MÖBIUS TRANSFORMATIONS, THE CARATHÉODORY METRIC, AND THE OBJECTS OF COMPLEX ANALYSIS AND POTENTIAL THEORY IN MULTIPLY CONNECTED DOMAINS

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Abstract. It is proved that the family of Ahlfors extremal mappings of a multiply connected region in the plane onto the unit disc can be expressed as a rational combination of two fixed Ahlfors mappings in much the same way that the family of Riemann mappings associated to a simply connected region can be expressed in terms of a single such map. The formulas reveal that this family of mappings extends to the double as a real analytic function of both variables. In particular, the infinitesimal Carathéodory metric will be expressed in strikingly simple terms. Similar results are proved for the Green's function, the Poisson kernel, and the Bergman kernel.

1. Introduction

Let f_b denote the Riemann mapping function associated to a point b in a simply connected planar domain $\Omega \neq \mathbb{C}$. Everyone knows that f_b is the solution to an extremal problem; it is the holomorphic map h of Ω into the unit disc such that $h'(b)$ is real and as large as possible. Everyone also knows that all the maps f_b can be expressed in terms of a single Riemann map f_a associated to a point $a \in \Omega$ via

$$f_b(z) = \lambda \frac{f_a(z) - f_a(b)}{1 - f_a(z)\overline{f_a(b)}}$$

where the unimodular constant λ is given by

$$\lambda = \frac{\overline{f_a'(b)}}{|f_a'(b)|}.$$

In this paper, I shall prove that the solutions to the analogous extremal problems on a finitely multiply connected domain in the plane, the Ahlfors mappings, can be expressed in terms of just *two* fixed Ahlfors mappings. Many similarities with the formula above in the simply connected case will become apparent and I will explore some of the algebraic objects that present themselves. A byproduct of these considerations will be that the infinitesimal Carathéodory metric on a multiply connected domain is a rather simple rational combination of two Ahlfors maps and one of their derivatives. I will explain an outlook which reveals a natural way to view the extremal functions involved in the definition of the Carathéodory metric "off the diagonal" in such a way that they extend to $\widehat{\Omega} \times \widehat{\Omega}$ where $\widehat{\Omega}$ is the double of Ω.

1991 *Mathematics Subject Classification.* 32H10.
Key words and phrases. Szegő kernel.
Research supported by NSF grant DMS-0072197.

I will also investigate the complexity of the classical Green's function and Bergman kernel associated to a multiply connected domain. In particular, it is proved in §6 that if Ω is a finitely connected domain in the plane such that no boundary component is a point, then there exist two Ahlfors maps f_a and f_b associated to Ω such that the Bergman kernel for Ω is given by

$$K(w,z)=\frac{f_a'(w)\overline{f_a'(z)}}{(1-f_a(w)\overline{f_a(z)})^2}\left(\sum_{j,k=1}^{N}H_j(w)\overline{H_k(z)}\right),$$

where the functions H_j are rational combinations of the two Ahlfors maps f_a and f_b. Future avenues of research include the problem of extending these results to finite Riemann surfaces and the problem to determine the way the rational functions that arise in these formulas depend on the domain.

2. The smooth case

To get started, we shall assume that Ω is a bounded n-connected domain in the plane with C^∞ smooth boundary consisting of n non-intersecting curves. (Later, we shall consider general n-connected domains such that no boundary component is a point.) Let $S(z,w)$ denote the Szegő kernel associated to Ω (see [2] or [8] for definitions and standard terminology in what follows).

Fix a point a in Ω so that the $n-1$ zeroes $a_1,\dots,a_{n-1}$ of $S(z,a)$ in the z-variable are distinct simple zeroes. (That such points a form an open dense subset of Ω was proved in [3].) Let a_0 be equal to a. I proved in [4, Theorem 3.1] that the Szegő kernel can be expressed in terms of the $n+1$ functions of one variable, $S(z,a)$, $f_a(z)$, and $S(z,a_i)$, $i=1,\dots,n-1$ via the formula

$$S(z,w)=\frac{1}{1-f_a(z)\overline{f_a(w)}}\sum_{i,j=0}^{n-1}c_{ij}S(z,a_i)\,\overline{S(w,a_j)} \tag{2.1}$$

where $f_a(z)$ denotes the Ahlfors map associated to (Ω,a) and the coefficients c_{ij} are given as the coefficients of the inverse matrix to the $n\times n$ matrix $[S(a_j,a_k)]$. A similar formula for the Garabedian kernel was proved in [5],

$$L(z,w)=\frac{f_a(w)}{f_a(z)-f_a(w)}\sum_{i,j=0}^{n-1}c_{ij}S(z,a_i)L(w,a_j), \tag{2.2}$$

where the constants c_{ij} are the same as the constants in (2.1).

