REAL ALGEBRAIC GEOMETRY OF REAL ALGEBRAIC JORDAN CURVES IN THE PLANE AND THE BERGMAN KERNEL

STEVEN R. BELL

Abstract. We characterize the space of restrictions of real rational functions to certain algebraic Jordan curves in the plane via the Dirichlet-to-Neumann map associated to the domain in the complex plane bounded by the curve and its Bergman kernel. The characterization leads to a partial fractions-like decomposition for such rational functions and new ways to describe such Jordan curves. The multiply connected case is also explored.

In honor of László's 70th!

1. Introduction

Shopping for a birthday gift for an old friend can be daunting, but sometimes you stumble onto a gift that seems just right. László Lempert is known for going back to something, and back again, until he has just the right way of thinking about it, even in an infinite dimensional setting. I will revisit some results in [2, 4, 3, 6], simplify proofs, and try to convince the reader that there might be better ways to think about them.

Suppose that γ is a C^∞ smooth real algebraic Jordan curve in the plane given by the zero set of a polynomial $p(x,y)$ of two real variables. Such a curve can be described locally by a parameterization of the curve given by a pair of smooth real analytic functions $(x(t), y(t))$ where t is a real parameter. By writing $z = x + iy$ and by letting t become a complex variable τ, a holomorphic map $z(\tau)$ defined on a neighborhood of an open line segment on the real line that maps the segment into the curve is obtained. We may assume that $z'(t)$ is nonvanishing and that $z(\tau)$ locally maps a neighborhood of the line segment one-to-one onto a collared neighborhood of the Jordan curve. The function

$$R(w) = z\left(\overline{z^{-1}(w)}\right),$$

is an antiholomorphic reflection function for the Jordan curve. Locally, it maps the inside of the curve to the outside, the outside to inside, and it is its own inverse. The curve is fixed by the map. The Schwarz function $S(z)$ associated to the curve is the unique holomorphic function defined on a neighborhood of the curve such that $S(z) = \bar{z}$ along the curve. It is given by

$$S(z) = \overline{R(z)}.$$

1991 *Mathematics Subject Classification.* 30C40.

A primary object of study in this paper is the field of functions that are the restrictions of rational functions $r(x, y)$ to the Jordan curve. We seek to expand such functions in a generalized partial fractions expansion. As is typical in problems in real algebra, we will gain insight by letting the variables and coefficients be complex. Indeed, by writing $z = x + iy$, we may express $r(x, y)$ as a quotient $P(z, \bar{z})/Q(z, \bar{z})$, where P and Q are polynomials with complex coefficients. The example of the function $1 - z\,\bar{z}$ on the unit circle alerts us to the danger that we must avoid the possibility that $Q(z, \bar{z}) \equiv 0$ on γ. Because the curve is defined by a polynomial equation, we are in danger of dividing by a polynomial that vanishes on the curve when we talk of rational functions of x and y restricted to the curve. We now define the space of functions that we intend to study carefully. Keep in mind that that the space will contain rational functions $r(x, y) = p(x, y)/q(x, y)$ where p and q are polynomials with real coefficients, where q is not identically zero on γ. That is why the term "real" algebraic geometry is used in the title.

A rational function $R(z, \bar{z})$ is given as a quotient $R(z, \bar{z}) = P(z, \bar{z})/Q(z, \bar{z})$ where $P(z, w)$ and $Q(z, w)$ are relatively prime complex polynomials (with complex coefficients). For the restriction of such a rational function to γ to be meaningful, we must assume that $Q(z, \bar{z})$ is not identically zero on the curve. We now consider the functions $P(z, S(z))$ and $Q(z, S(z))$, which agree with $P(z, \bar{z})$ and $Q(z, \bar{z})$ on γ. They are holomorphic on a collared neighborhood of the curve and their quotient agrees with the rational function on the curve away from the zeroes of the denominator. The zeroes of the denominator are isolated since $Q(z, \bar{z})$ is not identically zero on the curve. This shows that the rational function $R(z, \bar{z})$ is C^∞ smooth on the curve, except perhaps at finitely many pole-like singular points. Similarly, the restriction of a rational function that is not identically zero on the curve has at most finitely many isolated zeroes on the curve. We will call the space of such nondegenerate rational functions $R(z, \bar{z})$ restricted to the curve in this way $\mathcal{R}$. It is easy to verify that $\mathcal{R}$ is a field. Let $\mathcal{R}_s$ denote the subspace of $\mathcal{R}$ of functions without singularities on γ. Note that such functions are C^∞ smooth on γ.

The purpose of this paper is to show that, after a change of variables that is close to the identity, the spaces $\mathcal{R}_s$ and $\mathcal{R}$ can be described nicely in algebraic terms and that the descriptions give rise to new ways to represent the curve. The description also reveals that the algebra of such rational functions restricted to the curve can be viewed as a linear space in a manner reminiscent of the partial fractions decomposition of a rational function in the plane. The tools used to obtain these results are the Dirichlet-to-Neumann map associated to the domain Ω enclosed by γ and its Bergman kernel. The change of variables is given as a form of generalized Bergman coordinates.

