THE GREEN'S FUNCTION AND THE AHLFORS MAP

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ABSTRACT. The classical Green's function associated to a simply connected domain in the complex plane is easily expressed in terms of a Riemann mapping function. The purpose of this paper is to express the Green's function of a finitely connected domain in the plane in terms of a single Ahlfors mapping of the domain, which is a proper holomorphic mapping of the domain onto the unit disc that is the analogue of the Riemann map in the multiply connected setting.

1. INTRODUCTION

If one knows a Riemann map f associated to a simply connected domain $\Omega \neq \mathbb{C}$ in the complex plane, then the classical Green's function associated to Ω is given by

$$G(z, w) = -\ln \left| \frac{f(z) - f(w)}{1 - \overline{f(w)}\, f(z)} \right|. \tag{1.1}$$

The function on the right hand side of equation (1.1) can be realized as the real part of an antiderivative of a rational function composed with $(f(z), f(w), \overline{f(w)}\,)$. To see this, let

$$\phi(z) := \frac{1}{2} \ln \left| \frac{z - w}{1 - \bar{w}\, z} \right|^2,$$

and notice that

$$\frac{\partial \phi}{\partial z} = \frac{1 - |w|^2}{2(z - w)(1 - \bar{w}\, z)}.$$

Define a rational function R of three complex variables via

$$R(z, w, v) = -\frac{1 - wv}{(z - w)(1 - vz)},$$

and let $\alpha(z, w, \bar{w})$ denote a local antiderivative of $R(z, w, \bar{w})$ in the z variable defined via

$$\alpha(z, w, \bar{w}) = \int_{\gamma_z} R(\zeta, w, \bar{w})\; d\zeta,$$

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where, for z and w in the unit disc with $z \neq w$, γ_z is any curve that starts at 1, enters the unit disc, avoids w, and ends at z. Since ϕ is real valued and vanishes on the boundary, it follows that

$$\phi(z) = \phi(z) - \phi(1) = \text{Re } 2 \int_{\gamma_z} \frac{\partial \phi}{\partial \zeta} \, d\zeta.$$

Hence, the Green's function of Ω is given by

$$G(z,w) = \text{Re } \alpha(f(z), f(w), \overline{f(w)}\,). \tag{1.2}$$

Note that, although α is a multivalued function, its real part is single valued on the unit disc. (We shall give a more careful explanation of these ideas in §2 when we generalize them to the multiply connected setting.)

On a bounded multiply connected domain, an Ahlfors mapping f associated to a point a in the domain is the solution to the same extremal problem that determines the Riemann map on a simply connected domain, i.e., among all holomorphic functions mapping the domain into the unit disc, the Ahlfors map is the unique function such that $f'(a)$ is real and as large as possible. We shall give more details about the Ahlfors map in §2. (See [3] for a construction of the Ahlfors map in the spirit of the present work.)

In this paper, we address the philosophical question, "If one knows an Ahlfors mapping of a multiply connected domain onto the unit disc, then does one know the Green's function of the domain?" In fact, we show that if one knows an Ahlfors map f and *finitely many complex numbers*, then one knows the Green's function of the domain. We shall show that the Green's function can be expressed in terms of a finite sum of the real parts of finitely many functions which are antiderivatives of *algebraic* functions composed with the Ahlfors map. The finitely many basic functions involved will be defined similarly to the function Re $\alpha(f(z), f(w), \overline{f(w)}\,)$ appearing in equation (1.2), where the rational function R of three variables will be replaced by an algebraic function of three variables (see Theorem 1.1 below).

When viewed in the correct light, our formula is seen to replace the rational function inside equation (1.1) with an algebraic function (which, unfortunately, may depend on the domain under study).

To describe our main result, we must define three types of abelian functions. Given an algebraic function $A(z,w,v)$ of three complex variables, we shall say that α is a *Green antiderivative of type I* if α is defined by analytic continuation via the formula

$$\alpha(z,w,\bar{w}) = \int_{\gamma_z} A(\zeta,w,\bar{w}) \, d\zeta,$$

where, for z and w in the unit disc, γ_z is any curve that starts at 1, enters the unit disc, avoids finitely many points (which will be specified), and ends at z. In what follows, it will be known that, for fixed w in the unit disc, the algebraic function $A(\zeta,w,\bar{w})$ of ζ is well defined and analytic on a neighborhood of the unit circle and analytically continues nicely along curves like γ_z so that the integral makes sense. (Even though $\alpha(z,w,\bar{w})$ may be multivalued by our definition, when we take real parts later on in the paper, we will obtain single valued functions in a manner similar to that used to deduce equation (1.2).)