Given a point $w\in\Omega$, the Ahlfors map f_w associated to the pair (Ω,w) is a proper holomorphic mapping of Ω onto the unit disc. It is an n-to-one mapping (counting multiplicities), it extends to be in $C^\infty(\overline{\Omega})$, and it maps each boundary curve of Ω one-to-one onto the unit circle. Furthermore, $f_w(w)=0$, and f_w is the unique function mapping Ω into the unit disc maximizing the quantity $|f_w'(w)|$ with $f_w'(w)>0$. The Ahlfors map is related to the Szegő kernel and Garabedian kernel via (see [2, page 49])

$$f_w(z)=\frac{S(z,w)}{L(z,w)}. \tag{2.3}$$

When equations (2.1) and (2.2) are substituted into (2.3), we obtain the monstrosity

$$f_w(z) = \frac{f_a(z) - f_a(w)}{f_a(w)(1 - f_a(z)\overline{f_a(w)}\,)} \, \frac{\sum_{i,j=0}^{n-1} c_{ij} S(z, a_i) \, \overline{S(w, a_j)}}{\sum_{i,j=0}^{n-1} c_{ij} S(z, a_i) L(w, a_j)}.$$

Next, divide the numerator and the denominator of the second quotient in this expression by $S(z, a)S(w, a)$ and multiply the whole thing by one in the form of $\overline{S(w, a)}/\overline{S(w, a)}$ to obtain

$$f_w(z) = \frac{f_a(z) - f_a(w)}{f_a(w)(1 - f_a(z)\overline{f_a(w)}\,)} \left(\frac{\sum_{i,j=0}^{n-1} c_{ij} \frac{S(z,a_i)}{S(z,a)} \, \overline{S(w, a_j)}/\overline{S(w, a)}}{\sum_{i,j=0}^{n-1} c_{ij} \frac{S(z,a_i)}{S(z,a)} \, L(w, a_j)/S(w, a)} \right) \frac{\overline{S(w, a)}}{S(w, a)}.$$

It is not hard to show that $f_a(z)$ and quotients of the form $S(z, a_i)/S(z, a)$ and $L(z, a_i)/S(z, a)$ extend to the double $\widehat{\Omega}$ of Ω as meromorphic functions (see [6, p. 6]). Since the argument is quick and simple, we give it here. Let $R(z)$ denote the antiholomorphic reflection function on $\widehat{\Omega}$ which maps Ω into the reflected copy of Ω. Note that $f_a(z)$ is equal to $1/\overline{f_a(z)}$ on $b\Omega$, which is equal to $1/\overline{f_a(R(z))}$ there. Hence, the holomorphic function $f_a(z)$ on Ω and the meromorphic function $1/\overline{f_a(R(z))}$ on the complement of Ω in $\widehat{\Omega}$ both extend continuously up to $b\Omega$ and have the same values there. Hence f_a extends meromorphically to the double. Similar reasoning can be applied to the quotients as follows. The Garabedian kernel is related to the Szegő kernel via the identity

$$\frac{1}{i} L(z, a)T(z) = S(a, z) \qquad \text{for } z \in b\Omega \text{ and } a \in \Omega, \tag{2.4}$$

where $T(z)$ denotes the complex number of unit modulus pointing in the tangent direction at $z \in b\Omega$ chosen so that $iT(z)$ represents an inward pointing normal vector to the boundary. Hence, $S(z, a_i)/S(z, a)$ is equal to the conjugate of $L(z, a_i)/L(z, a)$ on the boundary and the same reasoning used above for f_a shows that $S(z, a_i)/S(z, a)$ extends to the double meromorphically. Similarly $L(z, a_i)/S(z, a)$ is equal to the conjugate of $S(z, a_i)/L(z, a)$ on the boundary and this shows that $L(z, a_i)/S(z, a)$ extends to the double meromorphically.

