THE FUNDAMENTAL ROLE OF THE SZEGŐ KERNEL IN POTENTIAL THEORY AND COMPLEX ANALYSIS

Steven R. Bell*

Abstract. We shall show that the Szegő and Bergman kernels associated to a finitely connected domain in the plane are generated by only *three* holomorphic functions of *one complex variable* of the form $H(z) = S(z, A)$ where $S(z, w)$ is the Szegő kernel and A is a fixed point in the domain. Many other important functions of potential theory and conformal mapping theory will be shown to be rational combinations of the same three basic functions.

1. Introduction. I showed in [4] that the Bergman and Szegő kernels associated to an n-connected domain in the plane are generated by $n+1$ basic holomorphic functions of *one complex variable.* I also showed that the Poisson kernel is generated by finitely many functions of one complex variable plus $n-1$ harmonic functions of one complex variable. Here we shall show that the Szegő and Bergman kernels associated to an n-connected domain in the plane such that no boundary component is a point are generated by only *three* holomorphic functions of *one complex variable.* We also show that the Poisson kernel is a rational combination of the same three holomorphic functions plus $n-1$ explicit harmonic functions of one complex variable. Furthermore, many other functions of potential theory and conformal mapping theory, including the classical functions $F_j' = 2(\partial\omega_j/\partial z)$, $j = 1, \ldots, n-1$, will be shown to be rational combinations of the three basic functions.

To be more precise, suppose that Ω is an n-connected domain in the plane such that no boundary component is a point. For the purpose of this introduction, assume that the boundary of Ω consists of n non-intersecting C^∞ smooth closed curves γ_j, $j = 1, \ldots, n$. (Later, we shall show how to totally eliminate all assumptions about boundary smoothness.) Let $S(z, w)$ denote the Szegő kernel associated to Ω and let $K(z, w)$ denote the Bergman kernel. We shall show that there exist three points A_1, A_2, and A_3 in Ω such that the Szegő and Bergman kernels are generated by the three functions $S(\cdot, A_j)$, $j = 1, 2, 3$, in the sense that $K(z, w)$ and $S(z, w)$ are given as rational combinations of $S(z, A_1)$, $S(z, A_2)$, and $S(z, A_3)$ and the conjugates of $S(w, A_1)$, $S(w, A_2)$, and $S(w, A_3)$. Furthermore, the Poisson kernel $p(z, w)$ associated to Ω can be written

$$R_0(z, w) + \sum_{k=1}^{n-1} R_k(w) h_k(z),$$

where $R_0(z, w)$ is a rational combination of $S(z, A_j)$ and $S(w, A_j)$, $j = 1, 2, 3$, and the conjugates of these six functions, and where $R_k(w)$ is a rational combination of

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$S(w, A_j)$, $j = 1, 2, 3$, and their conjugates, and where the h_k are explicit harmonic functions of one variable that extend C^∞ smoothly up to the boundary.

The points A_1, A_2, and A_3 are not particularly special. In fact, the set of points (A_1, A_2, A_3) with the properties above is open and dense in $\Omega \times \Omega \times \Omega$, and it can be said that the results above are true for a generic choice of A_1, A_2, and A_3.

For a generic choice of A_1, A_2, and A_3, we shall also show that there is a holomorphic polynomial $P(z, w)$ of two complex variables such that

$$P(\frac{S(z, A_1)}{S(z, A_3)}, \frac{S(z, A_2)}{S(z, A_3)}) \equiv 0$$

for $z \in \Omega$, and it follows from this that $S(z, A_1)/S(z, A_2)$ is an algebraic function of $S(z, A_2)/S(z, A_3)$. Therefore, $S(z, A_1)$ is an algebraic function of $S(z, A_2)$ and $S(z, A_3)$. When this result is combined with the results mentioned above, we deduce that the Szegő kernel $S(z, w)$ is a rational combination of algebraic functions of $S(z, A_2)$ and $S(z, A_3)$ and the conjugates of algebraic functions of $S(w, A_2)$ and $S(w, A_3)$. The same is true of the Bergman kernel.

The starting point for all of these results is the formula

$$(1.1) \quad S(z, w) = \frac{1}{1 - f_a(z)\overline{f_a(w)}} \left(c_0 S(z, a)\overline{S(w, a)} + \sum_{i,j=1}^{n-1} c_{ij} S(z, a_i)\, \overline{S(w, a_j)} \right)$$

that I proved in [4] which expresses the Szegő kernel in terms of an Ahlfors mapping $f_a(z)$ associated to a point $a \in \Omega$ and the n other functions of one variable given by $S(z, a)$, and $S(z, a_i)$, $i = 1, \dots, n-1$. The points $a_1, \dots, a_{n-1}$ are the zeroes of $S(z, a)$ in the z variable and the constants in the formula are given by $c_0 = 1/S(a, a)$, and c_{ij} are the coefficients of the inverse matrix to the non-singular matrix $[S(a_j, a_k)]$. (The only restriction on the point a is that it be chosen so that the zeroes of $S(z, a)$ are simple zeroes; this is true for all but finitely many a in Ω. We shall define the Ahlfors map and relate the zeroes of the Ahlfors map to the zeroes of the Szegő kernel and the pole of the Garabedian kernel in the next section.)

We take a moment here to sketch the main ideas used in the proofs of our results. We shall prove that for fixed points A_i and A_j in Ω, the functions of z of the form $S(z, A_i)/S(z, A_j)$ extend as meromorphic functions to the double of Ω. Furthermore, for a generic choice of points A_1, A_2, and A_3 in Ω, the extensions of the two functions

$$\frac{S(z, A_1)}{S(z, A_3)} \quad \text{and} \quad \frac{S(z, A_2)}{S(z, A_3)}$$

form a *primitive pair* for the double of Ω (see Farkas and Kra [9, page 249]) in the sense that they generate the field of meromorphic functions on the double of Ω. We shall show that the Ahlfors map f_a extends to the double of Ω as a meromorphic function. Since $S(z, a)/S(z, A_3)$ and $S(z, a_j)/S(z, A_3)$ extend to the double as meromorphic functions, it follows that the $n+1$ functions that appear in formula (1.1) can all be expressed as rational combinations of the three functions $S(z, A_1)$, $S(z, A_2)$, and $S(z, A_3)$. Hence $S(z, w)$ can be expressed as a rational combination of $S(z, A_1)$, $S(z, A_2)$, and $S(z, A_3)$ and the conjugates of $S(w, A_1)$, $S(w, A_2)$, and $S(w, A_3)$. We shall show that this conclusion remains valid if Ω is merely an n-connected domain such that no boundary component is a point.

