AHLFORS MAPS, THE DOUBLE OF A DOMAIN, AND COMPLEXITY IN POTENTIAL THEORY AND CONFORMAL MAPPING

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Abstract. We prove that the Bergman kernel function associated to a finitely connected planar domain can be expressed as a rational combination of two independent Ahlfors maps associated to the domain plus the derivative of one of the maps. Similar results are shown to hold for the Szegő and Poisson kernels and other objects of potential theory. These results generalize routinely to the case of a relatively compact domain in a Riemann surface with finitely many boundary components.

1. Introduction. The Bergman kernel associated to an n-connected domain Ω in the plane such that no boundary component is a point has long been known to extend to the double of Ω as a meromorphic differential. In this paper, we express the Bergman, Szegő, and Poisson kernels for Ω in terms of Ahlfors maps and this point of view reveals information about the complexity of the kernels and about the obstruction to the extendibility of the kernels to the double as *functions*. Our results give rise to some interesting questions in conformal mapping and potential theory.

To give a precise statement of our main results, we need to introduce some notation and terminology. Given a point a in Ω, let $f_a(z)$ denote the Ahlfors map associated to (a,Ω), which is a proper holomorphic map of Ω onto the unit disc (see [2, pages 47-52]). Ahlfors maps can be thought of as substitutes for the absent Riemann mapping functions in the multiply connected setting and the results of this paper reinforce this sentiment. Let $K(z,w)$ denote the Bergman kernel associated to Ω and let $S(z,w)$ denote the Szegő kernel. For a boundary component γ_j of $b\Omega$, let ω_j denote the harmonic measure function which is harmonic on Ω, has boundary values of one on γ_j, and which has boundary values of zero on the other boundary components. Let $F_j'(z) := 2(\partial\omega_j/\partial z)$. Let $G(w,z)$ denote the classical Green's function associated to Ω.

Theorem 1.1. *Suppose Ω is an n-connected domain in the plane such that no boundary component is a point. There exist points a and b in Ω and complex rational functions R and Q of four complex variables such that*

$$K(z,w) = f_a'(z)\overline{f_a'(w)}R(f_a(z), f_b(z), \overline{f_a(w)}, \overline{f_b(w)})$$

and

$$S(z,w) = S(z,a)S(a,w)Q(f_a(z), f_b(z), \overline{f_a(w)}, \overline{f_b(w)}).$$

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Furthermore, the functions F'_j can be expressed

$$F'_j(z) = f'_a(z) R_3(f_a(z), f_b(z))$$

where R_3 is a rational function of two complex variables. There exist $n-1$ points $a_1, \ldots, a_{n-1}$ in Ω such that the complex derivative of the Green's function can be expressed

$$\frac{\partial G}{\partial z}(w,z) = R_0(w,z) + \sum_{j=1}^{n-1} R_j(z) \frac{\partial G}{\partial z}(w, a_j),$$

where $R_0(z,w)$ is a rational combination of $S(z,a)$, $f_a(z)$, and $f_b(z)$, and $S(w,a)$, $f_a(w)$, and $f_b(w)$ and the conjugates of $S(w,a)$, $f_a(w)$, and $f_b(w)$, and where $R_j(z)$ is a rational combination of $S(z,a)$, $f_a(z)$, and $f_b(z)$.

We shall see that *most* points a and b in Ω satisfy the conditions of Theorem 1.1.

Theorem 1.1 reveals that the familiar formulas which hold in the simply connected case,

$$S(z,w) = \frac{c\, S(z,a)\overline{S(w,a)}}{1 - f_a(z)\overline{f_a(w)}} \quad \text{and} \quad K(z,w) = \frac{f'_a(z)\overline{f'_a(w)}}{\pi(1 - f_a(z)\overline{f_a(w)})^2},$$

where $c = 1/S(a,a)$ and where the Ahlfors map $f_a(z)$ is equal to the Riemann map at a, have direct analogues in the multiply connected setting. (In the simply connected case, there is no need for a second Riemann map in the formulas because all Riemann maps are rational combinations of a single Riemann map.)

Although these results sound like pure complex variable results, they have a bearing on the complexity of the solution operator to the classical Dirichlet problem. We shall show in the last section of this paper that the classical Poisson kernel can be expressed as a rational combination of the three functions $S(\cdot, a)$, f_a, and f_b and their conjugates plus $n-1$ explicit harmonic functions of one variable. In fact, we shall show that if the boundary of the domain is smooth enough that the Poisson kernel $p(w,z)$ associated to Ω is related to the Green's function via the standard identity,

$$p(w,z) = -\frac{i}{\pi} \frac{\partial}{\partial z} G(w,z) T(z),$$

where $T(z)$ is the complex number of unit modulus which represents the tangent vector at z to $b\Omega$ pointing in the direction of the standard orientation of the boundary, then the result about the Green's function in Theorem 1.1 yields that $p(w,z)$ can be written

$$p(w,z) = R_0(w,z) + \sum_{j=1}^{n-1} R_j(z) \frac{\partial G}{\partial z}(w, a_j),$$

where $R_0(z,w)$ is a rational combination of $S(z,a)$, $f_a(z)$, $f_b(z)$, $S(w,a)$, $f_a(w)$, and $f_b(w)$ and the conjugates of these six functions, and where $R_j(z)$ is a rational combination of $S(z,a)$, $f_a(z)$, and $f_b(z)$ and their conjugates.