It is proved in [7] that it is possible to choose a second Ahlfors map f_b so that f_a and f_b generate the field of meromorphic functions on $\widehat{\Omega}$. (Such a pair is called a *primitive pair*, see [1] and [9]). Hence, we have now shown that there exists a rational function on $\mathbb{C}^6$ such that

$$f_w(z) = \lambda(w) R(f_a(z), f_b(z), f_a(w), f_b(w), \overline{f_a(w)}, \overline{f_b(w)}) \tag{2.5}$$

where $\lambda(w)$ is the unimodular function given by

$$\lambda(w) = \overline{S(w, a)}/S(w, a).$$

This formula is very reminiscent of the formula for the Riemann maps mentioned at the beginning of this paper. It now becomes irresistible to drop the factor $\lambda(w)$ from equation (2.5) and to define a function $F(z, w)$ via

$$F(z, w) = f_w(z)/\lambda(w).$$

Let us call this function the *alternatively normalized Ahlfors map*. Under this normalization, the map $z \mapsto F(z, w)$ has a derivative at w with extremal modulus, however the argument of the derivative is $-\arg \lambda(w)$ there instead of zero. This family of extremal maps has the astonishing feature that it extends in a unique way to $\widehat{\Omega} \times \widehat{\Omega}$ as a complex rational function of $f_a(z)$, $f_b(z)$, and $f_a(w)$, $f_b(w)$, $\overline{f_a(w)}$, $\overline{f_b(w)}$. Furthermore, this extension is meromorphic in z and real analytic in w. One might also glimpse some semblance of an analogue of a Möbius function in these deliberations and we shall come back to this point later in the paper.

Another important consequence of formula (2.5) is that the infinitesimal Carathéodory metric can be expressed in terms of two Ahlfors maps. In fact, it is shown in [7, p. 344] that the quotient $f_b'(z)/f_a'(z)$ extends to be meromorphic on the double of Ω and is therefore a rational combination of $f_a(z)$ and $f_b(z)$. Hence, if we differentiate (2.5) with respect to z and take the modulus of the expression, we obtain that $|f_w'(z)|$ is given by $|f_a'(z)|$ times the modulus of a rational function of $f_a(z)$, $f_b(z)$, $f_a(w)$, $f_b(w)$, $\overline{f_a(w)}$, and $\overline{f_b(w)}$. We have shown that the infinitesimal Carathéodory metric is given by $\rho(z)|dz|$ where

$$\rho(z) = |f_a'(z)| \left| Q(f_a(z), f_b(z), f_a(w), f_b(w), \overline{f_a(w)}, \overline{f_b(w)}) \right|,$$

where Q is a rational function on $\mathbb{C}^6$.

Many questions present themselves at this point. The formula above for the infinitesimal Carathéodory metric almost looks exact. Might there exist special multiply connected domains where the Carathéodory metric could be computed as easily as it is in the unit disk? Another natural question to ask is whether or not similar formulas hold for finite Riemann surfaces. Ahlfors mappings are available in this setting, but the relationship between these maps and the kernel functions used in the proof in the planar case are not as straightforward. New methods of proof would have to be discovered.

3. The nonsmooth case

Suppose that Ω is merely an n-connected domain in the plane such that no boundary component is a point. It is well known that there is a biholomorphic mapping ϕ mapping Ω one-to-one onto a bounded domain Ω_a in the plane with smooth real analytic boundary. The standard construction yields a domain Ω_a that is a bounded n-connected domain with C^∞ smooth boundary whose boundary consists of n non-intersecting simple closed real analytic curves. Let subscript or superscript a's indicate that a kernel function or mapping is associated to Ω_a. Kernels without sub or superscripts are associated to Ω. It is well known that the function ϕ' has a single valued holomorphic square root on Ω (see [2, p. 43]). We define the Szegő kernel and Garabedian kernel associated to Ω via the natural transformation formulas,