The Bergman kernel $K(z,w)$ is related to the Szegő kernel via the identity

$$K(z,w) = 4\pi S(z,w)^2 + \sum_{i,j=1}^{n-1} A_{ij} F_i'(z)\overline{F_j'(w)}, \tag{1.2}$$

where the functions $F_i'(z)$ are classical functions of potential theory described as follows ([8, page 119], or see also [2, pages 94–96]). The harmonic function ω_j which solves the Dirichlet problem on Ω with boundary data equal to one on the boundary curve γ_j and zero on γ_k if $k \neq j$ has a multivalued harmonic conjugate. The function $F_j'(z)$ is a globally defined single valued holomorphic function on Ω which is locally defined as the derivative of $\omega_j + iv$ where v is a local harmonic conjugate for ω_j. The Cauchy-Riemann equations reveal that $F_j'(z) = 2(\partial\omega_j/\partial z)$. We note here that, although F_j' is locally the derivative of a holomorphic function, it is not the derivative of a holomorphic function defined on all of Ω; the prime in the notation is traditional. We shall also show that the functions $F_j'(z)$ are rational combinations of $S(z,A_1)$, $S(z,A_2)$, and $S(z,A_3)$ and, hence, it shall follow that the Bergman kernel is a rational combination of $S(z,A_1)$, $S(z,A_2)$, and $S(z,A_3)$ and the conjugates of $S(w,A_1)$, $S(w,A_2)$, and $S(w,A_3)$. We shall show that this conclusion remains valid when Ω is only assumed to be an n-connected domain such that no boundary component is a point (with absolutely no assumptions made about boundary regularity).

In §5 we shall apply similar ideas to analyze formula (7.5) from [4] which relates the Poisson kernel to the Szegő kernel to prove the claim made above about the complexity of the Poisson kernel. In the last section of the paper, we study the field of meromorphic functions on Ω generated by the functions of z given by $\{S(z,a) : a \in \Omega\}$. Our main results above show that this field is equal to the field generated by the three functions $S(z,A_1)$, $S(z,A_2)$, and $S(z,A_3)$ for a generic choice of the points A_1, A_2, and A_3 in Ω. We show that this field contains an astounding variety of objects of potential theory and conformal mapping, including all the major kernel functions and all proper holomorphic mappings from Ω onto the unit disk. Other finitely generated function fields and rings with similar universal properties will be described.

Before we proceed, we take a moment to state our main results carefully.

Theorem 1.1. *Suppose that Ω is an n-connected domain in the plane such that no boundary component is a point. There exist three points A_1, A_2, and A_3 in Ω such that $K(z,w)$ and $S(z,w)$ are given as rational combinations of $S(z,A_1)$, $S(z,A_2)$, and $S(z,A_3)$ and the conjugates of $S(w,A_1)$, $S(w,A_2)$, and $S(w,A_3)$. The functions $F_j'(z)$ are rational combinations of $S(z,A_1)$, $S(z,A_2)$, and $S(z,A_3)$, and so is every proper holomorphic map of Ω onto the unit disc. The Green's function $G(z,w)$ associated to Ω satisfies*

$$\frac{\partial G}{\partial w}(z,w) = r_0(z,w) + \sum_{k=1}^{n-1} r_k(w) h_k(z),$$

where $r_0(z,w)$ is a rational combination of $S(w,A_1)$, $S(w,A_2)$, and $S(w,A_3)$ and $S(z,A_1)$, $S(z,A_2)$, and $S(z,A_3)$ and the conjugates of $S(z,A_1)$, $S(z,A_2)$, and $S(z,A_3)$, and where $r_k(w)$ is a rational combination of $S(w,A_1)$, $S(w,A_2)$, and

$S(w, A_3)$, for each k, and where $h_k(z)$ is an explicit harmonic function of z given by

$$G_w(z, w_k) - \frac{S(z, w_k)L(z, w_k)}{S(w_k, w_k)},$$

where w_k is a fixed point in Ω.

If the boundary of Ω is smooth enough that the Poisson kernel $p(z, w)$ associated to Ω is related to the Green's function via the standard identity

$$p(z, w) = -\frac{i}{\pi}\frac{\partial}{\partial w}G(z, w)T(w)$$

(see §2 for the definition of T), then Theorem 1.1 will yield that the Poisson kernel can be written as $R_0(z, w) + \sum_{k=1}^{n-1} R_k(w)h_k(z)$, where the functions $R_0(z, w)$ and $R_k(w)$ are as we described above.

2. Some preliminary facts. Assume that Ω is an n-connected domain in the plane such that no boundary component is a point, and that the boundary of Ω consists of n non-intersecting C^∞ smooth closed curves γ_j, $j = 1, \ldots, n$. Suppose that γ_j is parameterized in the standard sense by $z_j(t)$, $0 \le t \le 1$. Let $T(z)$ be the C^∞ function defined on $b\Omega$ such that $T(z)$ is the complex number representing the unit tangent vector at $z \in b\Omega$ pointing in the direction of the standard orientation (meaning that $iT(z)$ represents the *inward pointing normal vector* at $z \in b\Omega$). This complex unit tangent vector function is characterized by the equation $T(z_j(t)) = z_j'(t)/|z_j'(t)|$.