We shall say that α is a *Green antiderivative of type II* if

$$\alpha(w,\bar{w}) = \int_\gamma A(\zeta,w,\bar{w})\ d\zeta,$$

where γ is a fixed curve that starts at 1, stays in the closed unit disc, avoids finitely many points (which will be specified), and terminates at another boundary point. It will be known that, for fixed w in the unit disc, the algebraic function $A(\zeta,w,\bar{w})$ of ζ is well defined and analytic on a neighborhood of the unit circle and analytically continues nicely along curves like γ.

We shall say that α is a *Green antiderivative of type III* if it is the antiderivative of an algebraic function which we may express via

$$\alpha(z) = \int_{\gamma_z} A(\zeta)\ d\zeta,$$

where $A(\zeta)$ is algebraic and, for z in the unit disc, γ_z is any curve that starts at 1, enters the unit disc, avoids finitely many points (which will be specified), and ends at z. It will be known that A is well defined and analytic on a neighborhood of the unit circle and analytically continues along curves like γ_z.

We can now describe our main results.

Theorem 1.1. *Suppose that Ω is an n-connected domain in the plane ($n>1$) such that no boundary component is a point and suppose that f is an Ahlfors mapping of Ω onto the unit disc. There is a Green antiderivative $\alpha(z,w,\bar{w})$ of type I, and Green antiderivatives $\alpha_j(w,\bar{w})$ of type II, and Green antiderivatives $\beta_j(z)$ of type III such that the Green's function associated to Ω is given by*

$$G(z,w) = Re\ \alpha(f(z),f(w),\overline{f(w)}\,) + \sum_{j=1}^{n-1}\Big(Re\ \alpha_j(f(w),\overline{f(w)}\,)\Big)\left(Re\ \beta_j(f(z))\right).$$

This theorem reveals that if we know an Ahlfors mapping to the disc and the finitely many coefficients that define the algebraic functions in the background, then we can determine the Green's function of the domain. It also gives us a recipe for extending the Green's function past the boundary whenever an Ahlfors map extends. Furthermore, it shows that the Green's function can be "zipped" down to a very small data set, namely, finitely many coefficients plus the boundary values of a single Ahlfors map. Since Ahlfors maps are easily computed numerically (see [2] or Chapter 26 of [3]), the ideas of this paper may lead to numerical recipes for the objects of potential theory.

Following in the tradition of Bergman, Garabedian, Grunsky, and Schiffer (see [13]), we have shown in [6, 7] that many of the classical kernel functions of complex analysis can be expressed in terms of *two* Ahlfors mappings. However, until now, the Green's function has eluded our assaults upon it. We got the impression from reading [13] that people were looking for the results of the present paper, but they were missing the algebraic functions that arise here from the connection between the Green's function, the Szegő kernel, and the field of meromorphic functions on the double.

Before we can state our next result, we must make some definitions. As is customary, let $\frac{\partial}{\partial z}$ denote the differential operator $\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and let $\frac{\partial}{\partial \bar{z}}$ denote $\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. Assume for the moment that Ω is a bounded domain bounded by n non-intersecting C^∞-smooth Jordan curves. Let γ_j, $j = 1, \dots, n$, denote the boundary curves of Ω, where γ_n is the outer boundary. The functions ω_j are the standard harmonic measure functions which are harmonic functions on Ω with boundary values given by one on γ_j and zero on γ_k with $k \neq j$. Let F_j' denote the holomorphic function given by $2(\partial/\partial z)\omega_j$. It is well known that the vector space $\mathcal{F}'$ of complex linear combinations of all the F_j' is $n-1$ dimensional and that $\{F_j'\}_{j=1}^{n-1}$ forms a basis for $\mathcal{F}'$. It is easy to show that a basis $\{u_j\}_{j=1}^{n-1}$ for $\mathcal{F}'$ can be found such that

$$\delta_{kj} = \int_{\gamma_k} u_j(w)\, dw,$$

for $j = 1, \dots, n-1$ and $k = 1, \dots, n-1$ where δ_{kj} denotes the Kronecker delta. Indeed, if we set $u_j = \sum_{m=1}^{n-1} \sigma_{jm} F_m'$, then we need

$$\sum_{m=1}^{n-1} \sigma_{jm} \int_{\gamma_k} F_m'\, dz = \delta_{kj},$$

and it is well known that the matrix of periods $\int_{\gamma_k} F_m'\, dz$ is non-singular. Since the outward normal derivative of ω_k is equal to $-iF_k'(z)T(z)$ at a boundary point z, where $T(z)$ is a complex number representing the unit tangent vector at $z \in b\Omega$ pointing in the direction of the standard orientation, it follows that $i[\sigma_{jm}]$ is a matrix of real coefficients, a fact that we shall need later on. (See [15] for a nice presentation of these ideas and [5, p. 11] where the functions u_j arise in the context of the Green's function.)