$$S(z, w) = \sqrt{\phi'(z)}\ S_a(\phi(z), \phi(w))\overline{\sqrt{\phi'(w)}}$$

and

$$L(z, w) = \sqrt{\phi'(z)}\ L_a(\phi(z), \phi(w))\sqrt{\phi'(w)}.$$

The Ahlfors map associated to a point $b \in \Omega$ is defined to be the solution to the extremal problem, $f_b : \Omega \to D_1(0)$ with $f_b'(b) > 0$ and maximal. It is easy to see that Ahlfors maps satisfy

$$f_b(z) = \lambda f^a_{\phi(b)}(\phi(z))$$

for some unimodular constant λ and it follows that $f_b(z)$ is a proper holomorphic mapping of Ω onto $D_1(0)$. It also follows that $f_b(z)$ is given by $S(z,b)/L(z,b)$ just as in the smooth case. Now it is easy to see that all the quotients that appeared in the proofs of the results of §2 are invariant under ϕ and the proofs carry over line for line. We may now state the following theorem.

Theorem 3.1. *Suppose that Ω is an n-connected domain in the plane such that no boundary component is a point. There exist two points a and b in Ω such that the alternatively normalized Ahlfors map $F(z,w)$ associated to Ω is a complex rational function of $f_a(z)$, $f_b(z)$, and $f_a(w)$, $f_b(w)$, $\overline{f_a(w)}$, $\overline{f_b(w)}$. Furthermore, the family of Ahlfors mappings is given by formula (2.5) and the infinitesimal Carathéodory metric is given by $\rho(z)|dz|$ where*

$$\rho(z) = |f_a'(z)| \left| Q(f_a(z), f_b(z), f_a(w), f_b(w), \overline{f_a(w)}, \overline{f_b(w)}) \right|,$$

where Q is a rational function on $\mathbb{C}^6$.

4. What is a Möbius transformation?

Here is one way to "invent" Möbius transformations. Let $p(z)$ denote an irreducible polynomial of one variable with no zeroes in the unit disc, i.e., let $p(z) = z-b$ where $|b|>1$. Notice that $p(1/\bar{z})$ is equal to $p(z)$ on the unit circle. Let $q(z)$ denote the polynomial obtained by multiplying the conjugate of $p(1/\bar{z})$ by the power of z needed to clear the poles in the unit disc, i.e., $q(z) = 1-z\bar{b}$. Since $|q(z)| = |\bar{z}p(1/\bar{z})|$, it follows that $|q(z)| = |p(z)|$ on the unit circle. Notice that $q(z)/p(z)$ is a Möbius transformation (let $b = 1/\bar{a}$ to make it look more standard).

It is shown in [7] that every proper holomorphic mapping of a smooth n-connected domain Ω onto the unit disk can be expressed as a rational combination of two Ahlfors maps f_a and f_b associated to points a and b in Ω. It is an interesting problem to determine just exactly which rational functions arise in this manner, and it is tempting to call some of these rational functions Möbius transformations. Here is one way to construct such a rational function. Let Δ^2 denote the unit bidisc. Let $p(z,w)$ denote an irreducible polynomial of two variables with no zeroes in the closure of Δ^2. Notice that $p(1/\bar{z}, 1/\bar{w})$ is equal to $p(z,w)$ on the distinguished boundary of Δ^2. Suppose N is the degree of $p(z,w)$ in z and M is the degree in w. Let $q(z,w)$ be the polynomial given by $z^N w^M$ times the conjugate of $p(1/\bar{z}, 1/\bar{w})$. Since $q(z,w)$ and $p(z,w)$ have the same modulus on the distinguished boundary of Δ^2, and since $|z^N w^M| = 1$ there, it follows that the modulus of $q(z,w)/p(z,w)$ is also one there. It follows that, if it is not constant, $q(f_a(z), f_b(w))/p(f_a(z), f_b(w))$ is a proper holomorphic mapping of Ω onto the unit disc.

More generally, the same construction can be carried out if $p(z,w)$ is an irreducible polynomial on $\mathbb{C}^2$ which does not vanish on the portion of the curve $z \mapsto (f_a(z), f_b(z))$ inside the closed unit bidisc. Can any proper map from Ω to the unit disc be expressed in a similar manner, perhaps as some kind of combination of these basic maps?