The Szegő kernel $S(z, w)$ associated to Ω is holomorphic in the first variable and antiholomorphic in the second on $\Omega \times \Omega$ and hermitian, i.e., $S(w, z) = \overline{S(z, w)}$. Furthermore, the Szegő kernel is in $C^\infty((\overline{\Omega} \times \overline{\Omega}) - \{(z, z) : z \in b\Omega\})$ as a function of (z, w) (see [2, page 100]). The Garabedian kernel $L(z, w)$ is related to the Szegő kernel via the identity

$$\text{(2.1)} \qquad \frac{1}{i}L(z, a)T(z) = S(a, z) \qquad \text{for } z \in b\Omega \text{ and } a \in \Omega.$$

For fixed $a \in \Omega$, the kernel $L(z, a)$ is a holomorphic function of z on $\Omega - \{a\}$ with a simple pole at a with residue $1/(2\pi)$. Furthermore, as a function of z, $L(z, a)$ extends to the boundary and is in the space $C^\infty(\overline{\Omega} - \{a\})$. In fact, $L(z, w)$ is in $C^\infty((\overline{\Omega} \times \overline{\Omega}) - \{(z, z) : z \in \overline{\Omega}\})$ as a function of (z, w) (see [2, page 102]). Also, $L(z, a)$ is non-zero for all (z, a) in $\overline{\Omega} \times \Omega$ with $z \neq a$ and $L(a, z) = -L(z, a)$ (see [2, page 49]).

Given a point $a \in \Omega$, the Ahlfors map f_a associated to the pair (Ω, a) is a proper holomorphic mapping of Ω onto the unit disc. It is an n-to-one mapping (counting multiplicities), it extends to be in $C^\infty(\overline{\Omega})$, and it maps each boundary curve one-to-one onto the unit circle. Furthermore, $f_a(a) = 0$, and f_a is the unique function mapping Ω into the unit disc maximizing the quantity $|f_a'(a)|$ with $f_a'(a)$ real and positive. The Ahlfors map is related to the Szegő kernel and Garabedian kernel via (see [2, page 49])

$$\text{(2.2)} \qquad f_a(z) = \frac{S(z, a)}{L(z, a)}.$$

Note that $f_a'(a) = 2\pi S(a,a) \neq 0$. Because f_a is n-to-one, f_a has n zeroes. The simple pole of $L(z,a)$ at a accounts for the simple zero of f_a at a. The other $n-1$ zeroes of f_a are given by the $(n-1)$ zeroes of $S(z,a)$ in $\Omega - \{a\}$. Let $a_1, a_2, \ldots, a_{n-1}$ denote these $n-1$ zeroes (counted with multiplicity). I proved in [3] (see also [2, page 105]) that, as a tends to a boundary curve γ_j, the $n-1$ zeroes $a_1, \ldots, a_{n-1}$ become distinct simple zeroes which separate and tend toward the $n-1$ distinct boundary components of Ω which are different from γ_j. Formula (1.1) is valid for any point $a \in \Omega$ such that the zeroes of $S(z,a)$ in the z variable are distinct simple zeroes.

If Ω is merely an n-connected domain in the plane such that no boundary component of Ω is a point, then it is well known that there is a biholomorphic mapping Φ mapping Ω one-to-one onto a bounded domain Ω^a in the plane with real analytic boundary. The standard construction yields a domain Ω^a that is a bounded n-connected domain with C^∞ smooth boundary whose boundary consists of n non-intersecting simple closed real analytic curves. Let superscript a's indicate that a kernel function is associated to Ω^a. Kernels without superscripts are associated to Ω. The transformation formula for the Bergman kernels under biholomorphic mappings gives

$$K(z,w) = \Phi'(z) K^a(\Phi(z), \Phi(w)) \overline{\Phi'(w)}. \tag{2.3}$$

It is well known that the function Φ' has a single valued holomorphic square root on Ω (see [2, page 43]). To avoid a discussion of the meaning of the Cauchy transform and the Szegő projection in non-smooth domains, we shall opt to *define* the Szegő kernel associated to Ω to be the function on $\Omega \times \Omega$ given by the natural transformation formula,

$$S(z,w) = \sqrt{\Phi'(z)}\; S^a(\Phi(z), \Phi(w)) \overline{\sqrt{\Phi'(w)}}. \tag{2.4}$$

Similarly, the Garabedian kernel is defined via

$$L(z,w) = \sqrt{\Phi'(z)}\; L^a(\Phi(z), \Phi(w)) \sqrt{\Phi'(w)}. \tag{2.5}$$

(When Ω has C^2 smooth boundary, the map Φ extends C^1 up to the boundary and these formulas are classical.) The Green's functions satisfy

$$G(z,w) = G^a(\Phi(z), \Phi(w)) \tag{2.6}$$

and the functions ω_j associated to harmonic measure satisfy

$$\omega_j(z) = \omega_j^a(\Phi(z)) \quad \text{and} \quad F_j'(z) = \Phi'(z) F_j^{a\prime}(\Phi(z)).$$

Finally, the Ahlfors map associated to a point $b \in \Omega$ is defined to be the solution to the extremal problem, $f_b : \Omega \to D_1(0)$ with $f_b'(b) > 0$ and maximal. It is easy to see that the Ahlfors map satisfies

$$f_b(z) = \lambda f^a_{\Phi(b)}(\Phi(z))$$

for some unimodular constant λ and it follows that $f_b(z)$ is a proper holomorphic mapping of Ω onto $D_1(0)$. Furthermore, the transformation formula (2.4) yields that $f_b(z)$ is given by $S(z,b)/L(z,b)$, and it can be verified that formula (1.1) holds in this more general setting. Furthermore, the transformation formulas reveal that (1.2) is also valid for Ω.

3. The Szegő kernel and the double of a domain. We now suppose that Ω is an n-connected domain in the plane such that no boundary component is a point. For the moment, we further assume that the boundary of Ω consists of n non-intersecting *real analytic* C^∞ smooth closed curves. Let $\widehat{\Omega}$ denote the double of Ω and let $R(z)$ denote the antiholomorphic involution on $\widehat{\Omega}$ which fixes the boundary of Ω. Let $\widetilde{\Omega} = R(\Omega)$ denote the reflection of Ω in $\widehat{\Omega}$ across the boundary. It is easy to see that an Ahlfors map f_a extends to be a meromorphic function on $\widehat{\Omega}$ because $f(z) = 1/\overline{f(z)}$ for $z \in b\Omega$ and, since $R(z) = z$ on $b\Omega$, it follows that

$$f_a(z) = 1/\overline{f_a(R(z))} \quad \text{for } z \in b\Omega.$$

The function on the left hand side of this formula is holomorphic on Ω and the function on the right hand side is meromorphic on $\widetilde{\Omega}$ and the two functions extend continuously to $b\Omega$ from opposite sides and agree on $b\Omega$. Hence, the function given by $f_a(z)$ on $\overline{\Omega}$ and $1/\overline{f_a(R(z))}$ on $\widetilde{\Omega}$ is meromorphic on $\widehat{\Omega}$.