Theorem 1.1 shall be seen to be a consequence of next two theorems.

Theorem 1.2. *Suppose Ω is an n-connected domain in the plane such that no boundary component is a point. Let $\widehat{\Omega}$ denote the double of Ω. The functions*

$$\frac{K(w,z)}{f_a'(w)\overline{f_a'(z)}} \quad \text{and} \quad \frac{S(w,z)}{S(w,a)S(a,z)}$$

extend to $\widehat{\Omega}\times\widehat{\Omega}$ as functions that are meromorphic in w and antimeromorphic in z.

Theorem 1.3. *Suppose Ω is an n-connected domain in the plane such that no boundary component is a point. There exist points a and b in Ω such that the two Ahlfors maps f_a and f_b extend meromorphcially to the double of Ω and form a primitive pair for the field of meromorphic functions on the double of Ω.*

The statement in the last theorem means that any meromorphic function on the double of Ω is a rational combination of f_a and f_b.

Theorem 1.2 generalizes routinely to the case of a relatively compact domain Ω in a Riemann surface that has finitely many boundary components, none of which are points. Indeed, the Bergman kernel on Ω is then viewed as a differential $(1,1)$ form $K(w,z)dw\wedge d\bar{z}$, and if we let $\alpha = df_a/f_a$ where f_a is an Ahlfors map, then the proof of Theorem 1.2 given in §3 can be interpreted to yield that

$$\frac{K(w,z)dw\wedge d\bar{z}}{\alpha(w)\wedge\overline{\alpha(z)}}$$

can be viewed as a *function* on $\Omega\times\Omega$ which extends to $\widehat{\Omega}\times\widehat{\Omega}$ to be meromorphic in w and antimeromorphic in z. The corresponding version of Theorem 1.1 that follows from this result is that the Bergman kernel associated to Ω is given as

$$K(w,z)dw\wedge d\bar{z} = R(f_a(w), f_b(w), \overline{f_a(z)}, \overline{f_b(z)})df_a(w)\wedge\overline{df_a(z)}$$

where R is a complex rational function of four variables. We do not give the details of the more general argument here because it is completely parallel to the argument in the planar case.

2. Ahlfors maps and the double of a domain. In order to dispense with repeating a long list of preliminary facts and formulas, I refer the reader to §2 of the paper [5] and the book [2] which it summarizes.

For the moment, assume that Ω is a bounded n-connected domain in the plane such that the boundary of Ω is given by n non-intersecting C^∞ smooth real analytic closed curves. Let $\widehat{\Omega}$ denote the double of Ω and let $R(z)$ denote the antiholomorphic involution on $\widehat{\Omega}$ which fixes the boundary of Ω. Let $\Omega^r = R(\Omega)$ denote the reflection of Ω across the boundary. (I like to think of Ω^r as being the“top half” of the double and Ω as being the “bottom half.”) Given a point a in Ω, the Ahlfors map f_a associated to (a,Ω) is a proper holomorphic map of Ω onto the unit disc which is an n-to-one branched covering map, which extends holomorphically past the boundary of Ω, and which maps each boundary curve of Ω one-to-one onto the unit circle (see [2, page 49]). The Ahlfors map f_a is given by

$$f_a(z) = \frac{S(z,a)}{L(z,a)}$$

where $S(z,a)$ is the Szegő kernel associated to Ω and $L(z,a)$ is the Garabedian kernel (see [2, page 24]). An elementary, but key, fact in what follows is that f_a extends to be a meromorphic function on $\widehat{\Omega}$. This is because $f_a(z) = 1/\overline{f_a(z)}$ for $z \in b\Omega$ and, since $R(z) = z$ on $b\Omega$, it follows that

$$f_a(z) = 1/\overline{f_a(R(z))} \quad \text{for } z \in b\Omega.$$

The function on the left hand side of this formula is holomorphic on Ω and the function on the right hand side is meromorphic on Ω^r and the two functions extend continuously to $b\Omega$ from opposite sides and agree on $b\Omega$. Hence, the function given by $f(z)$ on $\overline{\Omega}$ and $1/\overline{f(R(z))}$ on Ω^r is meromorphic on $\widehat{\Omega}$ (via a simple local argument using Morera's theorem).

Since $\widehat{\Omega}$ is a compact Riemann surface, the ring of meromorphic functions on $\widehat{\Omega}$ is generated by just two elements known as a primitive pair (see Farkas and Kra [8, page 249]). We shall now show that we may choose almost any two Ahlfors maps to serve as a primitive pair for $\widehat{\Omega}$. To see that two Ahlfors maps f_a and f_b form a primitive pair, it is only necessary to show that f_b separates the points in $f_a^{-1}(w_0)$ for some choice of w_0 such that the number of points in $f_a^{-1}(w_0)$ is equal to the order of f_a (see Ahlfors and Sario [1, pages 321-324]). We shall take w_0 to be equal to zero and we shall take a to be a point in Ω such that the zeroes of the Ahlfors map f_a in Ω are n distinct simple zeroes. All but possibly finitely many a in Ω fall into this category, see [2, page 105-108]. Let a_0 denote a (which is the zero of f_a at a) and let $a_1, \dots, a_{n-1}$ denote the other $n-1$ zeroes of f_a (which are also the $n-1$ zeroes of $S(z,a)$ in Ω). We emphasize that we are thinking of f_a and f_b as being defined on $\widehat{\Omega}$, and since $f_a(z) = 1/\overline{f_a(R(z))}$ for $z \in \Omega^r$, it follows that f_a has no zeroes in Ω^r. Let A_{ij} denote the set of points $b \in \Omega$ such that $f_b(a_i) = f_b(a_j)$, i.e., points b where $S(a_i,b)/L(a_i,b) = S(a_j,b)/L(a_j,b)$, i.e., where