The motivating example for what we are about to do is the field of restrictions of rational functions to the unit circle in the plane. The Schwarz function for the unit circle is $S(z) = 1/z$. If $R(z, \bar{z})$ is a function in $\mathcal{R}_s$, then $R(z, 1/z)$ is a rational function of z that agrees with $R(z, \bar{z})$ on the unit circle. Expanding

$R(z, 1/z)$ in partial fractions yields a decomposition

$$R(z,1/z) = P(z) + \sum_{i=1}^{n}\sum_{k=1}^{N_n} \frac{A_{ik}}{(z-a_i)^k} + \sum_{i=1}^{m}\sum_{k=1}^{M_m} \frac{B_{ik}}{(z-b_i)^k}$$

where $P(z)$ is a polynomial and where the poles a_i are inside the unit circle and the poles b_i are outside. (Note that we can think of a polynomial as having a pole at the point at infinity.) If we now replace z in the first double sum by $1/\bar{z}$ and note that the holomorphic poles of the individual terms are moved from inside the unit circle to antiholomorphic poles that are outside of the unit circle by this change, we obtain a decomposition of $R(z, \bar{z})$ on the unit circle as a sum $r_1(z) + \overline{r_2(z)}$ where r_1 and r_2 are rational functions with only poles outside the unit circle. This decomposition was studied by Peter Ebenfelt [9] in his studies of the Dirichlet problem with rational boundary data where he also noted that it solves the Dirichlet problem with rational boundary data $R(z, \bar{z})$. If $r_1(z) + \overline{r_2(z)}$ were identically zero on the unit circle, then the Schwarz reflection principle yields that $r_1(z)$ inside the unit circle extends $-\overline{r_2(1/\bar{z})}$ outside the unit circle to a bounded entire function. Hence the decomposition is unique up to the addition of a constant, and unique if we agree to include that constant in the r_1 term. Replacing z by $1/\bar{z}$ in the polynomial and the second double sum instead of the first leads to a similar decomposition where all the poles of both rational functions are *inside* the unit circle. Thus, we have shown that there are three ways to represent the restriction of a rational function to the unit circle: the poles inside decomposition, the poles outside decomposition, and as the restriction of a holomorphic rational function to the unit circle. (A fourth way is as the restriction of an antiholomorphic rational function to the unit circle, which can be seen by starting with the conjugate of the rational function instead and expressing it as the restriction of a complex rational function.) We thank the referee for pointing out that Philip Davis also used related ideas in Chapter 13 of [8] using the classical method of images of potential theory on the unit disc.

To accomplish similar decompositions on a more general algebraic Jordan curve, it will be important that its Schwarz function extend to the inside of the domain as a meromorphic function. This condition is equivalent to the domain being a quadrature domain with respect to area measure, as shown by Aharonov and Shapiro [1] (see also [13]). Gustafsson [11] later connected the study of area quadrature domains to the double of the domain in the multiply connected setting and we will use many of his ideas in what follows.

Area quadrature domains are dense among bounded smooth simply connected domains in a very strong sense, and so our first step is to make a change of variables that is C^∞ close to the identity to transform our curve to be the boundary of an area quadrature domain.

2. Step 1: A change of variables

Let γ and Ω be as in §1 and let $K(z,w)$ denote the Bergman kernel associated to Ω. Bergman coordinates that can be taken as close to the identity as desired are defined in [3] as follows. Let $\mathcal{K}$ denote the Bergman span associate to Ω, which is the complex linear span of all functions of the form

$$K_a^0(z) := K(z,a)$$

and

$$K_a^m(z) := \frac{\partial^m}{\partial \bar{w}^m} K(z,w)\bigg|_{w=a}$$

as a ranges over points in Ω and m ranges over all nonnegative integers. As shown in [3], we may find an element of $\mathcal{K}$ as close in $C^\infty(\overline{\Omega})$ to the function which is identically one as desired. If such an element is sufficiently close to one, it has an antiderivative $f(z)$ that is as close to z in $C^\infty(\overline{\Omega})$ as desired such that f is a conformal mapping of Ω one-to-one onto a nearby domain Ω_2. It is shown in [3] that such a domain Ω_2 is an area quadrature domain, which is therefore bounded by a smooth real algebraic curve by results of Aharonov and Shapiro that we will soon review. With this change of variables, we now focus on smooth real algebraic Jordan curves that are the boundary of an area quadrature domain.

The history of the study of area quadrature domains in the plane is long and glorious. The foundations of the subject go back to those papers of Aharonov and Shapiro [1] and Gustafsson [11] (see also [13], [14], and [12]). Many of the basic theorems that we will need are proved in [2, Chap. 22] using a similar philosophy to the techniques of this paper.

3. The main results

From now until we consider the multiply connected case, we will suppose that γ is a smooth real algebraic Jordan curve that bounds an area quadrature domain Ω. Then the Schwarz function $S(z)$ for γ extends meromorphically to Ω [1]. For z in γ, let $T(z)$ be equal to the complex unit tangent vector pointing in the direction of the standard orientation for the boundary $b\Omega$ of Ω.

The Dirichlet-to-Neumann (D-to-N) map takes a C^∞ smooth function on γ to the normal derivative of its Poisson extension to Ω. The following theorem is proved in [4, Theorem 1.5].