We shall now prove that for fixed points A_1 and A_2 in Ω, functions of z of the form $S(z, A_1)/S(z, A_2)$ extend as meromorphic functions to the double of Ω. Indeed, if we write the conjugate of formula (2.1), first using $a = A_1$ and then $a = A_2$, and divide the two resulting formulas, we see that $S(z, A_1)/S(z, A_2)$ is equal to the complex conjugate of $L(z, A_1)/L(z, A_2)$ when $z \in b\Omega$. Hence, $S(z, A_1)/S(z, A_2)$ is a meromorphic function on Ω which extends continuously up to $b\Omega$, and the conjugate of $L(R(z), A_1)/L(R(z), A_2)$ is a meromorphic function on $\widetilde{\Omega}$ which extends continuously up to $b\Omega$ from the "outside" of Ω and which agrees with $S(z, A_1)/S(z, A_2)$ on $b\Omega$. Hence, $S(z, A_1)/S(z, A_2)$ extends to $\widehat{\Omega}$ as a meromorphic function.

Choose A_1 in Ω so that the $n-1$ zeroes of $S(z, A_1)$ in the z variable are distinct and simple. Choose $A_3 \in \Omega$ so that the $n-1$ zeroes of $S(z, A_3)$ are distinct and simple and such that no zero of $S(z, A_3)$ is a zero of $S(z, A_1)$. That this is possible follows from the fact mentioned in §2 that the zeroes of the Szegő kernel separate into distinct simple zeroes that tend to different boundary components as A_3 tends to the boundary.

We now wish to show that it is possible to choose a point A_2 so that the meromorphic extensions of $S(z, A_1)/S(z, A_3)$ and $S(z, A_2)/S(z, A_3)$ to $\widehat{\Omega}$ form a primitive pair for $\widehat{\Omega}$ (meaning that they generate the field of meromorphic functions on the double of Ω, see Farkas and Kra [9, page 249]). Let $b_1, \ldots, b_{n-1}$ denote the zeroes of $S(z, A_1)$. Let $G(z)$ denote the meromorphic extension of $S(z, A_1)/S(z, A_3)$ to $\widehat{\Omega}$. The order of $G(z)$ as a meromorphic function on $\widehat{\Omega}$ is n because $S(z, A_1)/S(z, A_3)$ has $n-1$ zeroes in Ω and no zeroes on $b\Omega$, and the conjugate of $L(R(z), A_1)/L(R(z), A_3)$ has exactly one zero at $R(A_3)$ in $\widetilde{\Omega}$. For a fixed point A_2 in Ω, let $H(z)$ be defined to be the meromorphic extension of $S(z, A_2)/S(z, A_3)$ to $\widehat{\Omega}$.

We shall now prove that A_2 can be chosen so that $G(z)$ and $H(z)$ form a primitive pair. To do this, we need only show that $H(z)$ separates the n points in $\widehat{\Omega}$ where $G(z)$ vanishes, i.e., the points $b_1, \ldots, b_{n-1}$, and $R(A_3)$ (see [1, page 321-324]). Of course, we shall choose A_2 to be unequal to A_3, and so it follows that $H(R(A_3)) = 0$. Hence, our problem reduces to showing that H separates the points $b_1, \ldots b_{n-1}$ with non-zero values.

Let S_i denote the set of points w in $\overline{\Omega}$ where $S(b_i, w) = 0$. Since the Szegő kernel is not identically zero in w (see [2, page 49]), and since $S(b_i, w)$ extends antiholomorphically to a neighborhood of $\overline{\Omega}$, it follows that S_i is a finite set. Let $S_0 = \cup_{i=1}^{n-1} S_i$. (Think of S_0 as being the set of A_2 where $H(b_i)$ might not be non-vanishing for $i = 1, \ldots, n-1$.) By choosing $A_2 \in \Omega - S_0$, we are assured that $H(b_j) \neq 0$ for $j = 1, \ldots, n-1$.

Let S_{ij} denote the the set of points w in $\overline{\Omega}$ such that $S(b_i, w) = c_{ij} S(b_j, w)$ where $c_{ij} = S(b_i, A_3)/S(b_j, A_3)$ is a non-zero constant in $\mathbb{C}$. (Think of S_{ij} as being the set of points A_2 such that H might not separate b_i from b_j.) We shall now show that S_{ij} is a finite set.

The complex linear span of the set of functions of z given by $\mathcal{S} := \{S(z, w) : w \in \Omega\}$ is dense in $H^2(b\Omega)$ because if $h(z)$ is a function in $H^2(b\Omega)$ that is orthogonal to all the functions in the spanning set, then $h(w) = 0$ for all $w \in \Omega$ by the reproducing property of the Szegő kernel. Hence, it follows that any complex polynomial can be uniformly approximated on a compact subset of Ω by functions in $\mathcal{S}$. Let P be a polynomial of one complex variable such that $P(b_i) \neq c_{ij} P(b_j)$. Since it is possible to choose linear combinations of functions in the spanning set $\mathcal{S}$ which converge uniformly on compact subsets of Ω to P, it follows that the function of w given by $S(b_i, w) - c_{ij} S(b_j, w)$ cannot be identically zero in w on Ω. Since this function extends to be holomorphic on a neighborhood of $\overline{\Omega}$ as a function of w, it follows that S_{ij} is a finite set. Hence, the set

$$\{A_3\} \cup S_0 \cup (\cup_{i<j} S_{ij})$$

of points A_2 such that $H(z)$ might fail to separate the n points $R(A_3), b_1, \ldots, b_{n-1}$ as a function of z, is at most a finite subset of $\overline{\Omega}$. We assume from now on that A_2 is not in this finite set.