$$\frac{S(a_i,b)}{S(a_j,b)} = \frac{L(a_i,b)}{L(a_j,b)}.$$

The function on the left side of this last equality is antiholomorphic in b and the function on the right is holomorphic in b. Furthermore, the function on the right is not constant because $L(a_i,b)$ has a pole in b at $b = a_i$ and $L(a_j,a_i)$ is non-zero. Hence, the largest the set A_{ij} can be is a finite union of real analytic one real dimensional curve segments and points. The set $A := \cup_{i<j} A_{ij}$ where f_b might fail to separate the zeroes of f_a is at most a finite union of real analytic curve segments and points. If we choose b in $\Omega - A$, then f_a and f_b form a primitive pair, and Theorem 1.3 is proved in the case that Ω has smooth real analytic boundary components.

We remark here for future use that if f_a and f_b form a primitive pair for $\widehat{\Omega}$, then any function $H(w,z)$ which is meromorphic in w and antimeromorphic in z on $\widehat{\Omega} \times \widehat{\Omega}$ can be expressed as a rational combination of $f_a(w)$, $f_b(w)$, $\overline{f_a(z)}$, and $\overline{f_b(z)}$.

3. The Bergman kernel and the double of a domain. We shall continue to assume that Ω is a bounded n-connected domain in the plane such that the boundary of Ω is given by n non-intersecting C^∞ smooth real analytic closed curves, and we shall use the notation that we set up in the previous section. In particular, suppose that f_a and f_b are a primitive pair of Ahlfors maps as constructed in §2.

We next prove that the Bergman kernel $K(z,w)$ associated to Ω is such that the function Φ on $\Omega\times\Omega$ given by

$$\Phi(w,z) := \frac{f_a(w)K(w,z)\overline{f_a(z)}}{f_a'(w)\overline{f_a'(z)}}$$

extends as a single valued function to $\widehat{\Omega}\times\widehat{\Omega}$ which is meromorphic in w and antimeromorphic in z. The remark at the end of §2 then yields that there is a complex rational function R of four complex variables such that

$$\Phi(w,z) = R(f_a(w), f_b(w), \overline{f_a(z)}, \overline{f_b(z)}\,)$$

and the proof of the first formula in Theorem 1.1 in the case that Ω has smooth real analytic boundary will be complete.

Define the function $\Lambda(w,z)$ via

$$\Lambda(w,z) = -\frac{2}{\pi}\frac{\partial^2 G(w,z)}{\partial w \partial z}$$

where $G(w,z)$ is the classical Green's function for Ω (see [2, page 135]). The function Λ is a classical kernel and is usually denoted in the literature (see [7]) by L with anywhere between zero and three tildes and/or hats.

The manifold $\widehat{\Omega}\times\widehat{\Omega}$ is a union

$$(\Omega\times\Omega)\cup(\Omega\times\Omega^r)\cup(\Omega^r\times\Omega^r)\cup(\Omega^r\times\Omega),$$

of four large open connected sets plus a union of the five lower dimensional sets

$$(b\Omega\times\Omega)\cup(b\Omega\times\Omega^r)\cup(b\Omega\times b\Omega)\cup(\Omega\times b\Omega)\cup(\Omega^r\times b\Omega).$$

We shall define $\Phi(w,z)$ on each of the four large open sets and then show that the extensions "paste" together continuously along the lower dimensional boundary sets. The function Φ will be meromorphic in w and antimeromorphic in z on the large open sets and the only singular points on the lower dimensional sets will be found along the boundary diagonal $\{(w,z)\in(b\Omega\times b\Omega) : z=w\}$ where Φ has a simple pole in one variable when the other is held fixed. The Hartogs Theorem about separate analyticity can then be used locally along a thin open set containing the lower dimensional sets to show that Φ extends as claimed.

We have already defined $\Phi(w,z)$ on $\Omega\times\Omega$. We define $\Phi(w,z)$ on $\Omega\times\Omega^r$ to be

$$\frac{f_a(w)\Lambda(w,R(z))f_a(R(z))}{f_a'(w)f_a'(R(z))}.$$

We define $\Phi(w,z)$ on $\Omega^r\times\Omega$ to be

$$\frac{\overline{f_a(R(w))\Lambda(R(w),z)f_a(z)}}{\overline{f_a'(R(w))f_a'(z)}}.$$

We define $\Phi(w,z)$ on $\Omega^r\times\Omega^r$ to be

$$\frac{\overline{f_a(R(w))}K(R(z),R(w))f_a(R(z))}{\overline{f_a'(R(w))}f_a'(R(z))}.$$

We shall now derive some identities that will be used to show that these definitions have continuous extensions to the boundaries which match from all sides.

