DENSITY OF QUADRATURE DOMAINS IN ONE AND SEVERAL COMPLEX VARIABLES

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Abstract. We will demonstrate a new way to understand that quadrature domains are dense in the realm of smoothly bounded planar domains. The new outlook has the virtue of possibly generalizing to several complex variables, and we propose an avenue of research for the study of quadrature domains in several variables.

1. Introduction

Quadrature domains in the plane share many of the properties of the unit disc. The boundaries are algebraic curves and the Bergman and Szegő kernel functions associated to them are algebraic functions, and every finitely connected domain such that no boundary component is a point is biholomorphic to a quadrature domain. The ground breaking work of Aharonov and Shapiro [1] and Gustafsson [11] started an avalanche of results on these fascinating and useful domains. (See [12] for the current state of research.) Gustafsson proved that quadrature domains are dense in the category of domains bounded by finitely many continuous curves, and this result has been generalized to the C^∞-smooth category in [8, 9]. This density theorem opens the door to methods for "zipping" the classical kernel functions of complex analysis and potential theory down to very small data sets (see [8]). The change of variables given by the density theorem also yields a new way to understand Bergman coordinates and representative domains (see [10]). The purpose of this paper is to demonstrate an alternate way of looking at the density theorem that generalizes nicely to several complex variables. We shall also debate about what the correct definition of quadrature domain should be in several complex variables and make a conjecture about a class of domains that we shall call "one point Quadrature domains."

In this paper, by *quadrature domain* in the plane, we will mean a bounded domain Ω such that there exist finitely many points $\{w_j\}_{j=1}^N$ in the domain and non-negative integers n_j such that complex numbers c_{jk} exist satisfying

$$\int_\Omega h\ dA = \sum_{j=1}^N \sum_{k=0}^{n_j} c_{jk} h^{(k)}(w_j)$$

for every function h in the Bergman space $H^2(\Omega)$ of square integrable holomorphic functions on Ω. Here, dA denotes Lebesgue area measure.

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If Ω is a bounded domain in the plane, then the Bergman space of holomorphic functions in $L^2(\Omega)$ and the associated Bergman kernel $K(z,w)$ are well understood. Let $K^0(z,w)$ denote $K(z,w)$ and let $K^m(z,w) = \frac{\partial^m}{\partial \bar{w}^m} K(z,w)$. Let $\mathcal{K}$ denote the complex linear span of all functions $h(z)$ of z of the form $K^m(z,a)$ as a ranges over Ω and m ranges over all non-negative integers. We shall call $\mathcal{K}$ the *Bergman span* associated to Ω. If U is an open subset of Ω, let $\mathcal{K}_U$ denote the complex linear span of functions of z from $\{K(z,a) : a \in U\}$, and given a point a in Ω, let $\mathcal{K}_a$ denote the complex linear span of functions of z from $\{K^m(z,a) : m = 0,1,2,\dots\}$.

The following theorem is well known.

Theorem 1.1. *A bounded domain is a quadrature domain if and only if the constant function $h(z) \equiv 1$ belongs to the Bergman span associated to the domain.*

We will prove in this paper that, if a bounded domain is a quadrature domain, then not only are the constant functions in the Bergman span, but so are all complex polynomials. (In fact, so are derivatives of rational functions with no poles in the closure of the domain, i.e., rational functions with poles of order 2 or higher outside the closure of the domain.) Hence, the following silly theorem is true.

Theorem 1.2. *A bounded domain is a quadrature domain if and only if the complex polynomials belong to the Bergman span associated to the domain.*

In the last section of this paper, we propose that this last theorem might not be as silly in several complex variables, and should perhaps be the *definition* of quadrature domain in several complex variables. We also make a conjecture about "one point" Quadrature domains in several variables that might relate to the Jacobian Conjecture.

Given a bounded domain Ω bounded by finitely many non-intersecting C^∞-smooth Jordan curves, let $A^\infty(\Omega)$ denote the subspace of $C^\infty(\overline{\Omega})$ consisting of analytic functions. We will explain in §2 a new method to construct an analytic function $f(z)$ on Ω that is as close to the identity in $A^\infty(\Omega)$ as desired such that $f(\Omega)$ is a quadrature domain. Hence, there exists quadrature domains that are as C^∞ close to Ω as desired.