We now have functions G and H as defined above which form a primitive pair. We showed above that quotients of the form $S(z, a)/S(z, A_3)$ and $S(z, a_j)/S(z, A_3)$ where a and a_j are the points appearing in formula (1.1) extend as meromorphic functions to $\widehat{\Omega}$. We also showed that $f_a(z)$ extends meromorphically to $\widehat{\Omega}$. It follows that the $n+1$ functions that appear in formula (1.1) can all be expressed as rational combinations of $G(z)$ and $H(z)$, and hence, they are rational combinations of $S(z, A_1)$, $S(z, A_2)$, and $S(z, A_3)$. It follows that $S(z, w)$ can be expressed as a rational combination of $S(z, A_1)$, $S(z, A_2)$, and $S(z, A_3)$ and the conjugates of $S(w, A_1)$, $S(w, A_2)$, and $S(w, A_3)$.

Finally, we must show how to eliminate the boundary smoothness assumption. Formula (2.4) reveals that quotients of the form $S(z, b)/S(z, B)$ are invariant under biholomorphic changes of variables. To be precise, if $\Phi : \Omega_1 \to \Omega_2$ is biholomorphic, then

$$\frac{S_1(z, b)}{S_1(z, B)} = c \frac{S_2(\Phi(z), b')}{S_2(\Phi(z), B')}$$

where c is a non-zero constant and $b' = \Phi(b)$ and $B' = \Phi(B)$. Since any n-connected domain in the plane such that no boundary component is a point can be mapped biholomorphically to an n-connected domain with smooth real analytic boundary curves, the remarks at the end of §2 reveal that the formulas required in the proof above carry over to this more general setting. In particular, rational identities between quotients of the form $S(z, b)/S(z, B)$ on the smooth domain give rise to similar identities on the non-smooth domain, and the proof given above carries over word for word.

4. The harmonic measure functions and the Bergman kernel. Assume again that Ω is an n-connected domain in the plane such that no boundary component is a point and that the boundary of Ω consists of n non-intersecting C^∞ smooth closed curves. Let $\mathcal{F}'$ denote the vector space of functions given by the complex linear span of the set of functions $\{F_j'(z) : j = 1, \ldots, n-1\}$ mentioned in §2. It is a classical fact that $\mathcal{F}'$ is $n-1$ dimensional. As above, choose a point $a \in \Omega$ such that the $n-1$ zeroes of $S(z,a)$ are simple and denote these zeroes by $a_1, \ldots, a_{n-1}$. Notice that $S(z,a_i)L(z,a)$ is in $C^\infty(\overline{\Omega})$ because the pole of $L(z,a)$ at $z=a$ is cancelled by the zero of $S(z,a_i)$ at $z=a$. A theorem due to Schiffer ([10], see also [2, page 80]) states that the set of $n-1$ functions $\{S(z,a_i)L(z,a) : i = 1, \ldots, n-1\}$ form a basis for $\mathcal{F}'$. This result will allow us to see that the functions in $\mathcal{F}'$ are also in the field $\mathcal{R}$ generated by the three functions $S(z,A_1)$, $S(z,A_2)$, and $S(z,A_3)$ produced in §3. Indeed, the Ahlfors map $f_a(z)$ was shown to extend to the double of Ω in §3 and so $f_a(z)$ is in $\mathcal{R}$. Since $f_a(z) = S(z,a)/L(z,a)$ and since we know that $S(z,a)$ is in $\mathcal{R}$, it follows that $L(z,a)$ in is $\mathcal{R}$. Finally, since $S(z,a_i)$ is in $\mathcal{R}$, we see that $S(z,a_i)L(z,a)$ is in $\mathcal{R}$ and it follows that $\mathcal{F}' \subset \mathcal{R}$.

We may now deduce from (1.2) that $K(z,w)$ is generated by $S(z,A_1)$, $S(z,A_2)$, and $S(z,A_3)$ and the conjugates of $S(w,A_1)$, $S(w,A_2)$, and $S(w,A_3)$. The remarks made at the ends of §2 and §3 reveal that this result remains valid even if Ω is only assumed to be an n-connected domain such that no boundary component is a point.

5. The Poisson kernel. Assume again that Ω is an n-connected domain in the plane such that no boundary component is a point and that the boundary of Ω consists of n non-intersecting C^∞ smooth closed curves. Let the point a be chosen (as above) such that the $n-1$ zeroes $a_1, \ldots, a_{n-1}$ of $S(z,a)$ are distinct and simple. It is proved in [4, pp. 1358–1362] that the Poisson kernel $p(z,w)$ associated to Ω is given by

$$(5.1)\quad p(z,w) = 2\mathrm{Re}\left[\frac{S(z,w)L(w,a)}{L(z,a)} - \sum_{j=1}^{n-1}\mu_j(w)\int_{\zeta\in\gamma_j}\frac{S(z,\zeta)L(\zeta,a)}{L(z,a)}\,ds\right] + \frac{|S(w,a)|^2}{S(a,a)} + \sum_{j=1}^{n-1}(\omega_j(z) - \lambda_j(a))\mu_j(w)$$

where $\lambda_j(a)$ is a constant given by

$$\int_{\zeta\in\gamma_j}\frac{|S(\zeta,a)|^2}{S(a,a)}\,ds$$

and the μ_j are functions given by

$$\mu_j(w) = \sum_{k=1}^{n-1} B_{jk}S(a_k,w)S(w,a)$$

where the constants B_{jk} are given explicitly in [4]. Furthermore, the μ_j satisfy

$$\delta_{kj} = \int_{\gamma_k}\mu_j\,ds$$

(where δ_{kj} denotes the Kronecker delta).

Formula (5.1) together with results of §3 show that the Poisson kernel is given as a rational combination of $S(z,A_1)$, $S(z,A_2)$, and $S(z,A_3)$ and $S(w,A_1)$, $S(w,A_2)$, and $S(w,A_3)$ and their conjugates, plus the $n-1$ holomorphic functions of z given by

$$\nu_j(z) := \int_{\zeta\in\gamma_j} \frac{S(z,\zeta)L(\zeta,a)}{L(z,a)}\, ds$$

and their conjugates, plus the $n-1$ harmonic functions $\omega_j(z)$. We shall now work to reduce this list of functions to only three holomorphic functions and $n-1$ harmonic functions of one variable.