Notice that, because $\ln|f_a(z)|^2 = 0$ for $z \in b\Omega$, we may differentiate $\ln|f(z(t))|^2$ with respect to t when $z(t)$ parameterizes the boundary in the standard sense to obtain

$$\frac{f_a'(z(t))}{f_a(z(t))} z'(t) + \overline{f_a'(z(t))z'(t)/f_a(z(t))} = 0.$$

Dividing this equation by $|z'(t)|$ reveals that

$$\frac{f_a'(z)}{f_a(z)} T(z) = -\overline{f_a'(z)T(z)/f_a(z)} \qquad \text{for } z \in b\Omega, \tag{3.1}$$

where $T(z)$ is the complex unit tangent vector defined via the equation $T(z(t)) = z'(t)/|z'(t)|$. This identity will be used in tandem with the identity

$$\Lambda(w,z)T(z) = -K(w,z)\overline{T(z)} \qquad \text{for } w \in \Omega \text{ and } z \in b\Omega, \tag{3.2}$$

that is satisfied by the Bergman kernel and the kernel Λ (see [2, page 135]).

Because K and Λ extend smoothly to $b\Omega \times b\Omega$ minus the boundary diagonal, (3.2) remains valid if $w \neq z$ and w and z are both boundary points. We now assume that w and z are both boundary points, $w \neq z$, and we multiply (3.2) by $T(w)$ and use the identities,

$$\Lambda(z,w)T(w) = -K(z,w)\overline{T(w)}$$

and $\Lambda(w,z) = \Lambda(z,w)$, to obtain

$$T(z)K(z,w)\overline{T(w)} = T(w)K(w,z)\overline{T(z)} \quad \text{for } z,\ w \text{ in } b\Omega \text{ and } z \neq w. \tag{3.3}$$

We will now use identities (3.1), (3.2), and (3.3) to finally produce three more identities that will show that $\Phi(w,z)$ satisfies the claimed properties. We first assume that w is in Ω and z is in $b\Omega$. Divide (3.2) by (3.1) to obtain

$$\frac{\Lambda(w,z)f_a(z)}{f_a'(z)} = \frac{K(w,z)\overline{f_a(z)}}{\overline{f_a'(z)}}.$$

Next, divide by $f_a'(w)/f_a(w)$ to obtain

$$\frac{f_a(w)\Lambda(w,z)f_a(z)}{f_a'(w)f_a'(z)} = \frac{f_a(w)K(w,z)\overline{f_a(z)}}{f_a'(w)\overline{f_a'(z)}} \tag{3.4}$$

for $z \in b\Omega$ and $w \in \Omega$. This identity extends to hold for z and w in $b\Omega$ with $z \neq w$ by continuity. (Note that f_a' and f_a are non-vanishing and non-singular on $b\Omega$.)

If we divide (3.3) by (3.1) and by the conjugate of (3.1) with w in place of z, we obtain

$$\frac{f_a(z)K(z,w)\overline{f_a(w)}}{f_a'(z)\overline{f_a'(w)}} = \frac{f_a(w)K(w,z)\overline{f_a(z)}}{f_a'(w)\overline{f_a'(z)}} \tag{3.5}$$

when $w \in b\Omega$ and $z \in b\Omega$ with $z \neq w$.

Finally, if we interchange the variables in (3.4) and use the identities $\Lambda(w,z) = \Lambda(z,w)$ and $K(w,z) = \overline{K(z,w)}$ and then take the conjugate of the whole thing, we obtain one more identity,

$$\text{(3.6)} \qquad \frac{\overline{f_a(w)\Lambda(w,z)f_a(z)}}{\overline{f_a'(w)f_a'(z)}} = \frac{f_a(w)K(w,z)\overline{f_a(z)}}{f_a'(w)\overline{f_a'(z)}}$$

for $w \in b\Omega$ and $z \in \Omega$.

We now illustrate how (3.4), (3.5), and (3.6) can be used to show that $\Phi(w,z)$ extends as claimed to $\widehat{\Omega} \times \widehat{\Omega}$. For example, suppose that w is a fixed point in Ω and we want to show that the definitions of $\Phi(w,z)$ on $\{w\} \times \Omega$ and $\{w\} \times \Omega^r$ can be extended antiholomorphically in z across $\{w\} \times b\Omega$. Since $R(z) = z$ on $b\Omega$, formula (3.4) shows that Φ extends continuously in z to $b\Omega$ from both sides and that the extensions agree there. Hence $\Phi(w,z)$ extends antiholomorphically in z across $b\Omega$. Similarly, if w is a fixed point in Ω^r and we want to show that the definitions of $\Phi(w,z)$ on $\{w\} \times \Omega$ and $\{w\} \times \Omega^r$ can be extended antiholomorphically across $\{w\} \times b\Omega$, replace w in (3.4) by $R(w)$, take the conjugate, and note that $R(z) = z$ on $b\Omega$ to see that Φ extends. If $w \in b\Omega$, use (3.5) in to validate the extension in z. Similar reasoning using (3.6) and the conjugate of (3.5) reveals that $\Phi(w,z)$ extends in w if z is held fixed. The argument is the conjugate of the case when w is held fixed because of the symmetric properties of the kernels and we omit it. The statements about the Bergman kernel in Theorems 1.1 and 1.2 are now proved in case the boundary of Ω is given by smooth real analytic curves.

4. The Szegő kernel and the double of a domain. We shall continue to assume that Ω is a bounded n-connected domain in the plane such that the boundary of Ω is given by n non-intersecting C^∞ smooth real analytic closed curves, and we shall use the notation that we set up in the previous sections. In particular, suppose that f_a and f_b are a primitive pair of Ahlfors maps as constructed in §2.

