z)$ can be made close to the identity, mapping the domain to a nearby area quadrature domain.

It is an interesting extension of the ideas in this section to replace the function $\bar{z}$ in the Cauchy integral formula by $\overline{F(z)}$ where F is a biholomorphic mapping from the domain Ω to another domain with real analytic boundary. The Schwarz function gets replaced by a holomorphic extension of $\overline{F(z)}$ to a neighborhood of the boundary. The arguments can be repeated, line by line, to show that F can be closely approximated by a mapping $F(z) - H(z)$ that maps Ω to an area quadrature domain. As before, if we replace the rational functions $R(z)$ and $r_j(z)$ by rational functions $\rho(z)$ and $\rho_j(z)$ with higher order poles at a single point a in the domain, we obtain a biholomorphic mapping to a one point area quadrature domain that approximates $F(\Omega)$.

We now give the promised formula for h and H in the simply connected case. The formula will make more sense and look cleaner if we first formalize a way to obtain the Riemann sum that produces the rational function $R(z)$. Let $\theta(z)$ be a C^∞ real valued nonnegative radially symmetric function compactly supported in the unit disc such that $1 = \int_{\mathbb{C}} \theta\, dA$, and, for $\epsilon > 0$, let $\theta_\epsilon(z) = (1/\epsilon^2)\theta(z/\epsilon)$ be the usual approximation to the identity. Note that the averaging property for holomorphic functions implies that

$$h(z) = \frac{i}{2} \iint_\Omega h(w)\theta_\epsilon(z - w)\ dw \wedge d\bar{w}$$

if h is holomorphic on Ω and z is a point in Ω farther than a distance of ϵ from the boundary.

For ψ and φ in $C_0^\infty(\mathbb{C})$, define the convolution,

$$(\psi * \varphi)(z) = \frac{i}{2} \iint_{\mathbb{C}} \psi(z - w)\varphi(w)\ dw \wedge d\bar{w}.$$

Since $\Psi \in C_0^\infty(\Omega)$ and θ_ϵ is an approximation to the identity, it follows that $\theta_\epsilon * \Psi$ converges to Ψ in $C^\infty(\overline{\Omega})$ as $\epsilon \to 0$. By approximating this convolution by a Riemann sum, we can obtain a function

$$\Psi_\epsilon(z) = \sum_{k=1}^{N} c_k \theta_\epsilon(z - a_k)$$

in $C_0^\infty(\Omega)$ that is as close to Ψ in $C^\infty(\overline{\Omega})$ as desired. Define our new rational approximation to

$$\lambda(z) = \frac{1}{2\pi i} \iint_\Omega \frac{\Psi(w)}{w - z}\ dw \wedge d\bar{w}$$

on the boundary via

$$R(z) = \frac{1}{2\pi i} \iint_\Omega \frac{\Psi_\epsilon(w)}{w - z}\ dw \wedge d\bar{w} = \frac{1}{\pi} \sum_{k=1}^{N} \frac{c_k}{z - a_k}.$$

Let $\lambda_\epsilon = \Psi - \Psi_\epsilon$. We think of λ_ϵ as a compactly supported smooth "foam function" since it resembles a solid minus "bubbles" that closely fill out its interior. Note that, when restricted to the boundary, $\lambda(z) - R(z)$ is given by

$$\lambda(z) - R(z) = \frac{1}{2\pi i} \iint_\Omega \frac{\lambda_\epsilon(w)}{w - z}\ dw \wedge d\bar{w},$$

where λ_ϵ has compact support, and so the Poisson extension of $\lambda(z) - R(z)$ to Ω will be given by an integral with respect to w of the Poisson extension of $1/(w-z)$ in the z variable. The Poisson extension of $1/(w-z)$ to Ω as a function of z for fixed w in Ω is equal to

$$\frac{1}{w-z} - 2G_w(w, z),$$

where $G(w, z)$ is the Green's function for Ω, which, if $g(z)$ denotes a Riemann mapping of Ω onto the unit disc is given by

$$G(w, z) = \text{Ln} \left| \frac{g(z) - g(w)}{1 - \overline{g(z)}\, g(w)} \right|.$$