It is also shown in [4, p. 1362] that

$$p(a,w) = \frac{|S(w,a)|^2}{S(a,a)} + \sum_{j=1}^{n-1}(\omega_j(a)-\lambda_j(a))\mu_j(w) \tag{5.2}$$

where μ_j and λ_j are defined as above. This formula was proved in a rather round about fashion in [4]. Since I am going to use it as the cornerstone of my proof here, and since it can be proved from first principles rather easily, I take the liberty of giving a quick proof of it here. Let $G(z,w)$ denote the classical Green's function associated to Ω, i.e., the unique function such that $G(z,w)+\log|z-w|$ extends to be harmonic in z on Ω for each $w\in\Omega$ and such that $G(z,w)$ extends up to the boundary in z with zero boundary values for each $w\in\Omega$. The Poisson kernel is related to the normal derivative of the Green's function via

$$p(z,w) = \frac{1}{2\pi}\frac{\partial}{\partial n_w}G(z,w) \qquad z\in\Omega,\ w\in b\Omega,$$

where $(\partial/\partial n_w)$ denotes the normal derivative in the w variable. We may rewrite this last formula in the form

$$p(z,w) = -\frac{i}{\pi}\frac{\partial}{\partial w}G(z,w)T(w). \tag{5.3}$$

Let us use subscript w's to denote the partial derivative $(\partial/\partial w)$ and subscript $\bar w$'s to denote the partial derivative $(\partial/\partial\bar w)$. Since the Green's function vanishes as a function of w on $b\Omega$ when $z\in\Omega$, the tangential derivative of the Green's function in the w variable is zero. Let $\zeta(s)$ denote a parametrization of $b\Omega$ with respect to arc length s. We may write

$$0 = \frac{d}{ds}G(z,\zeta(s)) = \frac{\partial G}{\partial\zeta}(z,\zeta(s))\zeta'(s) + \frac{\partial G}{\partial\bar\zeta}(z,\zeta(s))\overline{\zeta'(s)},$$

and, from this, we can deduce that

$$G_w(z,w)T(w) = -G_{\bar w}(z,w)\overline{T(w)} = -\overline{G_w(z,w)T(w)} \tag{5.4}$$

for $w\in b\Omega$ and $z\in\Omega$. Notice that $G_w(a,w)$ is holomorphic as a function of w on $\Omega-\{a\}$ and has a simple pole in the w variable at $w=a$ with residue $1/2$. Formula (2.1) implies that the function

$$H(a,w) := \pi\frac{S(w,a)L(w,a)}{S(a,a)}$$

also satisfies

$$H(a,w)T(w) = -\overline{H(a,w)T(w)} \tag{5.5}$$

for $w \in b\Omega$ and $a \in \Omega$. Furthermore, since $L(w,a)$ has a simple pole at $w = a$ with residue $1/(2\pi)$ and $S(a,a) \neq 0$, it follows that $H(a,w)$ has a simple pole in the w variable at $w = a$ with residue $1/2$. We may combine (5.4) and (5.5) to see that $G_w(a,w) - H(a,w)$ may be viewed as a holomorphic function of w in $C^\infty(\overline{\Omega})$ satisfying

$$(G_w(a,w) - H(a,w))T(w) = -\overline{(G_w(a,w) - H(a,w))T(w)} \tag{5.6}$$

for $w \in b\Omega$, and $a \in \Omega$. It is known (see [2, p. 79-81]) that holomorphic functions that satisfy an identity like (5.6) are in the finite dimensional subspace of $L^2(b\Omega)$ of functions which are orthogonal to both holomorphic functions and antiholomorphic functions and, as such, are given by $h(w)T(w)$ for some $h \in \mathcal{F}'$. Hence, we may state that there exist constants $c_j(a)$ such that

$$G_w(a,w) = H(a,w) + \sum_{j=1}^{n-1} c_j(a) F_j'(w). \tag{5.7}$$

It is well known that the matrix of periods

$$A_{kj} = \int_{\gamma_k} F_j'(w)\ dw$$

is non-singular (see [2, pp. 81-82]), and so we may choose a basis $\{u_j\}_{j=1}^{n-1}$ for $\mathcal{F}'$ such that

$$\delta_{kj} = \int_{\gamma_k} u_j(w)\ dw.$$

Using this new basis, we may write

$$G_w(a,w) = H(a,w) + \sum_{j=1}^{n-1} b_j(a) u_j(w). \tag{5.8}$$

To determine the constants $b_j(a)$, we integrate (5.8) against $-(i/\pi)\omega_k(w)T(w)\,ds$ and use (5.3) and the most basic property of the Poisson kernel to obtain

$$\omega_k(a) = -i \int_{w\in\gamma_k} \frac{S(w,a)L(w,a)}{S(a,a)}\ dw - (i/\pi) b_j(a),$$

and hence,

$$b_j(a) = i\pi \left(\omega_k(a) + i \int_{w\in\gamma_k} \frac{S(w,a)L(w,a)}{S(a,a)}\ dw \right).$$

We can use (2.1) in the form $-iL(w,a)\,dw = \overline{S(w,a)}\,ds$ to transform this last equation into

$$b_j(a) = i\pi \left(\omega_k(a) - \lambda_k(a) \right).$$

We next multiply (5.8) by $-(i/\pi)T(w)$ and use (5.3) and (2.1) to obtain

$$p(a,w)=\frac{|S(w,a)|^2}{S(a,a)}+\sum_{j=1}^{n-1}(\omega_k(a)-\lambda_j(a))\,u_j(w)T(w). \tag{5.9}$$

Finally, it is easily verified that the functions $u_j(w)T(w)$ in (5.9) are equal to the functions $\mu_j(w)$ appearing in (5.2) and the proof of (5.2) is complete.

Notice that (5.8) can now be written

$$G_w(a,w)=\pi\frac{S(w,a)L(w,a)}{S(a,a)}+i\pi\sum_{j=1}^{n-1}(\omega_k(a)-\lambda_k(a))\,u_j(w). \tag{5.10}$$

This is the formula that we shall use most directly in what follows.