Let $S(z,w)$ and $L(z,w)$ denote the Szegő and Garabedian kernels associated to Ω, respectively. Let $\mu_j(z) = i\sum_{k=1}^{n-1} \sigma_{jk}\omega_k$ so that $2(\partial/\partial z)\mu_j = iu_j$. Notice that μ_j is a real valued function on $\overline{\Omega}$ because the coefficients $i\sigma_{jk}$ are real. Finally, let

$$\lambda_j(w) = \int_{\zeta \in \gamma_j} \frac{|S(w,\zeta)|^2}{S(w,w)}\, ds,$$

where ds denotes the arc length measure. It is proved in [4] that λ_j is in $C^\infty(\overline{\Omega})$ and has the same boundary values as ω_j.

When Ω is a finitely connected domain such that no boundary component is a point, we may define the functions ω_j, F_j', and the Szegő and Garabedian kernels via a biholomorphic mapping Φ which maps to a bounded domain $\widetilde{\Omega}$ with real analytic boundary curves as explained in [5]. The transformation rule for the Szegő kernel under biholomorphic mappings yields that the functions λ_j associated to a smooth domain should transform via the same formula as ω_j, i.e., $\lambda_j(z) = \tilde{\lambda}_j(\Phi(z))$, where it is understood that Φ takes the boundary curve γ_j to the corresponding boundary curve $\tilde{\gamma}_j$ of $\widetilde{\Omega}$. When Ω does not have smooth boundary, we take this to be the definition of λ_j. With these definitions, it was shown in [4] and [5] that the formulas we shall use in this work from [4] and [5] are valid.

Our next theorem shows that the Green's function is a rather simple combination of functions of *one complex variable*. The function α appearing in the theorem is the same as that appearing in Theorem 1.1.

Theorem 1.2. *Suppose that Ω is an n-connected domain in the plane ($n > 1$) such that no boundary component is a point and suppose that f is an Ahlfors mapping of Ω onto the unit disc. There is a Green antiderivative $\alpha(z, w, \bar{w})$ of type I such that the Green's function associated to Ω is given by*

$$G(z,w) = \mathrm{Re}\ \alpha(f(z), f(w), \overline{f(w)}\,) + 2\pi \sum_{j=1}^{n-1} (\omega_j(w) - \lambda_j(w))\mu_j(z).$$

We shall also see along the way that the functions $\omega_j(w)$, $\lambda_j(w)$, and $\mu_j(z)$ appearing in the formula in Theorem 1.2 are special functions that are related to Green antiderivatives of various sorts. We collect these results in the following theorem.

Theorem 1.3. *Suppose that Ω is an n-connected domain in the plane ($n > 1$) such that no boundary component is a point and suppose that f is an Ahlfors mapping of Ω onto the unit disc. The functions $\omega_j(w)$ and $\mu_j(w)$ are equal to $\mathrm{Re}\ \beta_j(f(w))$ and $\mathrm{Re}\ \tau_j(f(w))$, respectively, where β_j and τ_j are Green antiderivatives of type III. The functions $\lambda_j(w)$ are equal to $\psi_j(f(w), \overline{f(w)})$ where ψ_j is a real valued Green antiderivative of type II.*

Before we turn to the proofs of the theorems, we take some time to motivate the proof and give a more precise description of the special functions of types I, II, and III in the statement of the theorems.

2. Motivation of the Proof

As is typical in many arguments in conformal mapping theory, the subject of this paper allows us to assume our domain Ω is a bounded domain bounded by n non-intersecting *real analytic* Jordan curves. Indeed, an Ahlfors mapping composed with a biholomorphic mapping is a constant times an Ahlfors mapping. Furthermore, the simple transformation formulas under biholomorphic mappings for the Green's functions and all the other functions appearing in Theorems 1.1, 1.2, and 1.3 allow us to reduce to the case of real analytic boundary by the standard change of variables.

Let $G(z, w)$ denote the classical Green's function associated to Ω (with singularity $-\ln|z - w|$). Given a point $a \in \Omega$, let f_a denote the Ahlfors map associated to the pair (Ω, a). It is an n-to-one mapping (counting multiplicities) proper holomorphic mapping onto the unit disc, it extends holomorphically past the boundary of Ω, and it maps each boundary curve one-to-one onto the unit circle. Furthermore, $f_a(a) = 0$, and f_a is the unique function mapping Ω into the unit disc maximizing the quantity $|f_a'(a)|$ with $f_a'(a) > 0$. An Ahlfors map should be thought of as the replacement for the Riemann map in the non-simply connected setting. It shares all the properties of a Riemann map but one. Instead of being one-to-one, it is m-to-one for the smallest possible m.