A straightforward calculation shows that the Poisson extension of $1/(w - z)$ is therefore equal to

$$\left(\frac{1}{z-w} - \frac{g'(w)}{g(z) - g(w)} \right) + \left(\frac{g'(w)\, \overline{g(z)}}{1 - \overline{g(z)}\, g(w)} \right).$$

Let $K_1(z, w)$ denote the function in the first set of large parentheses, and let $K_2(z, w)$ denote the second. Note that $K_1(z, w)$ is holomorphic in z and w on $\Omega \times \Omega$ and $K_2(z, w)$ is antiholomorphic in z and holomorphic in w on $\Omega \times \Omega$. The functions h and H are given by

$$h(z) = \frac{1}{2\pi i} \iint_\Omega K_1(z, w) \lambda_\epsilon(w)\; dw \wedge d\bar{w},$$

and

$$\overline{H(z)} = \frac{1}{2\pi i} \iint_\Omega K_2(z, w) \lambda_\epsilon(w)\; dw \wedge d\bar{w}.$$

It is interesting to note that the mapping $z - H(z)$ to a nearby area quadrature domain can be expressed in terms of a Riemann map and a foam function. We now explore the formula for H further to understand it better and to come up with a completely new way to find mappings to nearby area quadrature domains. First, we split H into two parts, $H = H_1 + H_2$, guided by the fact that $\lambda_\epsilon = \Psi - \Psi_\epsilon$. Recall that $\Psi(z) = S(z)(\partial/\partial\bar{z})\chi(z)$. Define

$$\begin{aligned}
\overline{H_1(z)} &= \frac{1}{2\pi i} \iint_\Omega \frac{\overline{g(z)}\, g'(w)}{1 - \overline{g(z)}\, g(w)}\; \Psi(w)\; dw \wedge d\bar{w} \\
&= \frac{\overline{g(z)}}{2\pi i} \iint_\Omega \frac{\partial}{\partial \bar{w}} \left(\frac{g'(w)}{1 - \overline{g(z)}\, g(w)}\; \chi(w) S(w) \right)\; dw \wedge d\bar{w} \\
&= -\frac{\overline{g(z)}}{2\pi i} \int_{b\Omega} \frac{g'(w)}{1 - \overline{g(z)}\, g(w)}\; \bar{w}\; dw.
\end{aligned}$$

If we let G denote the inverse of the Riemann map g, we may further manipulate this last integral to obtain

$$\overline{H_1(z)} = -\frac{\overline{g(z)}}{2\pi i} \int_{C_1(0)} \frac{1}{1 - \overline{g(z)}\, \zeta}\; \overline{G(\zeta)}\; d\zeta,$$

where $C_1(0)$ denotes the unit circle parametrized in the standard sense. Next, note that $\zeta = 1/\bar{\zeta}$ on the unit circle and

$$d\zeta = \frac{-1}{\bar{\zeta}^2}\, d\bar{\zeta},$$

and use the residue theorem to obtain

$$\overline{H_1(z)} = \frac{\overline{g(z)}}{2\pi i}\int_{C_1(0)} \frac{1}{\bar{\zeta}(\bar{\zeta}-\overline{g(z)})}\,\overline{G(\zeta)}\; d\bar{\zeta} = \overline{g(z)}\;\frac{\overline{z-G(0)}}{\overline{g(z)}} = \overline{z-G(0)}.$$

We now turn to studying

$$\overline{H_2(z)} = \frac{1}{2\pi i}\iint_{\Omega} \frac{\overline{g(z)}\,g'(w)}{1-\overline{g(z)}\,g(w)}\;\Psi_\epsilon(w)\; dw\wedge d\bar{w}.$$