The main tools we will use to demonstrate the density of quadrature domains are the following two theorems.

Theorem 1.3. *Suppose Ω_1 and Ω_2 are bounded domains in the plane. Suppose further that $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping. Then Ω_2 is a quadrature domain if and only if f' belongs to the Bergman span associated to Ω_1.*

Avci proved half of Theorem 1.3, that f' is in the Bergman span if Ω_2 is a quadrature domain, in his unpublished Stanford PhD thesis [2]. We will prove this theorem in §3, where we will also prove that the word "biholomorphic" in the theorem can be replaced by "proper holomorphic." We will note in §3 that Theorem 1.3 can be combined with Theorem 1.2 to extend the conclusion of

Theorem 1.3 to read: Ω_2 is a quadrature domain if and only if $f'f^n$ belongs to the Bergman span associated to Ω_1 for $n = 0, 1, 2, \ldots$; this allows us to divide $f'f$ by f' when Ω_2 is a quadrature domain to see that f is a quotient of elements in the Bergman span. This reveals that f, as constructed in this paper, is one and the same with the Bergman coordinates constructed in [10].

Theorem 1.4. *Suppose that Ω is a bounded domain bounded by finitely many non-intersecting C^∞-smooth Jordan curves. Then the Bergman span $\mathcal{K}$ associated to Ω is dense in $A^\infty(\overline{\Omega})$. Furthermore, $\mathcal{K}_U$ is dense in $A^\infty(\overline{\Omega})$ for any open subset U of Ω, and $\mathcal{K}_a$ is dense in $A^\infty(\overline{\Omega})$ for any point $a \in \Omega$.*

Theorem 1.4 was proved in [3] in several complex variables. The proof there works fine in one variable, but we will give a short one variable proof in §3 using the techniques of [6, Chap. 28].

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2. Constructing quadrature domains close by

In this section, we show how Theorems 1.3 and 1.4 allow us to construct quadrature domains that are close to a given smooth domain. Suppose that Ω is a bounded domain bounded by n non-intersecting C^∞ smooth Jordan curves, $n > 1$. We first prove a key lemma. Let $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ denote the inner boundary curves of Ω and let γ_n denote the outer boundary.

Lemma 2.1. *There are $n-1$ points $a_1, a_2, \ldots, a_{n-1}$ in Ω such that the $(n-1) \times (n-1)$ matrix of periods $[\lambda_{ij}]$ given by*

$$\lambda_{ij} = \int_{\gamma_i} K(z, a_j)\ dz,$$

$i = 1, \ldots, n-1$, $j = 1, \ldots, n-1$, is non-singular.

Proof of the lemma. Suppose that $\det[\lambda_{ij}]$ is zero for all choices of points a_j. Let ω_j denote the harmonic measure functions (that are harmonic functions equal to one on γ_j and zero on the other boundary curves), and let $F'_j = 2\frac{\partial}{\partial z}\omega_j$. It is well known that the functions F'_j are in $A^\infty(\Omega)$ and that the matrix of periods $A_{ij} = \int_{\gamma_i} F'_j(z)\ dz$, $i, j = 1, \ldots, n-1$, is non-singular. Think of the quantity $\det[\lambda_{ij}]$ as being a function of $(a_1, \ldots, a_{n-1})$. The determinant is linear in the first column. We may find a function in the span of $\{K(z, a_1) : a_1 \in \Omega\}$ that is as close to the function F'_1 in the $C^\infty(\overline{\Omega})$ topology as desired. Hence, the determinant of the matrix gotten from $[\lambda_{ij}]$ by replacing the first column with $\int_{\gamma_i} F'_1(z)\ dz$ is zero for all choices of $a_2, \ldots, a_{n-1}$. This process can be repeated for successive columns to deduce the contradiction that the matrix of periods $[A_{ij}]$ has zero determinant. Hence, the lemma is proved.