Theorem 3.1. *The D-to-N map associated to a bounded smooth simply connected area quadrature domain Ω maps $\mathcal{R}_s$ onto $\mathcal{K}T + \overline{\mathcal{K}\,T}$. It is close to being one-to-one in the sense that, if two functions map to the same function, they must differ by a constant. Furthermore, the elements of the Bergman span in the image are uniquely determined.*

We remark that, sometimes the D-to-N map is viewed, not as φ to harmonic extension u to normal derivative $\frac{\partial u}{\partial n}$, but as tangential derivative $\frac{\partial \varphi}{\partial s} = \frac{\partial u}{\partial s}$ to normal derivative $\frac{\partial u}{\partial n}$. In this setting, the D-to-N map is one-to-one.

We will give a new streamlined proof of Theorem 3.1 here, but first, some old fashioned mathematical plagiarism. See [7] or [2] for more detailed descriptions of this quick review of classical formulas.

The Green's function $G(z,a)$ associated to Ω is the harmonic function of z on $\Omega - \{a\}$ that vanishes on the boundary and is such that $G(z,a) + \ln|z-a|$ has a removable singularity at $a \in \Omega$. The Bergman kernel and the Green's function are related via

$$K(z,w) = -\frac{2}{\pi}\frac{\partial^2 G(z,w)}{\partial z \partial \bar{w}},$$

and the complementary kernel to the Bergman kernel is defined via

$$\Lambda(z,w) = -\frac{2}{\pi}\frac{\partial^2 G(z,w)}{\partial z \partial w}.$$

The Bergman kernel is C^∞ smooth on $\overline{\Omega} \times \overline{\Omega}$ minus the boundary diagonal and $\Lambda(z,w)$ is C^∞ smooth on $\overline{\Omega} \times \overline{\Omega}$ minus the diagonal. The Bergman kernel is holomorphic in z and antiholomorphic in w; $\Lambda(z,w)$ is holomorphic in both variables on $\Omega \times \Omega$ minus the diagonal and has a double pole as a function of z at $z = w$ with principal part $-\frac{1}{\pi}(z-w)^{-2}$ there. Furthermore, $\Lambda(z,w) = \Lambda(w,z)$, and

$$K(z,w)T(z) = -\overline{\Lambda(z,w)}\,\overline{T(z)} \tag{3.1}$$

when z is in the boundary and $w \neq z$.

We will use notation similar to the Bergman span notation to denote $\Lambda_a^0(z) = \Lambda(z,a)$ and

$$\Lambda_a^m(z) := \left.\frac{\partial^m}{\partial w^m}\Lambda(z,w)\right|_{w=a},$$

and denote $G_a^0(z) = G(z,a)$ and

$$G_a^m(z) := \left.\frac{\partial^m}{\partial w^m}G(z,w)\right|_{w=a},$$

and

$$G_a^{\overline{m}}(z) := \left.\frac{\partial^m}{\partial \bar{w}^m}G(z,w)\right|_{w=a}.$$

Note that $G_a^m(z)$ has a pole of order m at $z = a$ as a function of z and that $\frac{\partial}{\partial \bar{z}}G_a^m(z)$ is a nonzero complex constant times the conjugate of $K_a^m(z)$ and $\frac{\partial}{\partial z}G_a^{\overline{m}}(z)$ is a nonzero complex constant times $K_a^m(z)$. Similarly, $\frac{\partial}{\partial z}G_a^m(z)$ is a nonzero complex constant times $\Lambda_a^m(z)$ and $\frac{\partial}{\partial \bar{z}}G_a^{\overline{m}}(z)$ is a nonzero complex constant times the conjugate of $\Lambda_a^m(z)$. Note also that differentiating (3.1) with respect to $\bar{w}$ yields that

$$K_w^m(z)T(z) = -\overline{\Lambda_w^m(z)}\,\overline{T(z)} \tag{3.2}$$

when z is in the boundary and $w \neq z$.

For the future, let $\mathbf{\Lambda}$ denote the complex linear span of the functions Λ_a^m as a ranges over Ω and m ranges over all nonnegative integers.

Proof of Theorem 3.1. If u is a harmonic function on Ω that extends smoothly up to an open curve segment on the boundary, u can be written locally there as $h + \overline{H}$, where h and H are holomorphic functions that are smooth up to the segment. The normal derivative of the holomorphic function h along the outward pointing normal direction at a point z in the segment is $-i\,h'(z)T(z)$. Similarly, the normal derivative of $\overline{H}$ is $i\,\overline{H'(z)}\,\overline{T(z)}$. Since $h'(z) = \partial u/\partial z$ and $\overline{H'(z)} = \partial u/\partial \bar{z}$, we conclude that the normal derivative of a harmonic function u is given by

$$\frac{\partial u}{\partial n} = -i\,\frac{\partial u}{\partial z}\,T(z) + i\,\frac{\partial u}{\partial \bar{z}}\,\overline{T(z)}. \tag{3.3}$$