Because of the definition of Ψ_ϵ in terms of the functions $\theta_\epsilon(z-a_k)$, and because of the averaging property for holomorphic functions and the fact that $dw\wedge d\bar{w} = -2i\,dA$, we see that

$$\overline{H_2(z)} = \frac{-1}{\pi}\sum_{k=1}^{N} c_k\;\frac{\overline{g(z)}\,g'(a_k)}{1-\overline{g(z)}\,g(a_k)},$$

and we conclude that $H_2(z)$ is a rational function of $g(z)$ that closely approximates $-H_1(z) = G(0)-z$. Let $a = G(0)$. We may now conclude that our mapping $f(z)$ to a nearby area quadrature is given by

$$f(z) = z - H(z) = z - H_1(z) - H_2(z) = z - (z-a) - H_2(z) = a - H_2(z),$$

where $H_2(z)$ is a rational function of $g(z)$. Hence, the inverse of the Riemann map from our area quadrature domain to the unit disc $f\circ G$ is a rational function, as expected, and we have revealed a way to compute that rational function. These observations lead us to a another new way to map a simply connected domain to a nearby area quadrature domain, as we now explain.

The residue theorem showed that

$$z = a + \frac{g(z)}{2\pi i}\int_{C_1(0)} \frac{G(\zeta)}{\zeta(\zeta-g(z))}\; d\zeta,$$

where $a = G(0)$, and we can use the facts that $\zeta = 1/\bar{\zeta}$ on the unit circle and $d\zeta = (-1/\bar{\zeta}^2)d\bar{\zeta}$ to obtain

$$z = a - \frac{g(z)}{2\pi i}\int_{C_1(0)} \frac{G(1/\bar{\zeta})}{(1-\bar{\zeta}g(z))}\; d\bar{\zeta}.$$

If Ω is a simply connected domain bounded by a smooth real analytic Jordan curve, then the Riemann map g and its inverse G extend holomorphically past the boundaries, and we may slide the unit circle in the integral to a smaller circle $C_r(0)$ to obtain

$$z = a - \frac{g(z)}{2\pi i}\int_{C_r(0)} \frac{G(1/\bar{\zeta})}{(1-\bar{\zeta}g(z))}\; d\bar{\zeta},$$

and finally, we may approximate this integral by a Riemann sum to obtain a rational function of $g(z)$ that approximates the function z uniformly on a neighborhood of the closure of Ω. The image of Ω under this map is an area quadrature domain because the inverse of its Riemann map is a rational function (see [12]). Another way to conclude that the image of Ω is an area quadrature is to note that g extends meromorphically to the double because $g(z) = 1/\overline{g(z)}$ on the boundary, and consequently the mapping from Ω, being a rational function of g also extends to the double, and Gustafsson [9] showed that this condition is equivalent to the image being an area quadrature domain.

Being able to slide the curve in the last integral from $C_1(0)$ to a smaller circle is highly related to the fact that the unit disc is a double quadrature domain with a Schwarz function $S(\zeta) = 1/\zeta$ and a unit tangent function $T(\zeta) = i\zeta$ that extend holomorphically to a neighborhood of the unit circle and meromorphically to the disc. We apply similar reasoning in the next section when we wish to approximate a domain by an arc length quadrature domain. We will also return to the problem of approximating by an area quadrature domain in the finitely connected setting with yet another way to generate such maps.

3. The Cauchy integral and arc length quadrature domains

In this section, we show how to construct nearby arc length quadrature domains to a smooth domain bounded by real analytic curves. The techniques are grounded on results presented in [6] about Szegő coordinates and arc length quadrature domains, but they are simpler and more constructive. The simplifications were inspired by observations made in [5] about a simplified and more constructive way to show that the Szegő span is dense in the space of holomorphic functions that are smooth up to the boundary on the unit disc. All of these results spring from the early papers by Gustafsson [10] on arc length quadrature domains in the multiply connected setting, by Shapiro and Ullemar [12] in the simply connected case, and the kernel function outlook of Avci in [2].