5. The Poisson kernel extends nicely to the double

Of course the Poisson kernel extends to the double by simple reflection. Here we show that it extends nicely in both variables and in terms of some special functions with geometric meaning.

Assume that Ω is a bounded n-connected domain in the plane with C^∞ smooth boundary consisting of n non-intersecting curves. Let $\gamma_1, \dots, \gamma_{n-1}$ denote the inner curves and let γ_n denote the outer curve.

The classical Poisson kernel for Ω is related to the normal derivative of the Green's function via

$$p(z,w) = \frac{1}{2\pi}\frac{\partial}{\partial n_w} G(z,w) \qquad z \in \Omega,\ w \in b\Omega,$$

where $(\partial/\partial n_w)$ denotes the normal derivative in the w variable. It is a standard fact that we may rewrite this last formula (see [2, pages 134-136]) in the form

$$p(z,w) = -\frac{i}{\pi}\frac{\partial}{\partial w} G(z,w) T(w).$$

It is proved in [4, p. 1367] (see also [6, p. 12] for an easier proof) that the derivative of the Green's function $G_w(z,w) := \frac{\partial}{\partial w} G(z,w)$ is given by

$$G_w(z,w) = \pi \frac{S(w,z)L(w,z)}{S(z,z)} + i\pi \sum_{j=1}^{n-1} \left(\omega_j(z) - \lambda_j(z)\right) u_j(w), \tag{5.1}$$

where the functions $\lambda_j(z)$ are given by

$$\lambda_j(z) = \int_{w \in \gamma_j} \frac{|S(w,z)|^2}{S(z,z)}\, ds,$$

the functions $\omega_j(z)$ are the harmonic measure functions, and the functions u_j are a basis for the linear span $\mathcal{F}'$ of the functions $F_j' := 2(\partial\omega_j/\partial z)$ normalized so that

$$\delta_{kj} = \int_{\gamma_k} u_j(w)\, dw.$$

We now show that the principal term $\frac{S(w,z)L(w,z)}{S(z,z)}$ in the expression for $G_w(z,w)$ has the interesting property that it extends to the double of Ω in the z variable as a real analytic function that is a rational combination of two Ahlfors maps $f_a(z)$ and $f_b(z)$ and their conjugates. Indeed, if we substitute equations (2.1) and (2.2) into this expression, we obtain that

$$\frac{S(w,z)L(w,z)}{S(z,z)} = T_1(z,w)T_2(z,w) \tag{5.2}$$

where

$$T_1(z,w) = \frac{(1 - |f_a(z)|^2) f_a(z)}{(f_a(w) - f_a(z))(1 - f_a(w)\overline{f_a(z)}\,)}$$

and

$$T_2(z,w) = \frac{\left(\sum_{i,j=0}^{n-1} c_{ij} S(w,a_i)\,\overline{S(z,a_j)}\right)\left(\sum_{i,j=0}^{n-1} c_{ij} S(w,a_i) L(z,a_j),\right)}{\sum_{i,j=0}^{n-1} c_{ij} S(z,a_i)\,\overline{S(z,a_j)}}.$$

The first term extends to the double as a real analytic function because f_a does. If we divide the numerator and denominator of the second term by $|S(z,a)|^2$, we observe that the numerator is a linear combination of functions which are given as products of $L(z,a_n)/S(z,a)$ times the conjugate of $S(z,a_m)/S(z,a)$ times $S(w,a_q)S(w,a_k)$. As mentioned previously, the functions $S(z,a_m)/S(z,a)$ and $L(z,a_n)/S(z,a)$ extend meromorphically to the double, and hence can be expressed as rational functions of two Ahlfors maps $f_a(z)$ and $f_b(z)$. The denominator is a linear combination of functions given as the product of $L(z,a_n)/S(z,a)$ times the conjugate of $S(z,a_m)/S(z,a)$. Hence, it has these properties, too, in the z variable.

We have shown that, for fixed w, the function of z given by $S(w,z)L(w,z)/S(z,z)$ is a rational combination of two Ahlfors maps $f_a(z)$ and $f_b(z)$ and their conjugates.