Given a rational function $R(z, \bar{z})$ representing a function $r(z)$ in $\mathcal{R}_s$, let $u(z)$ denote its harmonic extension to Ω. The function $R(z, S(z))$ extends $r(z)$ to Ω as a meromorphic function and the harmonic extension u is equal to $R(z, S(z))$ minus a finite linear combination of derivatives of the Green's function $G_a^m(z)$ at points in a in Ω chosen to subtract off the principal parts of the poles inside Ω. Note that $\frac{\partial u}{\partial \bar{z}}$ is therefore seen to be a finite linear combination of functions $\frac{\partial}{\partial \bar{z}} G_a^m(z)$, which is a linear combination of the conjugates of the functions $K_a^m(z)$. Similarly, $R(\,\overline{S(z)}, \bar{z})$ extends $r(z)$ as an antimeromorphic function in Ω and subtracting off a linear combination of functions $G_a^{\overline{m}}(z)$ to eliminate the poles yields that $\frac{\partial u}{\partial z}$ is a linear combination of $\frac{\partial}{\partial z} G_a^{\overline{m}}(z)$, which is a linear combination of $K_a^m(z)$. Now formula (3.3) yields the promised decomposition of the normal derivative of u in terms of the Bergman span.

Note that everything we have done so far is valid in the multiply connected setting. We now use the assumption that Ω is simply connected to pin down the uniqueness of the terms in the expression involving the Bergman span. If κ_1 and κ_2 are two elements of the Bergman span such that $\kappa_1 T = \overline{\kappa_2 T}$ on the boundary, then $\kappa_1 dz$ extends to the double via extension by $\overline{\kappa_2}\, d\bar{z}$ on the reflected side. The only holomorphic 1-form on a Riemann surface of genus zero is the zero form. Hence κ_1 and κ_2 are both zero on Ω. This is only possible if all the coefficients in the descriptions of the elements in the Bergman span are also zero. We will complete the proof when we show that the D-to-N map takes $\mathcal{R}_s$ *onto* the image space, which we will do by means of the lemma soon to follow. □

In a moment, we will define $k_a^m(z)$ to be a certain complex antiderivative of $K_a^m(z)$, and we will see that it has some remarkable properties in our present context. Let $\widehat{\Omega}$ denote the double of Ω and let $\widetilde{\Omega}$ denote the reflection of Ω in $\widehat{\Omega}$. If a is a point in Ω, let $\tilde{a}$ denote the point in $\widetilde{\Omega}$ represented by the reflection of a in $\widehat{\Omega}$.

Lemma 3.2. *Suppose Ω is a simply connected smooth area quadrature domain and $a \in \Omega$. A complex antiderivative $k_a^m(z)$ of $K_a^m(z)$ on Ω extends to the double as a meromorphic function with a pole of order $m+1$ at the reflection $\tilde{a}$ of a in the reflected side of Ω in the double. Furthermore, $k_a^m(z)$ is an algebraic function whose restriction to the boundary of Ω is a rational function of z and $\bar{z}$*

in $\mathcal{R}_s$. The normal derivative of $k_a^m(z)$ is $-iK_a^m(z)T(z)$. Similarly, the normal derivative of the conjugate of $k_a^m(z)$ is $i\,\overline{K_a^m(z)T(z)}$. Consequently, the D-to-N map takes $\mathcal{R}_s$ onto $\boldsymbol{\mathcal{K}}T + \overline{\boldsymbol{\mathcal{K}}\,T}$.

There are various ways of understanding this lemma, several of which are explored in [4] in a meandering style. Perhaps the best way to explain it is to note that formula (3.1) reveals that the holomorphic 1-form $K_a^0(z)\,dz$ extends to the double as a meromorphic 1-form κ_a^0 via extension by the conjugate of $-\Lambda_a^0(z)\,dz$ on the reflected side. This 1-form has a single residue free double pole at the point $\tilde{a}$ on the reflected side $\widetilde{\Omega}$. If we fix a point b in Ω, then the line integral

$$h(z) = \int_{\gamma_b^z} \kappa_a^0$$

along a curve γ_b^z connecting b to a point z in the double that avoids $\tilde{a}$ is well defined and defines a single valued holomorphic function on the double minus $\tilde{a}$ with a simple pole at $\tilde{a}$. Indeed, $\kappa_a^0 = dh$. We define the function k_a^0 to be the restriction of h to $\overline{\Omega}$. (There will be occasion to think of k_a^0 as equal to the meromorphic function h on $\widehat{\Omega}$ later in the paper.)

Similarly, (3.2) reveals that the holomorphic 1-form $K_a^m(z)\,dz$ extends to the double as a meromorphic 1-form κ_a^m via extension by the conjugate of $-\Lambda_a^m(z)\,dz$ on the reflected side. This 1-form has a single residue free pole of order $m+2$ at $\tilde{a}$ in the reflected side. Therefore the line integral

$$h(z) = \int_{\gamma_b^z} \kappa_a^m \tag{3.4}$$

along a curve γ_b^z as above is well defined and defines a single valued holomorphic function on the double minus $\tilde{a}$ with a pole of order $m+1$ at $\tilde{a}$. Let k_a^m be the restriction of this function to $\overline{\Omega}$. (As with k_a^0, we will later consider k_a^m to be given by the meromorphic function h on $\widehat{\Omega}$.)