To begin, suppose that Ω is a finitely connected bounded domain bounded by smooth real analytic nonintersecting curves. Since the complex unit tangent vector function $T(z)$ is a real analytic function on the boundary, there is a holomorphic extension $\tau(z)$ of it to a neighborhood of the boundary. As mentioned in §2, the Schwarz function $S(z)$ also extends to a neighborhood of the boundary. Hence, there is a neighborhood U of the boundary were both $S(z)$ and $\tau(z)$ are holomorphic. Note that, on the boundary, $S(z) = \bar{z}$ and the element of arc length ds can be written as

$$ds = T(z)\, d\bar{z} = \frac{1}{\overline{\tau(z)}}\, d\bar{z},$$

since $\tau(z) = T(z)$ is unimodular on the boundary. We remark that, since $S(z) = \bar{z}$ on the boundary, it follows that $S'(z)T(z) = \overline{T(z)}$ on the boundary. Hence, $S'(z) = \overline{T(z)^2}$ on the boundary and we conclude that $S'(z) = 1/\tau(z)^2$ on our neighborhood of the boundary.

Suppose $F(z)$ is a biholomorphic mapping of Ω to another domain with real analytic boundary. Then F extends holomorphically past the boundary. We now show how to closely approximate F by another holomorphic mapping that maps Ω to an arc length quadrature domain. (If we want to find an arc length quadrature domain that is *close* to Ω, we simply take $F(z)$ to be the identity function z.)

The technique we are about to describe uses the easy half of the fact proved in [6] that a conformal mapping takes Ω to an arc length quadrature domain if and only if the derivative of the map is the square of a function in the Szegő span of Ω. Let $S(z,w)$ denote the Szegő kernel associated to Ω and write $S_a(z)=S(z,a)$ to emphasize that $S(z,a)$ is a holomorphic function of z when a is held fixed in Ω. Since F' is nonvanishing on a neighborhood of the closure of Ω, and since F is biholomorphic, it is well known that there is a holomorphic branch of a square root of F' on Ω that extends holomorphically past the boundary (see [4, p. 53] for a proof). We choose one of the two branches and denote it by $\sqrt{F'}$.

Suppose h is a holomorphic function that extends holomorphically to a neighborhood of the closure of Ω. We now show how to closely approximate h by an element of the Szegő span via a method that is considerably more elementary than an older method using the Szegő projection in [3]. Since the Szegő kernel reproduces holomorphic functions, we may write

$$h(z)=\int_{w\in b\Omega} S(z,w)h(w)\,ds=\int_{w\in b\Omega} S(z,w)h(\,\overline{S(w)}\,)\,\frac{1}{\overline{\tau(w)}}\,d\bar{w}$$

for z in Ω. Since $S(z,w)$ is antiholomorphic in w and extends antiholomorphically past the boundary in w for each fixed z in Ω, all the functions in the integral are antiholomorphic on a neighborhood of the boundary and we may slide the boundary curve inward a small distance $\epsilon>0$ to a curve γ_ϵ. (Note that ϵ does not depend on z.) We may then approximate the integral

$$h(z)=\int_{\gamma_\epsilon} S(z,w)h(\,\overline{S(w)}\,)\,\frac{1}{\overline{\tau(w)}}\,d\bar{w} \tag{3.1}$$

by a Riemann sum to see that $h(z)$ can be approximated uniformly on a neighborhood of the closure of Ω by a term

$$\sum_{k=1}^{N} c_k S_{a_k}(z)$$

in the Szegő span.