We shall use the shorthand notation $G_z(z,w)$ and $G_{\bar z}(z,w)$ to denote $\frac{\partial G}{\partial z}(z,w)$ and $\frac{\partial G}{\partial \bar z}(z,w)$, respectively.

If h is a real valued harmonic function that extends smoothly up to the boundary of Ω and is a constant c_j on each boundary curve γ_j, then we may differentiate the identity $c_j \equiv h(z(t))$ with respect to t where $z(t)$ parameterizes γ_j in the standard sense to see that

$$h_z(z(t))z'(t) + h_{\bar z}(z(t))\overline{z'(t)} = 0.$$

Divide this equation by $|z'(t)|$ to obtain the identity

$$h_z(z)T(z) + h_{\bar z}(z)\overline{T(z)} = 0$$

for $z \in b\Omega$ (where we have taken $z = z(t)$ and $T(z) = z'(t)/|z'(t)|$ to be consistent with our previous use of the complex unit tangent function T). Apply this idea to the Green's function to see that

$$G_z(z,w)T(z) = -G_{\bar z}(z,w)\overline{T(z)} \quad \text{for } z \in b\Omega, w \in \Omega.$$

Note that $G_{\bar z} = \overline{G_z}$ so that

$$G_z(z,w)T(z) = -\overline{G_z(z,w)}\ \overline{T(z)} \quad \text{for } z \in b\Omega, w \in \Omega. \tag{2.1}$$

Suppose that f is an Ahlfors map associated to a point in Ω. The boundary identity $0 \equiv \ln |f|^2$ can be used just as we did for the Green's function above to see that f satisfies

$$\frac{f'(z)}{f(z)}T(z) = -\left(\overline{f'(z)}/\,\overline{f(z)}\right)\overline{T(z)} \quad \text{for } z \in b\Omega. \tag{2.2}$$

If we divide equation (2.1) by equation (2.2) and note that $f(z) = 1/\overline{f(z)}$ on the boundary (because $|f(z)| = 1$ there), we see that, for fixed $w \in \Omega$, the meromorphic function of z

$$\frac{G_z(z,w)}{f'(z)}$$

is equal to the complex conjugate of the meromorphic function

$$\frac{G_z(z,w)f(z)^2}{f'(z)}$$

on the boundary, and therefore it extends to the double of Ω in the z variable as a meromorphic function.

We proved in [6] that, given a proper holomorphic map f to the unit disc, there is another proper holomorphic mapping F to the unit disc (which can be taken to be an Ahlfors map) such that f and F extend to the double and generate all the meromorphic functions on the double, i.e., they form a primitive pair for the double. Any two such functions are algebraically dependent (see Farkas and Kra [12, p. 248]), so there is an irreducible polynomial P such that $P(f,F) \equiv 0$. This shows that F is an algebraic function of f. Since $\frac{G_z(z,w)}{f'(z)}$ extends to the double, it is a rational function of f and F, and hence, it is an algebraic function A of f alone. This shows that

$$G_z(z,w) = f'(z)A(f(z)) \tag{2.3}$$

for this fixed w. Suppose z_0 is a point in the boundary of Ω such that $f(z_0) = 1$. Note that f' is non-vanishing on the boundary of Ω. Let $Z = \{z \in \Omega : f'(z) = 0\}$ denote the branch locus of f (which is a finite set) and let $W = \{f(z) : z \in Z\}$ denote the image of the branch locus. Although A may be multivalued, $G_z(z,w)$ is single valued. Thus, equation (2.3) reveals that there is a germ (or function element) of $A(\zeta)$ at $\zeta = 1$ which is holomorphic near 1 such that when $f'(A \circ f)$ is continued along curves starting at z_0, the single valued $G_z(z,w)$ is obtained. Furthermore, this function continues to a neighborhood of $\overline{\Omega}$ as a single valued holomorphic function in z with a simple pole in z at the point w and removable singularities at each of the finitely many points in Z different from w. It will profit us here to consider the various branches of A that arise in this manner a little more carefully. Let L denote the union of the set of closed line segments in the unit disc $D_1(0)$ joining each point in W to the origin. Note that f is an unbranched covering of $\Omega \setminus f^{-1}(L)$ onto $D_1(0) \setminus L$. Since f is n-to-one (counting multiplicities) from Ω to $D_1(0)$, and since f maps each boundary curve of Ω one-to-one onto the unit circle, we deduce that the set $f^{-1}(L)$ divides Ω into n two-connected components, one for each boundary curve. Let $\mathcal{O}_k$ denote the component with the boundary curve γ_k as part of its boundary. Notice that there is a single valued branch of f^{-1} defined on $D_1(0) \setminus L$ which maps this set biholomorphically to $\mathcal{O}_k$. There are n (single valued) holomorphic functions $A_1, A_2, \ldots, A_n$ on $D_1(0) \setminus L$ minus $f(w)$ which are branches of A such that $G_z(z,w) = f'(z)A_k(f(z))$ on $\mathcal{O}_k$. Each A_k extends continuously up to $L \setminus W$ minus $f(w)$ from one-sided neighborhoods, but A_k might have algebraic singularities at points in $W \cup \{f(w)\}$.