I proved in [6] that, for fixed points A_1 and A_2 in Ω, the functions of z of the form $S(z,A_1)/S(z,A_2)$ extend as meromorphic functions to the double of Ω. This is such a short and simple argument that we repeat it here. Write the identity (see [2, page 24]),

$$S(z,A) = i\,\overline{L(z,A)T(z)},$$

which holds for A in Ω and z in $b\Omega$, using $A = A_1$ and then $A = A_2$, and then divide the two resulting formulas to see that $S(z,A_1)/S(z,A_2)$ is equal to the complex conjugate of $L(z,A_1)/L(z,A_2)$ when $z \in b\Omega$. Hence, $S(z,A_1)/S(z,A_2)$ is a meromorphic function on Ω which extends continuously up to $b\Omega$, and the conjugate of $L(R(z),A_1)/L(R(z),A_2)$ is a meromorphic function on Ω^r which extends continuously up to $b\Omega$ from the "outside" of Ω and which agrees with $S(z,A_1)/S(z,A_2)$ on $b\Omega$. Hence, $S(z,A_1)/S(z,A_2)$ extends to $\widehat{\Omega}$ as a meromorphic function, and therefore $S(z,A_1)/S(z,A_2)$ is equal to a rational combination of the primitive pair $f_a(z)$ and $f_b(z)$.

I proved in [4] that

$$\text{(4.1)} \quad S(z,w) = \frac{1}{1 - f_a(z)\overline{f_a(w)}} \left(c_0 S(z,a)\overline{S(w,a)} + \sum_{i,j=1}^{n-1} c_{ij} S(z,a_i)\,\overline{S(w,a_j)} \right),$$

where f_a is the Ahlfors mapping associated to a point $a \in \Omega$ and the points $a_1, \ldots, a_{n-1}$ are the zeroes of $S(z,a)$ in the z variable. The constants in the formula are given by $c_0 = 1/S(a,a)$, and c_{ij} are the coefficients of the inverse matrix to the non-singular matrix $[S(a_j, a_k)]$. The only restriction on the point a is that it be chosen so that the zeroes of $S(z,a)$ are simple zeroes; this is true for all but finitely many a in Ω. We shall now assume that this condition is met by the function f_a of our primitive pair. Since $S(z,a_i)/S(z,a)$ has been shown to be a rational combination of f_a and f_b, it now follows from (4.1) that the function

$$\Psi(z,w) := \frac{S(z,w)}{S(z,a)\overline{S(w,a)}}$$

is a rational combination of $f_a(z)$, $f_b(z)$, and the conjugates of $f_a(w)$ and $f_b(w)$. This completes the proof of the formula for the Szegő kernel given in Theorem 1.1 in case the boundary of the domain is given by smooth real analytic curves. It also shows that Ψ extends to $\widehat{\Omega} \times \widehat{\Omega}$ as claimed in Theorem 1.2 because the Ahlfors maps extend to the double.

5. The case of non-smooth boundary curves. We now suppose that Ω is merely an n-connected domain in the plane such that no boundary component is a point. It is well known that there is a biholomorphic mapping φ mapping Ω one-to-one onto a bounded domain Ω^a in the plane with smooth real analytic boundary. The standard construction yields a domain Ω^a that is a bounded n-connected domain with C^∞ smooth boundary whose boundary consists of n non-intersecting simple closed real analytic curves. Let superscript a's indicate that a kernel function is associated to Ω^a. Kernels without superscripts are associated to Ω. The transformation formula for the Bergman kernels under biholomorphic mappings gives

$$K(z,w) = \varphi'(z) K^a(\varphi(z), \varphi(w)) \overline{\varphi'(w)}. \tag{5.1}$$

It is well known that the function φ' has a single valued holomorphic square root on Ω (see [2, page 43]). We shall *define* the Szegő kernel and Garabedian kernel associated to Ω via the natural transformation formulas,

$$S(z,w) = \sqrt{\varphi'(z)}\; S^a(\varphi(z), \varphi(w)) \overline{\sqrt{\varphi'(w)}} \tag{5.2}$$

and

$$L(z,w) = \sqrt{\varphi'(z)}\; L^a(\varphi(z), \varphi(w)) \sqrt{\varphi'(w)}. \tag{5.3}$$

The Green's functions satisfy

$$G(z,w) = G^a(\varphi(z), \varphi(w)) \tag{5.4}$$

and the functions associated to harmonic measure satisfy

$$\omega_j(z) = \omega_j^a(\varphi(z)) \quad \text{and} \quad F_j'(z) = \varphi'(z) F_j^{a\prime}(\varphi(z)),$$

provided that we stipulate that the boundary components have been numbered so that φ maps the j-th boundary component of Ω to the j-th boundary component of Ω^a. Finally, the Ahlfors map associated to a point $b \in \Omega$ is defined to be the solution to the extremal problem, $f_b : \Omega \to D_1(0)$ with $f_b'(b) > 0$ and maximal. It is easy to see that Ahlfors maps satisfy

$$f_b(z) = \lambda f^a_{\varphi(b)}(\varphi(z)) \tag{5.5}$$

for some unimodular constant λ and it follows that $f_b(z)$ is a proper holomorphic mapping of Ω onto $D_1(0)$.