Because the argument is more straightforward in the simply connected case, we assume for the moment that Ω is simply connected. To approximate our mapping F by a mapping that takes Ω to an arc length quadrature, we let $h(z)=\sqrt{F'(z)}$ in the argument above to get an element $s(z)$ of the Szegő span that is close to h. If $s(z)$ is sufficiently close to h, then if will be nonvanishing on a neighborhood of the closure of Ω and $s(z)^2$ will have a holomorphic antiderivative $f(z)$ on a neighborhood of the closure of Ω that is uniformly close to F on that set, and if the approximations are taken sufficiently close, f will be a conformal mapping to a domain with real analytic boundary that is as close as desired to the image of Ω under F. Since f' is equal to the square of the element $s(z)$ of the Szegő span, it follows that $\widetilde{\Omega}=f(\Omega)$ is an arc length quadrature domain. Indeed, to see this, notice that if g is in the Hardy space associated to $\widetilde{\Omega}$, then

$$\int_{b\widetilde{\Omega}} g\ ds=\int_{b\Omega}|f'|(g\circ f)\ ds=\int_{b\Omega}\overline{s(z)}\ s(z)g(f(z))\ ds,$$

and since

$$\int_{b\Omega}\overline{S_{a_k}(z)}\ s(z)g(f(z))\ ds$$

evaluates $s(z)g(f(z))$ at a_k, we obtain a quadrature identity on $\widetilde{\Omega}$ involving the points $f(a_k)$ as a_k run over the points on γ_ϵ used in the Riemann sum. If we had taken $F(z)$ to be the function z, then f' would be close to one and $f(\Omega)$ would be close to Ω.

We now wish to show how to modify our approximation in the simply connected case above to obtain a *one point* arc length quadrature domain instead. Notice that the residue theorem yields that

$$\sum_{k=1}^{N}c_kS_{a_k}(z)=\int_{b\Omega}S(z,w)\overline{R(w)}\ d\bar{w},$$

where $R(w)$ is a holomorphic rational function in w given by

$$R(w)=-\frac{1}{2\pi i}\sum_{k=1}^{N}\frac{\overline{c_k}}{w-a_k}.$$

We may approximate $R(w)$ on the neighborhood of the boundary given by points within a distance of $\epsilon/2$ of the boundary by a rational function

$$\rho(w)=\sum_{k=1}^{M+1}\frac{b_k}{(w-a)^k}$$

with a single higher order pole at a point a inside γ_ϵ. Let

$$S_a^m(z)=\left.\frac{\partial^m}{d\bar{w}^m}S(z,w)\right|_{w=a}.$$

Note that

$$\sum_{k=0}^{M}\beta_kS_a^k(z)=\int_{b\Omega}S(z,w)\overline{\rho(w)}\ d\bar{w},$$

where the higher order Cauchy formula yields that $\beta_k=-2\pi i\,\overline{b_{k+1}}/k!$.

To see that

$$\sigma(z):=\sum_{k=1}^{N}c_kS_{a_k}(z)-\sum_{k=0}^{M}\beta_kS_a^k(z)$$

can be made uniformly small on a neighborhood of the closure of Ω, notice that

$$\sigma(z) = \int_{b\Omega} S(z,w)\left(\overline{R(w)} - \overline{\rho(w)}\right)\, d\bar{w},$$

and we may slide the integral along the boundary curves inside the domain to a curve γ within a distance of $\epsilon/4$ of the boundary. By approximating R by ρ on γ, and by noting the $S(z,w)$ is holomorphic in z and antiholomorphic in w on a neighborhood of $b\Omega \times \gamma$ in $\mathbb{C} \times \mathbb{C}$, we see that $\sigma(z)$ can be made uniformly small on a neighborhood of the closure of Ω. In this way, we get approximations in $C^\infty(\overline{\Omega})$. (Another way to see this is to write

$$\sigma(z) = \int_{b\Omega} S(z,w)\left(\overline{R(w)} - \overline{\rho(w)}\right)\overline{\tau(w)}\, ds,$$