If φ is real-valued in a neighborhood of a curve ν that starts at a and ends at b, then

$$2\int_\nu \varphi_z \, dz = \left(\int_\nu \varphi_x \, dx + \varphi_y \, dy\right) + i\left(\int_\nu \varphi_x \, dy - \varphi_y \, dx\right),$$

and so

$$\varphi(b) - \varphi(a) = 2\mathrm{Re}\left(\int_\nu \varphi_z \, dz\right).$$

Assume for the moment that z is a point in Ω that is close to the boundary point z_0 that we chose above so that $f(z_0) = 1$. Since the Green's function is real valued and vanishes on the boundary, we may use this idea to antidifferentiate via

$$G(z,w) = 2\text{Re}\,\left(\int_\Gamma G_\zeta(\zeta,w)\;d\zeta\right) = \text{Re}\,\left(\int_\Gamma 2f'(\zeta)A(f(\zeta))\;d\zeta\right),$$

where Γ is any curve that starts at the boundary point z_0, moves into Ω, stays away from w, and terminates at z. (Since z is close to the boundary, we don't have to worry about avoiding the points in Z.)

Now the change of variables formula shows that

$$G(z,w) = \text{Re}\;\alpha(f(z)),$$

where, writing $u = f(z)$,

$$\alpha(u) = \int_{f(\Gamma)} 2A(\zeta)\;d\zeta.$$

Note that α is a local antiderivative of the algebraic function $2A$ obtained by integrating along a curve that starts at the point 1 in the unit circle and moves into the unit disc and terminates at u. Now, the formula $G(z,w) = \text{Re}\;\alpha(f(z))$ can be extended to all of Ω by means of analytic continuation. To be more precise, using the notation set up above, if z is in $\mathcal{O}_k$, we may define α at $u = f(z)$ as follows. Let γ denote any curve that starts at 1 and enters the unit disc and travels to u in $D_1(0)\setminus L$ avoiding $f(w)$. Define

$$\alpha(u) = \int_\gamma 2A_k(\zeta)\;d\zeta.$$

We may now state that $\text{Re}\,\alpha(f(z))$ is well defined on $\mathcal{O}_k$ and that these functions extend continuously up to $f^{-1}(L)$ minus w and match up there to represent the harmonic function $G(z,w)$ of z on $\Omega\setminus\{w\}$. (Since the functions match up along $f^{-1}(L)\setminus Z$ minus w, we may define a multivalued version of $\alpha(u)$ globally by integrating a single germ of $2A$ at 1 (correctly chosen) along a curve that starts at 1 and avoids w and the finitely many points in W, where it is understood that we are integrating the analytic continuation of $2A$ along the curve as we go.)

We remark here that the harmonic measure functions ω_j can be dealt with in exactly the same way as we have treated the Green's function in this section. Indeed, the same reasoning that yielded equation (2.1) can be applied to ω_j to obtain

$$F_j'(z)T(z) = -\overline{F_j'(z)}\,\overline{T(z)}$$

on $b\Omega$, and we may proceed as above to see that $F_j'(z) = f'(z)A(f(z))$ where A is algebraic. In this way, we prove that ω_j is equal to $\text{Re}\,\beta_j(f(z))$ where β_j is a Green antiderivative of type III.

The hard part in what follows is to see how the Green antiderivative α in the formula for $G(z,w)$ obtained in this section varies as we allow w to vary.