We remark here that it is easy to see that the functions $\lambda_j(a)$ are in $C^\infty(\overline{\Omega})$ and are such that $\lambda_j(a)=1$ for $a\in\gamma_j$ and $\lambda_j(a)=1$ for $a\in\gamma_k$, $k\neq j$. Indeed, notice that (2.1) can be used to write

$$\lambda_j(a)=\frac{1}{iS(a,a)}\int_{w\in\gamma_j}S(w,a)L(w,a)\,dw. \tag{5.11}$$

The Residue Theorem can be used to transform this last equation to

$$\begin{aligned}\lambda_j(a)&=\frac{1}{iS(a,a)}\left[\frac{2\pi i}{2\pi}S(a,a)-\int_{w\in\Omega-\gamma_j}S(w,a)L(w,a)\,dw\right]\\ &=1-\frac{1}{iS(a,a)}\int_{w\in\Omega-\gamma_j}S(w,a)L(w,a)\,dw\end{aligned} \tag{5.12}$$

Since $1/S(a,a)$ is a function in $C^\infty(\overline{\Omega})$ that vanishes on $b\Omega$ (see [4, p. 1344]), formula (5.11) shows that λ_j is smooth up to γ_k if $k\neq j$ and vanishes on γ_k and formula (5.12) shows that λ_j is smooth up to γ_j and is equal to one there.

We may now begin the proof that the Poisson kernel can be written as

$$R_0(a,w)+\sum_{j=1}^{n-1}R_j(w)h_j(a),$$

where $R_0(a,w)$ is a rational combination of $S(a,A_1)$, $S(a,A_2)$, $S(a,A_3)$, $S(w,A_1)$, $S(w,A_2)$, and $S(w,A_3)$ and the conjugates of these six functions, and where $R_j(w)$ is a rational combination of $S(w,A_1)$, $S(w,A_2)$, and $S(w,A_3)$ and their conjugates for $j=1,\dots,n-1$, and where the functions $h_j(a)$ are harmonic on Ω and extend smoothly up to the boundary. The plan of the proof is as follows. We shall show that there exist points $w_1,w_2,\dots,w_{n-1}$ in Ω such that $\det[u_j(w_k)]\neq 0$. Then we shall plug $w=w_k$ into (5.10) for $k=1,\dots,n-1$ to obtain a system. Solving that system for the functions $(\omega_j(a)-\lambda_j(a))$ reveals that these functions are linear combinations of the functions of a given by $G_w(a,w_k)$ and

$$\frac{S(w_k,a)L(w_k,a)}{S(a,a)}$$

for $k = 1, \ldots, n-1$. We shall plug this information into (5.2) and note that all the terms on the right hand side besides the Green's function terms have been expressed as rational combinations of the three basic Szegő kernel functions to deduce that the Poisson kernel can be written as $R_0(a,w) + \sum_{j=1}^{n-1} R_j(w)G_w(a,w_j)$ where R_0 and R_j are functions of the desired form. Finally, since

$$\frac{S(a,w_j)L(a,w_j)}{S(w_j,w_j)}$$

has exactly the same singularity in a at w_j that $G_w(a,w_j)$ has, we may add and subtract this expression from $G_w(a,w_j)$ and set $h_j(a)$ to be equal to the harmonic function

$$G_w(a,w_j) - \frac{S(a,w_j)L(a,w_j)}{S(w_j,w_j)},$$

which is in $C^\infty(\overline{\Omega})$, to be able to write $p(a,w) = R_0(a,w) + \sum_{j=1}^{n-1} R_j(w)h_j(a)$ as desired.

To prove there exist points $w_1, w_2, \ldots, w_{n-1}$ in Ω such that $\det[u_j(w_k)] \neq 0$, it will suffice to prove that there exist points such that $\det[F_j'(w_k)] \neq 0$ because $\{u_j\}$ and $\{F_j'\}$ are both bases for $\mathcal{F}'$. Suppose that $\det[F_j'(w_k)] \equiv 0$ for all $w = (w_1, \ldots, w_{n-1})$ in $\Omega^{n-1} \subset \mathbb{C}^{n-1}$. Now think of the first $n-2$ variables as being fixed and let w_{n-1} vary. Expand the determinant along the bottom row to obtain a linear relation

$$0 = \sum_{j=1}^{n-1} c_j(w_1, \ldots, w_{n-2})F_j'(w_{n-1})$$

where $c_j(w_1, \ldots, w_{n-2})$ is the determinant of an $(n-2) \times (n-2)$ submatrix of $[F_j'(w_k)]$. Because the F_j' are linearly independent on Ω, we conclude that each c_j must be zero. Now think of $w_1, \ldots, w_{n-1}$ as being variables again. We have reduced our problem to showing that no determinant of an $(n-2) \times (n-2)$ submatrix of $[F_j'(w_k)]$ can vanish identically. We can repeat the argument above until we get down to the 1×1 submatrices of $[F_j'(w_k)]$, and it is clear that these cannot be zero on Ω because none of the F_j' can be identically zero. This completes the proof of the existence of the w_k (and shows, in fact, that there is a dense open set of w in Ω^{n-1} that have the desired property).

If we choose $w_1, \ldots, w_{n-1}$ as above and solve the linear system obtained by plugging $w = w_k$ into (5.10) for $k = 1, \ldots, n-1$, we see that $(\omega_j(a) - \lambda_j(a))$ is a linear combination of $G_w(a,w_k)$ and $S(w_k,a)L(w_k,a)/S(a,a)$ for $k = 1, \ldots, n-1$.

Let $h_j(a) = G_w(a,w_j) - S(a,w_j)L(a,w_j)/S(w_j,w_j)$. Since the singular part of $G_w(a,w_j)$ is $1/(a-w_j)$ and since this agrees with the principal part of

$$S(a,w_j)L(a,w_j)/S(w_j,w_j)$$

at $a = w_j$, we may view h_j as a harmonic function on Ω that is in $C^\infty(\overline{\Omega})$. This completes the last detail of the plan of the proof mentioned above and we may consider our result proved in case the boundary is smooth. To finish the proof of Theorem 1.1 in case the boundary is not smooth, we only need to check that identity (5.10) is invariant under biholomorphic mappings, but this follows immediately from the transformation formulas given at the end of §2.