and therefore, σ is seen to be equal to the Szegő projection of a function that can be made uniformly small in C^∞ of the boundary. Since the Szegő projection is a continuous operator from C^∞ of the boundary to $C^\infty(\overline{\Omega})$ (see [4, p. 15]), we may approximate via such a higher order term in the Szegő span.) Using this approximation in place of our original $s(z)$, we may obtain a one point arc length quadrature domain as the image. As mentioned earlier, Avci [2] proved that a simply connected one-point arc length quadrature domain must also be an area quadrature domain. Hence, we have found another way to approximate by a double quadrature domain.

As in the area quadrature domain case of §2, finding a nearby arc length quadrature domain in the multiply connected case is more technical, but all the tools needed to handle it are on the table. We refer the reader to [6] for the details. Results there also show how to find a nearby *double* quadrature domain.

We close this section by showing how the new method for approximating a function by linear combinations of the Szegő kernel has a bearing on the problem of approximating a domain by an area quadrature domain. Suppose that Ω is a bounded domain bounded by finitely many smooth nonintersecting real analytic curves, and for a fixed point a in Ω, let $L_a(z) = L(z,a)$ denote the Garabedian kernel associated to a, which is related to the Szegő kernel via the identity

$$\overline{S(z,a)} = \frac{1}{i} L(z,a)T(z) \tag{3.2}$$

for z in the boundary. The Garabedian kernel extends holomorphically past the boundary in z, has a simple pole in z at $z = a$ with residue $1/(2\pi)$, and is nonvanishing on $\overline{\Omega} - \{a\}$. The function

$$(z-a)L_a(z)$$

has a removable singularity at $z = a$ and extends holomorphically past the boundary. We my therefore approximate it uniformly on a neighborhood of the closure of Ω by an element $\sigma(z)$ in the Szegő span. Since identity (3.2) shows that terms of the form $S(z,b)/L(z,a)$ are equal to the conjugate of $L(z,b)/S(z,a)$ on the boundary, it follows that $\sigma(z)/L_a(z)$ extends meromorphically to the double of Ω. Now

$$a + \sigma(z)/L_a(z)$$

is a function that extends meromorphically to the double and extends to be holomorphic on a neighborhood of the closure of Ω that can be made as close to the identity mapping as desired. If we construct such a map that is one-to-one on a neighborhood of the closure, we obtain a mapping to a nearby area quadrature domain by Gustafsson's theorem [9]. It is interesting to note that, if we replace the function $z - a$ by $F(z) - F(a)$ in this argument, where $F(z)$ is a biholomorphic mapping that extends holomorphically past the boundary, we obtain a mapping that approximates F taking Ω to a smooth area quadrature domain.

4. Simplified proofs of two key density lemmas

The fact proved in §3 that a holomorphic function that extends holomorphically past the boundary of a domain bounded by real analytic curves can be approximated uniformly on a neighborhood of the closure by linear combinations of functions of z of the form $S(z, a)$, where $S(z, a)$ denotes the Szegő kernel associated to the domain, has a cousin that is equally useful and equally relevant to quadrature domain theory. The cousin is obtained by changing the kernel from Szegő $S(z, a)$ to Bergman $K(z, a)$. In this section, we show how the proof given in §3 can be modified to give a new proof of the cousin. Before we do that, we remark that, since a bounded domain with C^∞ smooth boundary is conformally equivalent to a domain with real analytic boundary via a conformal mapping that extends C^∞ smoothly to the boundary, and since holomorphic functions that extend holomorphically past the boundary are dense in the space of holomorphic functions that extend C^∞ smoothly up to the boundary, the transformation formulas for the Szegő and Bergman kernels under conformal mappings can be used to show that the density of the linear combinations of kernel functions on domains with real analytic boundaries that we prove here imply the density in the space of holomorphic functions that are C^∞ smooth up to the boundary on C^∞ smooth domains. These methods yield simpler proofs of the density lemmas than were given in [3] and [4] (in Chapter 9 for the Szegő kernel and p. 173 for the Bergman kernel), which use regularity properties of the Szegő projection and Bergman projection.