Formulas (5.1) and (5.5) now make it an easy matter to verify that the function $\Phi(w, z)$ defined in §3 satisfies the invariance property

$$\Phi(w, z) = \Phi^a(\varphi(w), \varphi(z)).$$

Formulas (5.2) and (5.5) reveal that the function $\Psi(z, w)$ defined in §4 also satisfies the invariance property

$$\Psi(z, w) = \Psi^a(\varphi(z), \varphi(w)).$$

We proved in §3 and §4 that Φ^a and Ψ^a are rational combinations of two Ahlfors maps associated to Ω^a. Since Ahlfors maps also pull back under φ via (5.5), it follows that Φ and Ψ are rational combinations of the two corresponding Ahlfors maps associated to Ω and the proof of the first two formulas in Theorem 1.1 is complete in the general case of non-smooth boundary.

We remark here that if the Bergman kernel is algebraic, then so is the Ahlfors map f_a (see [5]), and hence there is an irreducible polynomial $P(z, w)$ of two complex variables such that $P(z, f_a(z)) \equiv 0$ on Ω. Differentiate this identity with respect to z to see that $f_a'(z)$ is a rational function of f_a and z. Hence, the formula in Theorem 1.1 reveals that an algebraic Bergman kernel $K(z, w)$ is a rational combination of $f_a(z)$, $f_b(z)$, and z and the conjugates of $f_a(w)$, $f_b(w)$, and w.

6. Other objects of potential theory and Ahlfors maps. We shall now assume that Ω is a bounded n-connected domain in the plane such that the boundary of Ω is given by n non-intersecting C^∞ smooth real analytic closed curves. The classical functions F_j' satisfy the identity

$$F_j'(z)T(z) = -\overline{F_j'(z)T(z)} \tag{6.1}$$

for $z \in b\Omega$ (see [2, page 80]). Divide this equation by formula (3.1) to obtain

$$\frac{f_a(z)F_j'(z)}{f_a'(z)} = \frac{\overline{f_a(z)F_j'(z)}}{\overline{f_a'(z)}}$$

for $z \in b\Omega$, where f_a is an Ahlfors map. We may now replace z by $R(z)$ in the right hand side of this last equation to see that $f_a(z)F_j'(z)/f_a'(z)$ extends to the double of Ω as a meromorphic function. Hence, it can be written as a rational combination of two Ahlfors maps, and the formula for F_j' in Theorem 1.1 is seen to be true in case Ω has a smooth real analytic boundary. The transformation formulas for F_j' and the Ahlfors maps in §5 can now be used to show that a formula of the form

$$\frac{f_a(z)F_j'(z)}{f_a'(z)} = R_4(f_a(z), f_b(z))$$

can be pulled back under conformal mappings; hence the formula is valid on a general n-connected domain such that no boundary component is a point.

It is interesting to note that when the formulas for $K(z,w)$ and F_j' in Theorem 1.1 are inserted in the identity

$$K(z,w) = 4\pi S(z,w)^2 + \sum_{i,j=1}^{n-1} A_{ij} F_i'(z)\overline{F_j'(w)}$$

(see [7, page 119] or [2, pages 94–96]), we obtain that

$$\frac{S(z,w)^2}{f_a'(z)\overline{f_a'(w)}} = R_4(f_a(z), f_b(z), \overline{f_a(w)}, \overline{f_b(w)}\,)$$

for some complex rational function R_4. This shows that $S(z,w)^2/(f_a'(z)\overline{f_a'(w)})$ extends to $\widehat{\Omega} \times \widehat{\Omega}$ as a single valued function which is meromorphic in z and antimeromorphic in w. This formula is analogous to the identity

$$S(z,w) = \frac{\sqrt{f_a'(z)}\,\overline{\sqrt{f_a'(w)}}}{2\pi(1 - f_a(z)\overline{f_a(w)})}$$

which holds in the simply connected case (where f_a would be a Riemann map).

We next turn to the study of the Poisson kernel associated to Ω. Notice that the identity $S(w,z) = -iL(w,z)T(z)$, which holds for $w \in \Omega$ and $z \in b\Omega$, when multiplied by its conjugate, yields that

$$S(z,w)L(z,w)T(z) = -\overline{S(z,w)L(z,w)T(z)} \tag{6.2}$$

for w in Ω and $z \in b\Omega$. A similar identity holds for the derivative of the Green's function with respect to z,

$$\frac{\partial G}{\partial z}(w,z)T(z) = -\overline{\frac{\partial G}{\partial z}(w,z)T(z)}, \tag{6.3}$$

for $w \in \Omega$ and $z \in b\Omega$ (see [2, page 134]).

Choose a point a in Ω so that the $n-1$ zeroes of $S(z,a)$ in the z variable are simple zeroes (see [2, page 105-108]) and let $a_1, \ldots, a_{n-1}$ denote these zeroes. By choosing a to be sufficiently close to the boundary, we may assume that the a_j are as close to the boundary as we wish. Let γ_n denote the outer boundary of Ω and let γ_j, $j = 1, \ldots, n-1$ denote the inner boundary curves of Ω. We shall show at the end of this section that the $(n-1)\times(n-1)$ matrix $[F_j'(a_k)]$ (where $j = 1, \ldots, n-1$ and $k = 1, \ldots, n-1$) is non-singular.