3. Proof of Theorem 1.2

Assume that Ω is a bounded domain in the plane bounded by n non-intersecting real analytic curves. We shall need to use some formulas proved in [4] that relate the Poisson kernel to the Szegő kernel $S(z,w)$ and the Garabedian kernel $L(z,w)$. Before we write the formulas, we recall some basic facts about the Szegő and Garabedian kernels on a domain with real analytic boundary (proofs of which can be found in [3]). The kernel $S(z,w)$ extends holomorphically past the boundary in z for each fixed w in Ω. It extends meromorphically past the boundary in z for each fixed w in $b\Omega$; in fact, it extends holomorphically past $b\Omega \setminus \{w\}$ and has only a simple pole at the point w. Furthermore $S(z,w) \neq 0$ if $z \in b\Omega$ and $w \in \Omega$. If $w \in b\Omega$, then $S(z,w)$ has exactly $n-1$ simple zeroes in z, one on each boundary curve different from the one containing the point w. The kernel $L(z,w)$ has a simple pole in z at the point $w \in \Omega$. It extends holomorphically past the boundary in z for each fixed w in Ω. It extends meromorphically past the boundary in z for each fixed w in $b\Omega$; in fact, it extends holomorphically past $b\Omega\setminus\{w\}$ and has only a simple pole at the point w. Furthermore $L(z,w) \neq 0$ if $z, w \in \Omega$ with $z \neq w$. If $w \in b\Omega$, then $L(z,w)$ has exactly $n-1$ simple zeroes in z, one on each boundary curve different from the one containing the point w (and these zeroes agree with those of the Szegő kernel). Finally, $S(z,w)$ is in C^∞ of $\overline{\Omega} \times \overline{\Omega}$ minus the boundary diagonal $\{(z,z) : z \in b\Omega\}$ and $L(z,w)$ is in C^∞ of $\overline{\Omega} \times \overline{\Omega}$ minus the diagonal $\{(z,z) : z \in \overline{\Omega}\}$.

It is proved in [4] that

$$G_z(z,w) = \pi \frac{S(z,w)L(z,w)}{S(w,w)} + i\pi \sum_{j=1}^{n-1} (\omega_j(w) - \lambda_j(w))u_j(z), \tag{3.1}$$

where the functions ω_j, λ_j and u_j were defined in §1. For an alternate and more natural proof of this identity, see [5, pp. 10–12] (where it must be noted that the indices j and k are one and the same). It is also proved in [4] that it is possible to choose a point a in Ω so that the $n-1$ zeroes of $S(z,a)$ in the z variable are distinct and simple. Choose such a point a and let $a_1, a_2, \dots, a_{n-1}$ denote these simple zeroes. For convenience, let a_0 denote a. It is proved in [4] that the Szegő kernel and Garabedian kernels can be expressed via

$$S(z,w) = \frac{1}{1 - \overline{f(w)}\, f(z)} \left(c_{00} S(z,a)\overline{S(w,a)} + \sum_{j,k=1}^{n-1} c_{jk} S(z,a_j)\overline{S(w,a_k)} \right) \tag{3.2}$$

and

$$L(z,w) = \frac{f(w)}{f(z) - f(w)} \left(c_{00} S(z,a)L(w,a) + \sum_{j,k=1}^{n-1} \bar{c}_{jk} S(z,a_j)L(w,a_k) \right). \tag{3.3}$$

To shorten some expressions in what follows, we define coefficients $c_{0j} = 0$ and $c_{j0} = 0$ when $j \neq 0$. Let f denote the Ahlfors map f_a associated to the point a.

We now take the principal term

$$X(z,w) := \pi \frac{S(z,w)L(z,w)}{S(w,w)}$$

from equation (3.1) and replace the Szegő and Garabedian kernels by the expressions in equations (3.2) and (3.3). Expand this large expression to see that $X(z,w)$ is a linear combination of rational functions of $f(z)$, $f(w)$, and $\overline{f(w)}$ times terms of the form

$$\frac{S(z,a_p)S(z,a_q)\overline{S(w,a_r)}L(w,a_s)}{\sum_{j,k=0}^{n-1} c_{jk} S(w,a_j)\overline{S(w,a_k)}}. \tag{3.4}$$

We shall next use the fact that certain combinations of functions extend to the double as meromorphic functions (in a manner similar to what was done in [7] to study the Carathéodory metric and the Poisson kernel). In particular

$$\frac{S(z,a_p)S(z,a_q)}{f'(z)} \tag{3.5}$$

and functions of the form $S(w,a_r)/S(w,a)$ and $L(w,a_s)/S(w,a)$ all extend meromorphically to the double. To see that $S(w,a_r)/S(w,a)$ extends to the double, use the standard identity