Suppose now that Ω is a bounded simply connected domain bounded by a smooth real analytic Jordan curve. Let $K_w(z) = K(z, w)$ denote the Bergman kernel associated to Ω. If h is a holomorphic function on a neighborhood of the closure of Ω, we wish to approximate h uniformly on a domain containing the closure of Ω by linear combinations of the Bergman kernel as we did in §3 with the Szegő kernel. The Schwarz function $S(z)$ associated to Ω extends to be holomorphic on a neighborhood of the boundary and $S(z) = \bar{z}$ on the boundary. There is a holomorphic function H, also holomorphic on a neighborhood of the closure such that $h(z) = H'(z)$ there. Let $\langle \cdot, \cdot \rangle_\Omega$ denote the L^2 inner product with respect to area measure dA in the form

$$\langle u, v\rangle_\Omega = \iint_\Omega u\,\overline{v}\ dA = \iint_\Omega u(w)\ \overline{v(w)}\ (\frac{i}{2}dw \wedge d\bar{w}).$$

We now use the reproducing property of the Bergman kernel

$$h(z) = \langle h, K_z\rangle_\Omega$$

to write

$$\begin{aligned} h(z) &= \iint_\Omega h(w)\overline{K_z(w)}\ (\frac{i}{2}dw \wedge d\bar{w}) \\ &= \iint_\Omega \frac{\partial}{\partial w}\left[H(w)\overline{K_z(w)}\right]\ (\frac{i}{2}dw \wedge d\bar{w}) \\ &= \frac{i}{2}\int_{b\Omega} H(w)\overline{K_z(w)}\ d\bar{w} \\ &= \frac{i}{2}\int_{b\Omega} H(\,\overline{S(w)}\,)\overline{K_z(w)}\ d\bar{w}. \end{aligned}$$

Since the integrand of the last integral is antiholomorphic, we may slide the boundary curve inward and approximate the integral by a finite Riemann sum, thereby approximating h uniformly on a neighborhood of the closure of Ω by a function in the Bergman span.

Suppose now that Ω is a bounded domain bounded by $N \geq 1$ smooth real analytic nonintersecting curves. If h is a holomorphic function on a neighborhood of the closure of Ω, there is a holomorphic function H, also holomorphic on a neighborhood of the closure, and complex constants c_j such that

$$h(z) = H'(z) + \sum_{j=1}^{N-1} c_j F_j'(z),$$

where $F_j' = 2(\partial\omega_j/\partial z)$ is 2 times the derivative of the harmonic measure function ω_j that is harmonic on Ω, one on the j-th inner boundary curve γ_j, and zero on the rest of the boundary curves. Now,

$$\begin{aligned} h(z) = \langle h, K_z\rangle_\Omega &= \iint_\Omega h(w)\overline{K_z(w)}\ (\frac{i}{2}dw \wedge d\bar{w}) \\ &= \iint_\Omega \frac{\partial}{\partial w}\left[\left(H(w) + 2\sum_{j=1}^{N-1} c_j\omega_j(w)\right)\overline{K_z(w)}\right]\ (\frac{i}{2}dw \wedge d\bar{w}) \\ &= \frac{i}{2}\int_{b\Omega} H(w)\overline{K_z(w)}\ d\bar{w} + i\sum_{j=1}^{N-1} c_j \int_{\gamma_j} \overline{K_z(w)}\ d\bar{w} \\ &= \frac{i}{2}\int_{b\Omega} H(\,\overline{S(w)}\,)\overline{K_z(w)}\ d\bar{w} + i\sum_{j=1}^{N-1} c_j \int_{\gamma_j} \overline{K_z(w)}\ d\bar{w} \end{aligned}$$

and because the integrands of all the integrals are antiholomorphic, we may slide the boundary curves inward and approximate the integrals by finite Riemann sums.