We note that we have defined the functions k_a^m on Ω via

$$k_a^m(z) = \int_{\gamma_b^z} K_a^m(w)\ dw,$$

and k_a^m are therefore seen to be homorphic in z and antiholomorphic in a on Ω, and normalized by the property that they all vanish at b.

That restrictions to the boundary of Ω of meromorphic functions on the double of an area quadrature domain are rational functions of z and $\bar{z}$ was shown by Björn Gustafsson in [11]. We describe his argument here. Since $S(z) = \bar{z}$ on the boundary, the function $S(z)$ extends to the double of Ω as a meromorphic function G_1 via extension by the function $\bar{z}$ on the reflected side. The conjugate identity, $z = \overline{S(z)}$ on $b\Omega$, yields that the function z extends to the double as a meromorphic function G_2 via extension by the function $\overline{S(z)}$ on the reflected side. By taking a point p sufficiently close to the point at infinity in the Riemann sphere, we may arrange that $G_1^{-1}(p)$ consists of distinct points of multiplicity one inside Ω. Since $G_2(z)$ separates these points, G_1 and G_2 form a primitive pair for the double, and therefore generate the field of meromorphic functions (see Farkas and Kra [10, p. 249]). Hence, meromorphic functions on the double are rational combinations G_1 and G_2, and hence, functions of z and $\bar{z}$ when restricted to the boundary of Ω. Primitive pairs are always algebraicly dependent, and hence there is an irreducible polynomial $P(z,w)$ such that $P(z,S(z)) \equiv 0$ on Ω. Hence, $S(z)$ is an algebraic function and all meromorphic functions on Ω that extend meromorphically to the double are algebraic. It also follows that $b\Omega$ is in the set where $P(z,\bar{z}) = 0$, and Gustafsson explores the nature of this algebraic curve further in [11]. This completes the proof of the lemma.

Let $\mathbf{k}$ denote the complex linear span of all the functions k_a^m defined by the integral (3.4) as a ranges over points in Ω and m ranges over all nonnegative integers.

The following theorem now follows by noting that the normal derivative of k_a^m is $-iK_a^m T$ and that two functions on the boundary that map to the same function via the D-to-N map must differ by a constant.

Theorem 3.3. *On a smooth simply connected area quadrature domain Ω, a rational function $R(z,\bar{z})$ in $\mathcal{R}_s$ can be decomposed as a constant plus a uniquely determined element of $\mathbf{k}+\overline{\mathbf{k}}$. Thus,*

$$\mathcal{R}_s = \mathbb{C} + \mathbf{k} + \overline{\mathbf{k}}.$$

It is interesting to note that the complex polynomials are in the Bergman span associated to an area quadrature domain (see [2, p. 119]). Hence, we deduce that $\mathbf{k}$ contains all antiderivatives of complex polynomials that vanish at b, i.e., all polynomials that vanish at b. Furthermore, it can be shown that all complex rational functions with residue free poles outside $\overline{\Omega}$ that vanish at b are also in $\mathbf{k}$.

With this theorem behind us, we can compare it to the decomposition of rational functions of z and $\bar{z}$ restricted to the unit circle we described as our motivator early on and see that they are one and the same result!

Another pleasant moment arises when we realize that Theorem (3.3) gives a solution to the Dirichlet problem in terms of the Bergman kernel on a smooth simply connected area quadrature domain when the boundary data is real and rational. A silly way to solve the Dirichlet problem would be to solve it with real rational data, appeal to the fact that real rational functions on the boundary are an algebra that separates points and are therefore dense among continuous functions, then use sup norm estimates to get general solutions from special solutions. Going back to such foundational considerations is a hobby of the author. See [5] for what happens when one allows oneself to get carried away with such thoughts.

We remark here that all the arguments we have used up to this point can be repeated using the kernel $\Lambda(z,w)$ in place of $K(z,w)$ to yield the following theorems. Recall that $\mathbf{\Lambda}$ denotes the complex linear span of the functions Λ_a^m as a ranges over Ω and m ranges over all nonnegative integers.

Theorem 3.4. *The D-to-N map associated to a smooth area quadrature domain maps $\mathcal{R}_s$ onto $\mathbf{\Lambda}T+\overline{\mathbf{\Lambda}\,T}$. It is close to being one-to-one in the sense that, if two functions map to the same function, they must differ by a constant. Furthermore, the elements of the span $\mathbf{\Lambda}$ of the complementary kernel to the Bergman kernel in the image are uniquely determined.*

Actually, Theorem (3.4) could also be deduced from Theorem (3.1) by merely applying the identity (3.2) to the terms in the decomposition.

The same reasoning that led to the definitions of the antiderivatives k_a^m yield that antiderivatives of Λ_a^m exist and have special properties. Indeed, let η_a^m be the meromorphic 1-form on $\widehat{\Omega}$ defined as $\Lambda_a^m(z)\,dz$ on Ω and as $-\overline{K_a^m(z)}\,d\bar{z}$ on $\widetilde{\Omega}$, and define

$$h(z)=\int_{\gamma_{\tilde{b}}^z}\eta_a^m, \tag{3.5}$$

where $\tilde{b}$ is the reflection of the point b used in equation (3.4) and $\gamma_{\tilde{b}}^z$ is a curve starting at $\tilde{b}$ in $\widehat{\Omega}$ and ending at z which avoids the point a. Define λ_a^m to be the restriction of h to $\overline{\Omega}$ and let $\boldsymbol{\lambda}$ denote the complex linear span of λ_a^m as a ranges over Ω and m ranges over all nonnegative integers. The next theorem follows by the same reasoning we applied to the k_a^m.