We now claim that functions of the form

$$S(w,a_q)S(w,a_k)/f_a'(w)$$

extend meromorphically to the double of Ω. Indeed, since identity (2.4) yields that $S(w,a_q)S(w,a_k)T(w)$ is equal to the conjugate of $-L(w,a_q)L(w,a_k)T(w)$ for w in the boundary and since $T(w)f_a'(w)/f_a(w)$ is equal to the conjugate of $-T(w)f_a'(w)/f_a(w)$, we may use similar reasoning to that in [5, p. 202] and divide these two expressions to see that $S(w,a_q)S(w,a_k)f_a(w)/f_a'(w)$ extends meromorphically to the double. Since $f_a(w)$ extends to the double, the claim is proved. Now, if we were to divide the large expression for $S(w,z)L(w,z)/S(z,z)$ above by $f_a'(w)$, we would deduce that

$$\frac{S(w,z)L(w,z)}{f_a'(w)S(z,z)}$$

is a rational combination of the two functions $f_a(w)$ and $f_b(w)$, and the four functions $f_a(z)$ and $f_b(z)$ and their conjugates.

It is proved in [6] that there exist $n-1$ points w_j in Ω such that the functions

$$(\omega_k(z)-\lambda_k(z))$$

are linear combinations of functions of z of the form

$$G_w(z,w_j)-\pi\frac{S(w_j,z)L(w_j,z)}{S(z,z)}.$$

The function $G_w(z,w_j)$ is harmonic in z on $\Omega-\{w_j\}$ and vanishes on the boundary. Hence, it extends to the double as a harmonic function with two singular points. The function $S(w_j,z)L(w_j,z)/S(z,z)$ has been shown to extend to the double. Since $F_j'(w)T(w)$ is equal to the conjugate of $-F_j'(w)T(w)$ on the boundary, the same reasoning used above yields that functions of the form

$$F_j'(w)/f_a'(w)$$

extend meromorphically to the double of Ω. We may now state that $G_w(z,w)$ is given as $f_a'(w)$ times a rational combination of the two functions $f_a(w)$ and $f_b(w)$, and the four functions $f_a(z)$ and $f_b(z)$ and their conjugates plus a linear combination of the functions $G_w(z,w_j)$ times rational combinations of $f_a(w)$ and $f_b(w)$. In symbols,

$$\begin{aligned} G_w(z,w) \quad &= \quad f_a'(w)R_0(f_a(w),f_b(w),f_a(z),f_b(z),\overline{f_a(z)},\overline{f_b(z)}\,) \\ &\quad + f_a'(w)\sum_{j=1}^{n-1}G_w(z,w_j)R_j(f_a(w),f_b(w)) \end{aligned}$$

where the functions R_0 and R_j are rational. All the functions that comprise $G_w(z,w)$ extend nicely to the double except $f_a'(w)$.

The results of this section can be generalized to n-connected domains with nonsmooth boundaries in the same way as in §3, but we shall not do this here.

6. Linearizing the Green's function and Bergman kernel

In the simply connected case, the Green's function is related to a Riemann map $f(z)$ by the very simple formula

$$G(z,w) = \ln \left| \frac{f(z)-f(w)}{1-f(z)\overline{f(w)}} \right|.$$

In the multiply connected setting, the Green's function is also related to Ahlfors maps, but it is not clear if the Green's function can be expressed naturally in terms of maps. We saw some tantalizing evidence in the last section that there might be such an expression. In this section, I give some further evidence which leads me to believe that maybe such an expression exits. This evidence fits nicely into the subject matter of this paper because a genuine Möbius transformation is a key ingredient.