These density theorems are closely connected to the problems of approximating domains by area quadrature domains, arc length quadrature domains, and double quadrature domains, and they can be taken as the starting point of the theory (see [6] and Chapter 22 of [4]). We now demonstrate yet another way to approximate a domain by a nearby area quadrature domain. Step one is to approximate the function 1 by a linear combination $\kappa_0(z) = \sum_{k=1}^{N} c_k K(z, a_k)$ in the Bergman span. Step two is to approximate $z\kappa_0(z)$ by another linear combination $\kappa_1(z)$ of the same kind. Now $f(z) = \kappa_1(z)/\kappa_0(z)$ is a holomorphic function close to the identity map. We claim that it extends meromorphically to the double, and therefore maps the given domain to a nearby area quadrature domain. Indeed, the identity

$$\overline{K(z,a)T(z)} = -\Lambda(z,a)T(z)$$

that relates the Bergman kernel to the complementary kernel $\Lambda(z, a)$ (see [4, p. 186]) shows that κ_1/κ_0 is equal to the conjugate of a quotient of similar functions λ_1/λ_0 on the boundary where the function λ_0 is gotten from κ_0 by replacing each c_k by $\overline{c_k}$ and the Bergman kernel $K(z, a_k)$ by $\Lambda(z, a_k)$. Similarly for λ_1. Since $\Lambda(z, a)$ is holomorphic in z on $\Omega - \{a\}$ with a double pole at $z = a$, this shows that $f(z)$ extends meromorphically to the double and $f(\Omega)$ is therefore an area quadrature domain. It seems that ways to approximate by quadrature domains are as abundant as quadrature domains themselves.

References

[1] Aharonov, D. and H. S. Shapiro, *Domains on which analytic functions satisfy quadrature identities*, Journal D'Analyse Mathématique **30** (1976), 39–73.

[2] Avci, Y., Quadrature identities and the Schwarz function, Stanford University PhD thesis, 1977.

[3] Bell, S., *Unique continuation theorems for the $\bar\partial$-operator and applications*, J. of Geometric Analysis **3** (1993), 195–224.

[4] ———, *The Cauchy transform, potential theory, and conformal mapping*, 2nd Edition, CRC Press, Boca Raton, 2016.

[5] ———, *The adjoint of a composition operator via its action on the Szegő kernel*, Analysis and Math. Physics (2018), DOI 10.1007/s13324-018-0215-y.

[6] Bell, S., Gustafsson, B., and Sylvan, Z., *Szegő coordinates, quadrature domains, and double quadrature domains*, Computational Methods and Function Theory **11** (2011), No. 1, 25–44.

[7] P. Davis, *The Schwarz function and its applications*, Carus Mathematical Monographs **17**, Math. Assoc. of Amer., 1974.

[8] Ebenfelt, B. Gustafsson, D. Khavinson, and M. Putinar, *Quadrature domains and their applications*, Operator Theory: Advances and Applications **156**, Birkhäuser, Basel, 2005.

[9] Gustafsson, B., Quadrature domains and the Schottky double, *Acta Applicandae Math.* **1** (1983), 209–240.

[10] Gustafsson, B., Applications of half-order differentials on Riemann surfaces to quadrature identities for arc-length, *Journal D'Analyse Math.* **49** (1987), 54–89.

[11] Shapiro, H. S., The Schwarz function and its generalization to higher dimensions, Univ. of Arkansas Lecture Notes in the Mathematical Sciences, Wiley, New York, 1992.

[12] Shapiro, H. and C. Ullemar, *Conformal mappings satisfying certain extremal properties and associated quadrature identities*, Research Report TRITA-MAT-1986-6, Royal Inst. of Technology, 40 pp., 1981.

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