6. Finitely generated function fields and rings important to potential theory. Suppose that Ω is an n-connected domain in the plane such that no boundary component is a point. Let $\mathcal{S}$ denote the field of meromorphic functions on Ω generated by the functions of z given by $\{S(z,a) : a \in \Omega\}$. Our main results above show that this field is equal to the field generated by the three functions $S(z,A_1)$, $S(z,A_2)$, and $S(z,A_3)$ for a generic choice of the points A_1, A_2, and A_3 in Ω. We have shown that this field contains many objects of potential theory and conformal mapping, including the functions of z given by $K(z,a)$ where $K(z,w)$ denotes the Bergman kernel and a is any point in Ω. Furthermore, $\mathcal{S}$ also contains the functions $F_1',\dots,F_{n-1}'$ associated to the harmonic measure functions. We now note that formulas (1.1) and (1.2) reveal that $\mathcal{S}$ also contains the functions of z given by

$$\frac{\partial^m}{\partial\bar{w}^m}K(z,w) \quad \text{and} \quad \frac{\partial^m}{\partial\bar{w}^m}S(z,w)$$

where m is a positive integer and w is any fixed point in Ω. It follows from this via Theorem 2 of [6] that $\mathcal{S}$ contains *all* proper holomorphic mappings from Ω onto the unit disk (see also [5,7]). The field generated by $\mathcal{S}$ together with conjugates of functions in $\mathcal{S}$ is generated by only three elements plus their conjugates, and it contains all functions mentioned above plus all functions of the form $\{p(a,z) : a \in \Omega\}$ where $p(a,z)$ denotes the Poisson kernel.

Another interesting field is the field of all algebraic functions of functions in $\{S(z,a) : a \in \Omega\}$. The results of §3 show that this field is equal to the set of all algebraic functions of just two elements $S(z,A_1)$ and $S(z,A_2)$. Of course this field also contains all the other functions of potential theory contained by $\mathcal{S}$.

The techniques of §3 can also be applied to the Bergman kernel. Indeed, the Bergman kernel satisfies the identity

$$\Lambda(w,z)T(z) = -K(w,z)\overline{T(z)} \qquad \text{for } w\in\Omega \text{ and } z\in b\Omega \tag{6.1}$$

(see [2, page 135]) where Λ is a classical kernel function given by

$$\Lambda(z,w) = -\frac{2}{\pi}\frac{\partial^2 G(z,w)}{\partial z\partial w}.$$

Using identity (6.1) for the Bergman kernel in place of (2.1) for the Szegő kernel allows us to reason exactly as above to deduce that there is a dense open subset of points (A_1,A_2,A_3) in $\Omega\times\Omega\times\Omega$ such that the field of meromorphic functions on the double of Ω is generated by a primitive pair of functions given as the meromorphic extensions to the double of the two functions

$$\frac{K(z,A_1)}{K(z,A_3)} \quad \text{and} \quad \frac{K(z,A_2)}{K(z,A_3)}.$$

Let $K_0(z,w)$ denote the Bergman kernel $K(z,w)$ and let $K_m(z,w)$ denote the function $(\partial^m/\partial\bar{w}^m)K(z,w)$. It was proved in [5] that there exists a finite subset $\mathcal{A}$ of Ω and a positive integer N such that the Bergman kernel $K(z,w)$ is given as a rational combination of functions from the finite set of functions of z,

$$\{K_m(z,b) \,:\, b\in\mathcal{A},\ 0\le m\le N\},$$

and the finite set of functions of w,

$$\{\overline{K_m(w,b)} \,:\, b \in \mathcal{A},\ 0 \le m \le N\}.$$

It was also proved in [5] that functions of the form $K_m(z,b)/f_a'(z)$ extend meromorphically to the double of Ω, where f_a denotes the Ahlfors map associated to a point a in Ω. Hence, it follows that $K(z,A_3)/f_a'(z)$ is a rational combination of $K(z,A_1)/K(z,A_3)$ and $K(z,A_2)/K(z,A_3)$ and therefore, that $f_a'(z)$ is a rational combination of $K(z,A_1)$, $K(z,A_2)$, and $K(z,A_3)$. It now follows that all the functions of the form $K_m(z,b)$ are rational combinations of $K(z,A_1)$, $K(z,A_2)$, and $K(z,A_3)$, and since these functions generate the Bergman kernel, it follows that $K(z,w)$ is a rational combination of $K(z,A_1)$, $K(z,A_2)$, and $K(z,A_3)$, and the conjugates of $K(w,A_1)$, $K(w,A_2)$, and $K(w,A_3)$. It was also proved in [5] that the functions $F_1',\dots,F_{n-1}'$ are rational combinations of functions of the form $K_m(z,b)$, and we may also state now that $F_1',\dots,F_{n-1}'$ are rational combinations of $K(z,A_1)$, $K(z,A_2)$, and $K(z,A_3)$.

The same techniques that we applied to the Szegő kernel can also be applied to the Bergman kernel to show that $K(z,A_1)$ is an algebraic function of $K(z,A_2)$ and $K(z,A_3)$. Hence, it follows that the Bergman kernel $K(z,w)$ is an algebraic function of $K(z,A_2)$ and $K(z,A_3)$ and the conjugates of $K(w,A_2)$ and $K(w,A_3)$. It is interesting to note that the field of all algebraic functions of functions in $\{K(z,a) : a \in \Omega\}$ is equal to the set of all algebraic functions of just two elements $K(z,A_2)$ and $K(z,A_3)$, and that this field does contain the functions of z given by $S(z,w)$. Hence, the field of algebraic functions of functions in $\{K(z,a) : a \in \Omega\}$ is *equal to* the field of algebraic functions of functions in $\{S(z,a) : a \in \Omega\}$.

We have shown that the field $\mathcal{K}$ of meromorphic functions on Ω generated by set of functions of z given by $\{K(z,w) \,:\, w \in \Omega\}$ contains many objects of conformal mapping and potential theory. It appears not to be as universal as the field $\mathcal{S}$, however, because although (1.2) shows that $\mathcal{K}$ contains $S(z,w)^2$, I haven't been able to show that it contains $S(z,w)$.

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Mathematics Department, Purdue University, West Lafayette, IN 47907 USA
E-mail address: bell@math.purdue.edu