Theorem 3.5. *On a smooth area quadrature domain Ω, a rational function $R(z,\bar{z})$ in $\mathcal{R}_s$ can be decomposed as a constant plus a uniquely determined element of*

$$\boldsymbol{\lambda}+\overline{\boldsymbol{\lambda}}.$$

It is interesting to note that, by the way we have defined them via (3.4) and (3.5), $k_a^m(z)=-\overline{\lambda_a^m(z)}$ on the boundary, and this is another way to see that both functions extend meromorphically to the double. Also, the argument principle applied to the boundary relation yields that the one zero of k_a^0 at b balances the one pole of λ_a^0 at a and, because zeroes of k_a^0 get counted positively on the left and zeroes of λ_a^0 count negatively on right, there can be no other zeroes of k_a^0 in $\overline{\Omega}$ and no zeroes of λ_a^0 in $\overline{\Omega}$. Consequently, k_a^0/λ_a^0 is seen to be equal to a unimodular constant times the product of the Riemann map that maps b to the origin and the Riemann map that maps a to the origin, and k_b^0/λ_b^0 is a unimodular constant times the square of the Riemann map that maps b to the origin. It is not hard to use the transformation formula for the Bergman kernel under conformal mappings and the formula for the Bergman kernel of the unit disc to see that

$$k_a^0(z)=\frac{f(z)\overline{f'(a)}}{\pi(1-f(z)\,\overline{f(a)})},$$

where f is the Riemann map mapping Ω one-to-one onto the unit disc with $f(b)=0$ and $f'(b)>0$. Similarly,

$$\lambda_a^0(z)=\frac{f'(a)}{\pi(f(a)-f(z))}.$$

Theorem 3.4 and 3.5 are to the previous two as choosing poles inside the unit circle is to choosing them outside in the motivator decomposition of restrictions of rational function to the unit disc.

Lemma (3.2) also carries over naturally.

Lemma 3.6. *On a simply connected smooth area quadrature domain Ω, a complex antiderivative $\lambda_a^m(z)$ of $\Lambda_a^m(z)$ on Ω extends to the double as a meromorphic function. It is holomorphic on the reflected side and has a pole of order $m+1$ at $z=a$. Furthermore, $\lambda_a^m(z)$ is an algebraic function whose restriction to the boundary of Ω is a rational function of z and $\bar{z}$. Finally, the normal derivative of $\lambda_a^m(z)$ is $-i\Lambda_a^m(z)T(z)$ and so the D-to-N map takes $\mathcal{R}_s$ onto $\mathbf{\Lambda}T+\overline{\mathbf{\Lambda}\;T}$.*

Finally, notice that, by identity (3.2), the terms $\overline{K_a^m T}$ in Theorem 3.1 can be rewritten as $-\Lambda_a^m T$, and following the rest of the argument through in the proof of Theorem 3.3 yields that

$$\mathcal{R}_s=\mathbb{C}+\mathbf{k}+\boldsymbol{\lambda}.$$

Note that a rational function $R(z,\bar{z})$ extends to be a meromorphic function on the double via the formula $R(z,S(z))$, since $S(z)$ extends to the double. The elements $\mathbf{k}+\boldsymbol{\lambda}$ in the decomposition are now seen to be completely determined by the principal parts of the poles of $R(z,S(z))$ in Ω and the reflection of Ω. The poles inside determine the $\boldsymbol{\lambda}$ element, and the poles on the reflected side determine the $\mathbf{k}$ element. Hence, it is a genuine analogue of a partial fractions decomposition. It corresponds to the simple observation that the restrictions of rational functions to the unit circle are the same as the restrictions of complex rational functions. This is the point where it would be more natural to define the functions k_a^m and λ_a^m as the meromorphic functions on the double to which they extend. Then the formula for $R(z,\bar{z})$ on the boundary extends to a formula expressing $R(z,S(z))$ on the double and it is clear how the poles determine the elements in the expansion.

Finally, we remark that all the results of this paper extend to allow pole-like singularities of rational functions restricted to the boundary. Indeed, formulas (3.1) and (3.2) are valid when both z and w are in the boundary $z\neq w$ because the Green's function identities also extend to $b\Omega\times b\Omega$ minus the boundary diagonal. Hence, $\mathcal{R}$ can be decomposed as a vector space sum of the antiderivatives of elements in the extended Bergman span (which include base points a on the boundary), and conjugates of such functions. It is interesting to note that

$$T(z)K(z,w)\,\overline{T(w)}=-\overline{T(z)\Lambda(z,w)T(w)}=T(w)K(w,z)\;\overline{T(z)},$$

when both z and w are on the boundary, and so the terms involving $\overline{K_w^m(z)}\,\overline{T(z)}$ can be converted to terms $K_w^m(z)T(z)$ when w is in the boundary.