Define a function H on $\Omega \times \overline{\Omega}$ via

$$H(w,z) := \frac{\partial G}{\partial z}(w,z) - \frac{S(z,w)L(z,w)}{S(w,w)}.$$

Notice that $H(w,z)$ is holomorphic in z on $\Omega - \{w\}$. In fact, we may view $H(w,z)$ as being holomorphic in z on all of Ω because the singular parts $1/(2\pi(z-w))$ of the two functions at $z = w$ in the difference exactly cancel. Furthermore, (6.2) and (6.3) reveal that H satisfies the identity

$$H(w,z)T(z) = -\overline{H(w,z)T(z)} \tag{6.4}$$

for $w \in \Omega$ and $z \in b\Omega$. We may now divide (6.4) by an instance of (6.2) using $w = a$ to obtain

$$\frac{H(w,z)}{S(z,a)L(z,a)} = \overline{H(w,z)}/(\overline{S(z,a)L(z,a)}) \tag{6.5}$$

for $w \in \Omega$ and $z \in b\Omega$. Define

$$Q_w(z) := H(w,z)/(S(z,a)L(z,a)).$$

Identity (6.5) shows that Q_w extends to the double of Ω as a meromorphic function of z for each w in Ω and that the extension satisfies

$$Q_w(z) = \overline{Q_w(R(z))}. \tag{6.6}$$

Notice that, as a function on $\widehat{\Omega}$, $Q_w(z)$ might have simple poles at the points $a_1, \dots, a_{n-1}$ and the points $R(a_1), \dots, R(a_{n-1})$. We shall now subtract off terms to eliminate the singularities of Q_w on $\widehat{\Omega}$ and thereby obtain a holomorphic function on $\widehat{\Omega}$, i.e., a constant.

Observe that the function

$$q_j(z) := \frac{F_j'(z)}{S(z,a)L(z,a)}$$

extends to the double of Ω via (6.1) and the same reasoning we used to see that Q_w does. Furthermore, q_j also satisfies

$$q_j(z) = \overline{q_j(R(z))} \tag{6.7}$$

on $\widehat{\Omega}$. We now consider the function

$$\mathcal{M}(w,z) := Q_w(z) - \sum_{j=1}^{n-1} c_j(w) q_j(z)$$

where the complex numbers $c_j(w)$ are defined to solve the system,

$$0 = H(w, a_k) - \sum_{j=1}^{n-1} c_j(w) F_j'(a_k), \quad k = 1, \dots, n-1.$$

These values of $c_j(w)$ have been chosen so that $\mathcal{M}(w,z)$ has no singularities in z on $\overline{\Omega}$. We shall now show that $\mathcal{M}(w,z)$ has no singularities in z in Ω^r either, and it is therefore constant in z.

We now assume that a has been chosen to be close enough to the boundary so that the a_j are close enough to the boundary to fall in coordinate charts with the following properties. There is a chart mapping the unit disc to a neighborhood of a boundary point of Ω in $\widehat{\Omega}$ in such a way that the real line maps into $b\Omega$ and $-i/2$ gets mapped to a_j. In this coordinate, the reflection function $R(z)$ is given by $R(z)=\bar{z}$ and we may write (6.6) and (6.7) in the form

$$Q_w(z)=\overline{Q_w(\overline{z})} \quad \text{and} \quad g_j(z)=\overline{q_j(\overline{z})}.$$

It follows that, in the special coordinate, the residue of the simple pole of Q_w at $-i/2$ is the *conjugate* of the residue of the simple pole of Q_w at the reflected point $i/2$ which corresponds to $R(a_j)$. The same holds for the functions q_j.

Notice that $\mathcal{M}(w,z)$ satisfies

$$\mathcal{M}(w,z)=\overline{Q_w(R(z))}-\sum_{j=1}^{n-1} c_j(w)\overline{q_j(R(z))}$$

when $z\in b\Omega$ and so $\mathcal{M}(w,z)$ is equal to the conjugate of

$$Q_w(R(z))-\sum_{j=1}^{n-1}\overline{c_j(w)}q_j(R(z)) \quad \text{when } z\in\Omega^r. \tag{6.8}$$

Because of the fact mentioned above about the conjugate residues in the special coordinate chart and because $c_j(w)$ is given a conjugate in formula (6.8), we may state that $\mathcal{M}(w,z)$ has removable singularities in z at the points $R(a_j)$. Hence, $\mathcal{M}(w,z)$ is *holomorphic* in z on $\widehat{\Omega}$, and hence constant. Plugging in $z=a$ reveals that this constant is *zero* (by virtue of the pole of $L(z,a)$ at $z=a$). We have therefore proved that

$$\frac{\partial G}{\partial z}(w,z)=\frac{S(z,w)L(z,w)}{S(w,w)}-\sum_{j=1}^{n-1}c_j(w)F_j'(z)$$

where $c_j(w)$ is a linear combination of

$$\frac{\partial G}{\partial z}(w,a_j)-\frac{S(a_j,w)L(a_j,w)}{S(w,w)}.$$

Theorem 1.1 yields that $S(z,w)$ can be expressed as a rational combination of $S(z,a)$, $f_a(z)$, and $f_b(z)$, and conjugates of $S(w,a)$, $f_a(w)$, and $f_b(w)$. Since, $f_a(z)=S(z,a)/L(z,a)$, it follows that $L(z,a)$ is a rational combination of $S(z,a)$, $f_a(z)$, and $f_b(z)$. Similarly, $L(z,a_i)=S(z,a_i)/f_{a_i}(z)$. Since $f_{a_i}(z)$ is an Ahlfors map, it extends to the double and is therefore a rational combination of f_a and f_b. It follows that $L(z,a_i)$ is also a rational combination of $S(z,a)$, $f_a(z)$, and $f_b(z)$. Next, the identity (see [4, page 1368]),