Suppose that Ω is a multiply connected domain with C^∞-smooth boundary and let $f(z)$ denote an Ahlfors map associated to (Ω, a) that has simple zeroes. Let $\mathcal{L}(z,w)$ denote the function

$$\ln \left| \frac{f(z)-f(w)}{1-\overline{f(z)}f(w)} \right|.$$

We want to investigate the boundary behavior of the quotient $G(z,w)/\mathcal{L}(z,w)$ as z and w are both allowed to approach the boundary. First assume that z is a fixed point in Ω and let w approach the boundary. Both the numerator and the denominator extend C^∞ smoothly in w to the boundary and the Hopf Lemma reveals that both terms vanish to first order along the boundary. Hence, the quotient extends C^∞-smoothly up to the boundary in w and the limit is given (by L'Hôpital's rule) as the quotient of the normal derivatives $(\partial/\partial n_w)$ in the w variable. Recall that

$$\frac{\partial}{\partial n_w} G(z,w) = -2iG_w(z,w)T(w).$$

Since $\mathcal{L}$ is also a real valued harmonic function that vanishes on the boundary, the same reasoning that yields this identity can be applied to the normal derivative of $\mathcal{L}(z,w)$ to obtain

$$\frac{\partial}{\partial n_w}\mathcal{L}(z,w) = -2i\mathcal{L}_w(z,w)T(w) = \frac{f'(w)(1-|f(z)|^2)T(w)}{i(f(w)-f(z))(1-\overline{f(z)}f(w))}.$$

Notice the similarity of this expression with $T_1(z,w)$ in formula (5.2). We may now divide these two normal derivatives and use (5.1) and (5.2) to obtain that

$$\frac{(\partial/\partial n_w)G(z,w)}{(\partial/\partial n_w)\mathcal{L}(z,w)} = \frac{if(z)}{f'(w)}T_2(z,w) + T_3(z,w)$$

where

$$T_2(z,w) = \frac{\left(\sum_{i,j=0}^{n-1} c_{ij} S(w,a_i)\,\overline{S(z,a_j)}\right)\left(\sum_{i,j=0}^{n-1} c_{ij} S(w,a_i) L(z,a_j),\right)}{\sum_{i,j=0}^{n-1} c_{ij} S(z,a_i)\,\overline{S(z,a_j)}}.$$

and

$$T_3(z,w) = \frac{i(f(w)-f(z))(1-\overline{f(z)}f(w))}{f'(w)(1-|f(z)|^2)} \sum_{j=1}^{n-1} i\pi\left(\omega_j(z)-\lambda_j(z)\right)u_j(w).$$

Although this formula is painful to look at, a moment of suffering reveals that the right hand side can be written as a sum of simple terms to yield that

$$\frac{(\partial/\partial n_w)G(z,w)}{(\partial/\partial n_w)\mathcal{L}(z,w)} = \sum_j \mu_j(z)h_j(w) \tag{6.1}$$

where each of the functions μ_j extend to the double as real analytic functions and each of the functions $h_j(w)$ extend to the double as meromorphic functions. (Note that here we have used the fact proved earlier that u_j/f' extends to the double as a meromorphic function.) I view this formula as a linearization or polarization of the Poisson kernel. I take this opportunity to state a theorem.

Theorem 6.1. *Suppose that Ω is a bounded finitely connected domain in the plane with C^∞ smooth boundary. The Green's function associated to Ω satisfies an identity of the form*

$$\frac{\partial}{\partial w}G(z,w) = \frac{\partial}{\partial w}\mathcal{L}(z,w)\sum_j \mu_j(z)h_j(w)$$

where each μ_j extends to be real analytic on the double of Ω and each h_j extends to be meromorphic on the double of Ω.

We note that we have proved this identity when z is in Ω and w is in the boundary of Ω, but since the functions of w in the expression are all meromorphic, the identity extends to hold for all w in $\overline{\Omega}$.

We now continue to deal with equation (6.1) and we assume that w is back in the boundary and we let the z variable tend to a boundary point other than w to obtain

$$\frac{(\partial^2/\partial n_z\partial n_w)G(z,w)}{(\partial^2/\partial n_z\partial n_w)\mathcal{L}(z,w)} = \frac{(\partial^2/\partial\bar{z}\partial w)G(z,w)}{(\partial^2/\partial\bar{z}\partial w)\mathcal{L}(z,w)} = \frac{if(z)}{f'(w)}T_2(z,w) + T_4(z,w),$$

where $T_2(z,w)$ is as above, but $T_4(z,w)$ is given by

$$T_4(z,w) = \frac{i(f(w)-f(z))(1-\overline{f(z)}f(w))}{f'(w)}\sum_{j=1}^{n-1}\nu_j(z)u_j(w),$$