4. Special descriptions of the boundary of an area quadrature domain

We are now in a position to apply our results to describing the boundary curves of a simply connected smooth area quadrature domain. If a is a point on the boundary, the function

$$\frac{1}{S(z)-\bar{a}}$$

has a simple pole at a, and so there is a constant A such that

$$R(z)=\frac{1}{S(z)-\bar{a}}-\frac{A}{z-a}$$

has a removable singularity at a. Note that, because the boundary of Ω cannot be a line, $R(z)$ is a function in $\mathcal{R}_s$ that is not the zero function, and so can be expressed as a sum $c+k_1+\overline{k_2}$ where c is a constant and k_1 and k_2 are elements in $\mathbf{k}$ that are not both zero. Hence, the Jordan curve is given by the set where

$$\frac{1}{\bar{z}-\bar{a}}-\overline{k_2(z)}=\frac{A}{z-a}+c+k_1(z),$$

where k_1 and k_2 are algebraic functions that are holomorphic on Ω whose smooth boundary values are rational functions of z and $\bar{z}$. Using the other decompositions lead to interesting variations, which we leave to the reader.

5. Back to step 1

The change of variables introduced in §2 has some interesting properties that we now discuss.

The main theorems of this paper can be thought of as being invariant under changes of Bergman coordinates. Indeed, as shown in [3], a biholomorphic mapping between area quadrature domains is an algebraic function whose restriction to the boundary is rational in z and $\bar{z}$ because it and its inverse extend to the doubles as meromorphic functions. Hence, the mapping is a birational map when restricted to the boundaries which preserves all the algebraic properties of the decompositions described in §3.

Area quadrature domains are such that elements of their Bergman span are algebraic functions whose restrictions to the boundary are rational in z and $\bar{z}$ because they extend to the double as meromorphic functions. Furthermore, the function $T(z)$ on the boundary is such that $T(z)^2$ extends to the double as a meromorphic function, and $T(z)^2$ is therefore a rational function of z and $\bar{z}$ on the boundary. Consequently the D-to-N map on an area quadrature domain takes rational functions of z and $\bar{z}$ to real algebraic functions.

The procedure described in Step 1 to find Bergman coordinates close to the identity can easily be adapted to prove the same result in the multiply connected setting. It is worth noting here that there is a very elementary procedure for producing the change of variables in the simply connected case, as shown in [6]. Indeed, suppose Ω is a bounded domain with smooth real analytic boundary. Then a Riemann mapping function $F:\Omega\to D_1(0)$ extends holomorphically past the boundary. Approximate the inverse F^{-1} by a polynomial $P(z)$ on a neighborhood of the closed unit disc. Shapiro and Aharonov [1] proved that simply connected area quadrature domains are given as images of the unit disc under rational conformal mappings. Hence the image of the unit disc under P is an area quadrature domain nearby to Ω and the change of variables is $P \circ F$. Shapiro and Ullemar [14] proved that a simply connected domain Ω is a quadrature domain with respect to arclength if and only if the derivative of the inverse of the Riemann map is the square of a rational function. We can approximate a square root of the nonvanishing $(F^{-1})'$ by a polynomial $p(z)$, and let $P(z)$ be a polynomial antiderivative of $p(z)^2$ to obtain a domain nearby to Ω given by the image of the unit disc under P that is a quadrature domain with respect to both area and arclength. This domain has all the properties we admire in an area quadrature domain plus the property that $T(z)$ extends to the double as a meromorphic function. Hence, it is a rational function of z and $\bar{z}$ on the boundary and we have a domain whose D-to-N map takes rational functions of z and $\bar{z}$ to rational functions of z and $\bar{z}$.

6. The multiply connected case

Assume that Ω is a bounded n-connected smooth area quadrature domain. Many of the same arguments we have used in the simply connected setting can be carried through without change, and others with only minor modifications involving the harmonic measure functions ω_j and the associated holomorphic functions F_j', $j = 1, \dots, n-1$. We sketch the main arguments here.

Let γ_n denote the outer boundary curve of Ω and let γ_j, $j = 1, \dots, n-1$ denote the $n-1$ inner boundary curves. The function ω_j is the harmonic function on Ω with boundary values equal to one on γ_j and zero on the other boundary curves. The function F_j' is a holomorphic function on Ω equal to $2(\partial\omega_j/\partial z)$. If one considers a locally defined harmonic conjugate function v_j for ω_j, then F_j' is the complex derivative of $\omega_j + iv_j$. Even though v_j is only locally defined, F_j' is globally defined. However, even though we use a prime in the notation, F_j' is not the complex derivative of a function defined on Ω when $n > 1$.

It is well know that the matrix of periods of the F_j' on the $n-1$ inner boundary curves is nonsingular. Hence, there are unique constants c_j and a unique holomorphic function $k_a^m(z)$ on Ω such that

$$(k_a^m)'(z) = K_a^m(z) + \sum_{j=1}^{n-1} c_j F_j'(z). \tag{6.1}$$

The functions k_a^m can be defined by line integrals as in §3. Let $\mathbf{k}$ denote the complex linear span of the functions k_a^m.