$$\overline{S(z,w)} = \frac{1}{i} L(z,w)T(z),$$

which holds for $z \in b\Omega$ and $w \in \Omega$, to see that $S(w,a_r)/S(w,a)$ is equal to the conjugate of $L(w,a_r)/L(w,a)$ on the boundary. Hence, it extends to the double as a meromorphic function. Similarly, $L(w,a_s)/S(w,a)$ is equal to the conjugate of $S(w,a_s)/L(w,a)$ on the boundary, and so it extends to the double. To see that $S(z,a_p)S(z,a_q)/f'(z)$ extends, we use identity (2.2) and the fact that $\overline{f(z)} = 1/f(z)$ on $b\Omega$. Indeed, (3.5) is equal to the conjugate of $f(z)^2 L(z,a_p)L(z,a_q)/f'(z)$ on the boundary, and so it extends meromorphically to the double. We may now multiply (3.4) by unity three times, once in the form $f'(z)/f'(z)$, and in the form $S(w,a)/S(w,a)$, and in the form the conjugate of $S(w,a)/S(w,a)$. Next, we distribute the terms $1/f'(z)$, $1/S(w,a)$ and the conjugate of $1/S(w,a)$ into the expression in strategic places in order to rewrite (3.4) in the form

$$f'(z)\, \frac{\frac{S(z,a_p)S(z,a_q)}{f'(z)} \; \frac{\overline{S(w,a_r)}}{\overline{S(w,a)}} \; \frac{L(w,a_s)}{S(w,a)}}{\sum_{j,k=0}^{n-1} c_{jk} \frac{S(w,a_j)}{S(w,a)} \; \frac{\overline{S(w,a_k)}}{\overline{S(w,a)}}}. \tag{3.6}$$

Every quotient in this last expression extends to the double as a meromorphic or antimeromorphic function. As mentioned earlier, we proved in [6] that the field of meromorphic functions on the double is generated by f and another Ahlfors map F. Hence, we have shown that $X(z,w)$ is equal to $f'(z)$ times a rational function of $f(z)$, $F(z)$, $f(w)$, $F(w)$, $\overline{f(w)}$, and $\overline{F(w)}$. Since f and F are algebraically dependent, F is an algebraic function of f and we conclude that

$$X(z,w) = f'(z)A(f(z), f(w), \overline{f(w)}\,),$$

where $A(\zeta, u, v)$ is an algebraic function of three variables. Let z_0 denote the point in the outer boundary of Ω such that $f(z_0) = 1$. We may now repeat the argument in §2 and integrate equation (3.1) in the first variable along a curve Γ that starts at z_0, avoids w and the finitely many points in the branch locus of f, and terminates at z. Note that, since $2(\partial/\partial z)\mu_j = iu_j$, and since μ_j is real valued and vanishes on the outer boundary, it follows that Re $\int_\Gamma iu_j dz = \mu_j(z)$, and we obtain the formula for the Green's function in the statement of Theorem 1.2. This completes the proof in case Ω has real analytic boundary. In the more general case of the statement of the theorem, we note that all the elements in the formula transform under a biholomorphic mapping to a domain with real analytic boundary and so the formula holds in this more general setting.

4. Proof of Theorem 1.1

As explained in §2, we may assume that Ω is a bounded domain bounded by n non-intersecting real analytic Jordan curves. We proved in §3 that

$$G_z(z, w) = f'(z)A(f(z), f(w), \overline{f(w)}\,) + i\pi \sum_{j=1}^{n-1} (\omega_j(w) - \lambda_j(w))u_j(z), \tag{4.1}$$

where A is algebraic. Recall that

$$u_j = \sum_{k=1}^{n-1} \sigma_{jk} F'_k,$$

where the coefficients are such that $i\sigma_{jk}$ is real. Hence, the sum

$$i\pi \sum_{j=1}^{n-1} (\omega_j(w) - \lambda_j(w))u_j(z)$$

can be rearranged to appear in the form

$$\sum_{j=1}^{n-1} v_j(w)F'_j(z),$$

where v_j is a real valued function that is a linear combination over the real numbers of the real valued functions $\omega_j - \lambda_j$. Let z_0 be the point on the outer boundary such that $f(z_0) = 1$, and let ν_k denote a curve that starts at z_0, enters Ω and avoids $f(w)$ and the branch points of f and terminates at the boundary point z_k on γ_k such that $f(z_k) = 1$. Integrate equation (4.1) with respect to dz along ν_k and take the real part, noting that Re $\int_{\nu_k} F'_j(z)\, dz = \omega_j(z_k) - \omega_j(z_0) = 1$ if $j = k$ and zero otherwise, to obtain

$$0 = \text{Re} \int_{\nu_k} f'(z)A(f(z), f(w), \overline{f(w)}\,)\; dz + v_k(w).$$

Finally, we may use the change of variables formula as in §2 to see that

$$v_k(w) = -\text{Re} \int_{f(\nu_k)} A(\zeta, f(w), \overline{f(w)}\,)\; d\zeta,$$

where $f(\nu_k)$ is a curve that starts at 1, enters the unit disc, avoids the point $f(w)$ and the finitely many points in the image of the branch locus of f, and terminates back at 1. This shows that v_k is equal to $\operatorname{Re}\alpha_k(f(w),\overline{f(w)}\,)$ where α_k is a Green antiderivative of type II.