where $\nu_j(z)$ is equal to the limit of $i\pi\left(\omega_j(z)-\lambda_j(z)\right)/(1-|f(z)|^2)$ as z tends to the boundary. Since $(\partial^2/\partial\bar{z}\partial w)G(z,w)$ is equal to $K(w,z)$ and since $(\partial^2/\partial\bar{z}\partial w)\mathcal{L}(z,w)$ is equal to

$$\frac{f'(w)\overline{f'(z)}}{(1-f(w)\overline{f(z)})^2},$$

we deduce that

$$K(w,z) = \frac{f'(w)\overline{f'(z)}}{(1-f(w)\overline{f(z)})^2}\left(\sum_j \sigma_j(z)H_j(w)\right)$$

where the sum is finite and each function $H_j(w)$ extends to be meromorphic on the double of Ω. We may assume that this sum has been collapsed so that the functions $H_j(w)$ are linearly independent on Ω. We can now exploit the hermitian property of the Bergman kernel to easily deduce that the σ_j functions are actually linear combinations of the conjugates of the H_j. Hence, we have proved that

$$K(w,z) = \frac{f'(w)\overline{f'(z)}}{(1 - f(w)\overline{f(z)})^2} \left(\sum_{j,k=1}^{N} H_j(w)\overline{H_k(z)} \right).$$

We have only shown that this identity holds on the boundary, but it is clear that it extends to the inside of the domain because all the functions that appear in the identity are meromorphic. We can now use the fact proved in [7] that the field of meromorphic functions on the double is generated by two Ahlfors maps to be able to state that the functions H_j are rational combinations of two Ahlfors maps. We have operated under the assumption that Ω has smooth boundary. Finally, if Ω does not have smooth boundary, we can map to a domain with smooth boundary and use the fact that the terms in the expression for $K(z,w)$ transform under biholomorphic mappings to obtain the following theorem.

Theorem 6.2. *Suppose that Ω is a finitely connected domain in the plane such that no boundary component is a point. There exist two points a and b in Ω such that the Bergman kernel associated to Ω is given by*

$$K(w,z) = \frac{f_a'(w)\overline{f_a'(z)}}{(1 - f_a(w)\overline{f_a(z)})^2} \left(\sum_{j,k=1}^{N} H_j(w)\overline{H_k(z)} \right),$$

where the functions H_j are rational combinations of the two Ahlfors maps f_a and f_b.

There are many interesting questions that present themselves at this point. The rational functions that appear in the formula in Theorem 6.2 most likely satisfy an invariance property under biholomorphic mappings and have algebraic geometric significance. The functions λ_j have many interesting properties. I wonder if they might be expressible as rational combinations of two Ahlfors maps and their conjugates. I also wonder if the Green's function can be shown to have similar finite complexity to all the other kernel functions that have been studied in this paper, modulo some logarithmic expressions. It is also a safe bet that many of the results in this paper extend to the case of finite Riemann surfaces. I leave these investigations for the future.

References

[1] L. V. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton Univ. Press, Princeton, 1960.

[2] S. Bell, *The Cauchy transform, potential theory, and conformal mapping*, CRC Press, Boca Raton, 1992.

[3] ———, *The Szegő projection and the classical objects of potential theory in the plane*, Duke Math. J. **64** (1991), 1–26.

[4] ———, *Complexity of the classical kernel functions of potential theory*, Indiana Univ. Math. J. **44** (1995), 1337–1369.

[5] ———, *Finitely generated function fields and complexity in potential theory in the plane*, Duke Math. J. **98** (1999), 187–207.

[6] ———, *The fundamental role of the Szegő kernel in potential theory and complex analysis*, Journal fr die reine und angewandte Mathematik **525** (2000), 1–16.

[7] ———, *Ahlfors maps, the double of a domain, and complexity in potential theory and conformal mapping*, Journal d'Analyse Mathematique **78** (1999), 329–344.

[8] S. Bergman, *The kernel function and conformal mapping*, Math. Surveys **5**, AMS, Providence, 1950.

[9] H. M. Farkas and I. Kra, *Riemann Surfaces*, Springer-Verlag, New York, 1980.

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