Because ω_j is constant on boundary curves, its tangential derivative

$$\frac{\partial\omega_j}{\partial z}\, T(z) + \frac{\partial\omega_j}{\partial\bar{z}}\, \overline{T(z)}$$

is zero on the boundary, and we deduce the well-known identity

$$F_j'(z)\, dz = -\overline{F_j'(z)}\, d\bar{z}$$

on the boundary. Furthermore, the normal derivative of ω_j is given by

$$-i\frac{\partial\omega_j}{\partial z}\,T(z)+i\frac{\partial\omega_j}{\partial\bar z}\,\overline{T(z)}=-2i\frac{\partial\omega_j}{\partial z}\,T(z)=-iF_j'T(z).$$

The part of the new proof we gave of Theorem 3.1 in the simply connected case that showed that the image of the rational functions $\mathcal{R}_s$ under the D-to-N map is equal to $\boldsymbol{\mathcal{K}}T+\overline{\boldsymbol{\mathcal{K}}T}$ is valid in the multiply connected setting.

Theorem 6.1. *The D-to-N map associated to a bounded smooth finitely connected area quadrature domain Ω maps $\mathcal{R}_s$ into $\boldsymbol{\mathcal{K}}T+\overline{\boldsymbol{\mathcal{K}}\,T}$. It is close to being one-to-one in the sense that, if two functions map to the same function, they must differ by a constant.*

Given a rational function $R(z,\bar z)$ in $\mathcal{R}_s$, let φ denote its harmonic extension to Ω and represent the normal derivative

$$\frac{\partial\varphi}{\partial n}=\kappa_1T+\overline{\kappa_2T},$$

where κ_1 and κ_2 are in the Bergman span. Since the normal derivative of k_a^m is $-i(k_a^m)'T$, formula (6.1) yields that there are elements k_1 and k_2 in $\mathbf{k}$ and constants c_j such that the normal derivative of

$$k_1+\overline{k_2}+\sum_{j=1}^{n-1}c_j\omega_j$$

is equal to the normal derivative of φ. Hence, φ and this expression differ by a constant and we conclude that $R(z,\bar z)$ and $k_1+\overline{k_2}$ differ by a constant on each boundary component. Thus, the same result we obtained in the simply connected case holds in the multiply connected setting, with that added stipulation that the constants involved might be different on different boundary curves. We can also use the Bergman kernel to solve the Dirichlet problem with real rational boundary data if we are willing to toss in the functions ω_j. These ideas are explored further in [4]

7. Similar results on nonquadrature domains

We close by remarking that many of the results of this paper extend naturally to the setting of general bounded domains with smooth boundary if we replace the space of rational functions of z and $\bar z$ restricted to the boundary by the space of functions on the boundary that extend meromorphically to the double of the domain, as shown in [4].

References

[1] Aharonov, D. and H. S. Shapiro, *Domains on which analytic functions satisfy quadrature identities*, Journal D'Analyse Mathématique **30** (1976), 39–73.

[2] Bell, S., *The Cauchy transform, potential theory, and conformal mapping*, 2nd Edition, CRC Press, Boca Raton, 2016.

[3] Bell, Steven R., *Bergman coordinates*, Studia Math. **176** (2006), 69–83.

[4] Bell, S., *The Dirichlet and Neumann and Dirichlet-to-Neumann problems in quadrature, double quadrature, and non-quadrature domains*, Analysis and Mathematical Physics **5** (2015), 113–135.

[5] Bell, S. and Reyna de la Torre, L., *Something about Poisson and Dirichlet*, Chapter 1 in Encyclopedia of Complex Analysis, Steven G. Krantz, editor, Taylor and Francis, in press.

[6] Bell, S., Gustafsson, B., and Sylvan, Z., *Szegő coordinates, quadrature domains, and double quadrature domains*, Computational Methods and Function Theory **11** (2011), No. 1, 25–44.

[7] Bergman, S., *The kernel function and conformal mapping*, Mathematical Surveys, Vol. 5 (second edition), American Math. Society, Providence, 1970.

[8] P. Davis, *The Schwarz function and its applications*, Carus Mathematical Monographs **17**, Math. Assoc. of Amer., 1974.

[9] Ebenfelt, P., *Singularities encountered by the analytic continuation of solutions to Dirichlet's problem*, Complex Variables Theory Appl. **20**, (1992), 75–92.

[10] Farkas, H. and I. Kra, *Riemann Surfaces*, Springer-Verlag, New York, 1980.

[11] Gustafsson, B., *Quadrature domains and the Schottky double*, Acta Applicandae Math. **1** (1983), 209–240.

[12] Gustafsson, B., *Applications of half-order differentials on Riemann surfaces to quadrature identities for arc-length*, Journal D'Analyse Math. **49** (1987), 54–89.

[13] Shapiro, H. S., *The Schwarz function and its generalization to higher dimensions*, Univ. of Arkansas Lecture Notes in the Mathematical Sciences, Wiley, New York, 1992.

[14] Shapiro, H. and C. Ullemar, *Conformal mappings satisfying certain extremal properties and associated quadrature identities*, Research Report TRITA-MAT-1986-6, Royal Inst. of Technology, 40 pp., 1981.

Mathematics Department, Purdue University, West Lafayette, IN 47907

Email address: bell@math.purdue.edu