Now, when we integrate

$$G_z(z,w)=f'(z)A(f(z),f(w),\overline{f(w)}\,)+\sum_{j=1}^{n-1}\operatorname{Re}\alpha_j(f(w),\overline{f(w)}\,)F_j'(z)$$

with respect to z along a curve Γ as we did in §3, we obtain

$$G(z,w)=\operatorname{Re}\alpha(f(z),f(w),\overline{f(w)}\,)+\sum_{j=1}^{n-1}\operatorname{Re}\alpha_j(f(w),\overline{f(w)}\,)2\omega_j(z).$$

We showed in §2 that $\omega_j(z)$ is equal to $\operatorname{Re}\beta_j(f(z))$ where β_j is a Green antiderivative of type III. This completes the proof of Theorem 1.1.

5. Proof of Theorem 1.3

We continue to assume that Ω is a bounded domain bounded by n non-intersecting real analytic Jordan curves. Integrate equation (4.1) with respect to dz around one of the inner boundary curves, say γ_k. Note that, because the Poisson kernel is given by $P(w,z)=(-i/\pi)G_z(z,w)T(z)$, it follows that

$$\int_{\gamma_k}G_z(z,w)\ dz=i\pi\omega_k(w).$$

Also, recall that $\int_{\gamma_k}u_k(z)\ dz=1$ and $\int_{\gamma_k}u_j(z)\ dz=0$ if $j\neq k$. Hence,

$$\omega_k(w)=\frac{1}{i\pi}\int_{\gamma_k}f'(z)A(f(z),f(w),\overline{f(w)}\,)\ dz+(\omega_k(w)-\lambda_k(w)),$$

and consequently

$$\lambda_k(w)=\frac{1}{i\pi}\int_{\gamma_k}f'(z)A(f(z),f(w),\overline{f(w)}\,)\ dz.$$

We may now use the change of variables formula to see that

$$\lambda_k(w)=\frac{1}{i\pi}\int_{C_1}A(\zeta,f(w),\overline{f(w)}\,)\ d\zeta,$$

where C_1 denotes the unit circle taken in the standard sense. This shows that $\lambda_k(w)$ is equal to Re $\psi_k(f(w),\overline{f(w)}\,)$ where ψ_k is a Green antiderivative of type II. (In fact, the real part can be dropped here because $\psi_k(f(w),\overline{f(w)}\,)$ is real valued on Ω.)

We proved in §2 that $\omega_j(z)$ is equal to $\operatorname{Re}\beta_j(f(z))$ where β_j is a Green antiderivative of type III. Since μ_k is a linear combination of the functions ω_j, $j=1,\dots,n-1$, it is equal to $\operatorname{Re}\tau_k(f(z))$ where τ_k is a Green antiderivative of type III. This completes the proof of Theorem 1.3.

6. Extension of the Green's function

A fascinating consequence of the formulas in this paper is that the Green's function associated to a domain with an algebraic Ahlfors map can be harmonically continued in both variables along curves in the complex plane that avoid finitely many points. At those finitely many points, the singular behavior is limited by the nature of an antiderivative of an algebraic function composed with an algebraic function. Such extension behavior was intimated by results of Ebenfelt [11] and Khavinson and Shapiro [16] on extension of solutions to the Dirichlet problem. We remark that Aharonov and Shapiro [1] proved that the Ahlfors map associated to a quadrature domain in the plane must be algebraic, and Gustafsson [14] proved that quadrature domains are dense among all bounded finitely connected domains in the plane bounded by Jordan curves. In fact, it was proved in [8] (see also [9]) that, given a bounded finitely connected domain bounded by C^∞ smooth Jordan curves, it is possible to find a quadrature domain that is arbitrarily C^∞ close by which is biholomorphic via a mapping which is arbitrarily C^∞ close to the identity map. Thus, it is possible to make very subtle changes in a smooth domain so that the Green's function continues harmonically to the whole complex plane minus finitely many points.

In general, we have shown that the Green's function extends harmonically to the set where the Ahlfors map extends holomorphically (minus a discrete set of points where algebraic singularities might arise). We would be very curious to know what the Green's function is for the domain $\{z : |z + (1/z)| < r\}$, where r is a real constant greater than 2 (see [10] for why these domains might be important). Could it be an elementary function in the sense of Liouville? It may be the simplest Green's function associated to a multiply connected domain, but we are willing to bet that no Green's function associated to a multiply connected domain can be an elementary function.

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