$$L(z,w)=\frac{f_a(w)}{f_a(z)-f_a(w)}\left(c_0S(z,a)L(w,a)+\sum_{i,j=1}^{n-1}\bar{c}_{ij}S(z,a_i)L(w,a_j)\right),$$

shows that $L(z,w)$ is a rational combination of $S(z,a)$, $f_a(z)$, and $f_b(z)$, and $S(w,a)$, $f_a(w)$, and $f_b(w)$. Finally, since $F_j'(z)/(S(z,a)L(z,a))$ extends meromorphically to $\widehat{\Omega}$, it follows that $F_j'(z)$ is a rational combination of $S(z,a)$, $f_a(z)$, and $f_b(z)$. We may now state that

$$\frac{\partial G}{\partial z}(w,z) = R_0(w,z) + \sum_{j=1}^{n-1} R_j(z)\frac{\partial G}{\partial z}(w,a_j),$$

where $R_0(z,w)$ is a rational combination of $S(z,a)$, $f_a(z)$, and $f_b(z)$, and $S(w,a)$, $f_a(w)$, and $f_b(w)$ and the conjugates of $S(w,a)$, $f_a(w)$, and $f_b(w)$, and where $R_j(z)$ is a rational combination of $S(z,a)$, $f_a(z)$, and $f_b(z)$.

In case the boundary of Ω is not smooth, note that the functions Q_w and q_j are invariant under biholomorphic mappings and so an identity of the form $Q_w - \sum_{j=1}^{n-1} c_j(w)q_j \equiv 0$ is also invariant and the arguments above carry over.

When the boundary of Ω is sufficiently smooth, the Poisson kernel $p(w,z)$ is related to the Green's function via

$$p(w,z) = -\frac{i}{\pi}\frac{\partial}{\partial z}G(w,z)T(z),$$

and since $T(z) = iS(a,z)/L(z,a)$, our result about the Green's function yields that $p(w,z)$ can be written

$$p(w,z) = R_0(w,z) + \sum_{j=1}^{n-1} R_j(z)\frac{\partial G}{\partial z}(w,a_j),$$

where $R_0(z,w)$ is a rational combination of $S(z,a)$, $f_a(z)$, and $f_b(z)$, and $S(w,a)$, $f_a(w)$, and $f_b(w)$ and the conjugates of these six functions, and where $R_j(z)$ is a rational combination of $S(z,a)$, $f_a(z)$, and $f_b(z)$ and their conjugates.

To complete the arguments in this section, we need to prove that the $n-1$ by $n-1$ matrix $[F_j'(a_k)]$ (where $j = 1,\ldots,n-1$ and $k = 1,\ldots,n-1$) is non-singular. Schiffer proved that the complex linear span $\mathcal{F}'$ of $\{F_j' : j = 1,\ldots,n-1\}$ is equal to the complex linear span of $\{S(z,a)L(z,a_j) : j = 1,\ldots,n-1\}$ (see [2, page 80]). Since $\mathcal{F}'$ is an $n-1$ dimensional vector space, there is a non-singular matrix M which describes the change of bases. Let $S'(z,a)$ denote the partial derivative $(\partial/\partial z)S(z,a)$ and notice that $S'(a_k,a) \neq 0$ because a has been chosen so that the zeroes of $S(z,a)$ are simple zeroes. Next, observe that

$$S(z,a)L(z,a_j) = \begin{cases} 0, & \text{if } z = a_k,\ k \neq j \\ \frac{1}{2\pi}S'(a_j,a), & \text{if } z = a_j, \end{cases}$$

and hence the matrix $[S(a_k,a)L(a_k,a_j)]$ is non-singular. Since this matrix is related to $[F_j'(a_k)]$ via matrix multiplication by M, it follows that $[F_j'(a_k)]$ is non-singular and the proof of the formulas for the Green's function and Poisson kernel given above is complete.

We close by mentioning a tantalizing question that the results of this paper raises concerning the complexity of the Green's function. Might it hold that the Green's function can be written as

$$G(w,z) = R(w,z) + \ln|Q(w,z)|$$

where R and Q are rational combinations of $f_a(w)$, $f_b(w)$, $f_a(z)$, $f_b(z)$, and their conjugates? The Bergman kernel is a constant times $(\partial^2/\partial w\partial\bar{z})G(w,z)$ and although it might appear at first sight that differentiating the expression above for G with respect to w and $\bar{z}$ would lead to terms involving both f_a' and f_b', the offending f_b' terms can be eliminated by noting that (3.1) implies that

$$\frac{f_a'(z)f_b(z)}{f_a(z)f_b'(z)}$$

extends to the double of Ω as a meromorphic function. Hence $f_a'(z)/f_b'(z)$ extends to the double as a meromorphic function and it follows that

$$f_b'(z) = f_a'(z)R_1(f_a(z), f_b(z))$$

where R_1 is a complex rational function. One would obtain a formula for the Bergman kernel which is consistent with the formula in Theorem 1.1.

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