We may now construct a quadrature domain close to Ω as follows. Let a_j denote the points given by Lemma 2.1. Suppose that H is a function in the

Bergman span that is close to the constant function 1 in $A^\infty(\Omega)$. The periods of such a function are small. Hence, there exist small coefficients c_j so that the periods of $H(z) - \sum_{j=1}^{n-1} c_j K(z, a_j)$ are zero. Denote this last function, which is a function in the Bergman span which is also close to 1 in $A^\infty(\Omega)$, by $F(z)$. Let f be a complex antiderivative of F where we choose the constant of integration so that $f(z)$ is close to the identity in $A^\infty(\Omega)$. We want f to be so close to the identity that f is one-to-one on Ω and $f(\Omega)$ is as close as we desire to Ω. (We may run this argument backwards from here to determine just how close we want close to mean and how small we want small to mean earlier in the paragraph.)

Now according to Theorem 1.3, $f(\Omega)$ is a quadrature domain that is C^∞ close to Ω.

The case of a simply connected domain Ω is handled the same way, but the argument is simpler because we do not need the lemma or the points a_j.

3. Proofs of the main theorems

We first prove Theorem 1.3. Suppose Ω_1 and Ω_2 are bounded domains in the plane, and suppose that $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping. Assume that f' belongs to the Bergman span associated to Ω_1; we will show that Ω_2 is a quadrature domain. Let h be a function in the Bergman space $H^2(\Omega_2)$ (which is the subspace of $L^2(\Omega_2)$ consisting of analytic functions). Let the L_2 inner product on Ω_j be denoted $\langle u, v\rangle_j$. Since $|f'(z)|^2$ is equal to the real Jacobian of the conformal mapping f viewed as a mapping from $\mathbb{R}^2$ to itself, it is easy to verify that

$$\int_{\Omega_2} h\ dA = \langle h, 1\rangle_2 = \langle f'(h \circ f), f'\rangle_1, \tag{3.1}$$

and since f' is assumed to be in the Bergman span, and since the derivatives of the Bergman kernel in the second variable yield the corresponding derivatives of an analytic function when paired with one in the inner product, this last inner product is equal to a complex linear combination of values and derivatives of $f'(h \circ f)$ at finitely many points, giving a quadrature identity that reveals Ω_2 to be a quadrature domain.

To prove the reverse implication, suppose that Ω_2 is a quadrature domain, and let g be any function in $H^2(\Omega_1)$. Let $F = f^{-1}$ and note that

$$\langle g, f'\rangle_1 = \langle F'(g \circ F), 1\rangle_2 = \int_{\Omega_2} F'(g \circ F)\ dA, \tag{3.2}$$

and this last quantity is equal to a fixed linear combination of values and derivatives of $F'(g \circ F)$ at finitely many points in Ω_2, which is equal to a fixed linear combination of values and derivatives of g at finitely many points in Ω_1. Thus, f' has the same effect as an element of the Bergman span associated to Ω_1 when paired with an element of $H^2(\Omega_1)$, and we conclude that these two functions must be one and the same.

To generalize Theorem 1.3 with the words "proper holomorphic" in place of "biholomorphic," we use the tools described in [6, Chap. 16]. If $f : \Omega_1 \to \Omega_2$ is proper holomorphic, then it has a mapping degree m and there exist m local inverses $F_1, \ldots, F_m$ to f defined locally on Ω_2 minus the image of the branch locus of f. If g is in $H^2(\Omega_1)$, then $\sum_{k=1}^m F_k'(g \circ F_k)$ is a function in $H^2(\Omega_2)$ (with removable singularities at the images of the branch points of f). Equation (3.1) becomes

$$\int_{\Omega_2} h\, dA = \langle h, 1\rangle_2 = \frac{1}{m}\langle f'(h \circ f), f'\rangle_1,$$

and Equation (3.2) becomes

$$\langle g, f'\rangle_1 = \langle \sum_{k=1}^m F_k'(g \circ F_k), 1\rangle_2,$$

and the argument proceeds exactly the same. The only point that requires extra care is where one notes that when one evaluates a derivative of $\sum_{k=1}^m F_k'(g \circ F_k)$ at a point a in Ω_2, one gets a finite linear combination of g and its derivatives evaluated at points in $f^{-1}(a)$. This fact is proved in [5].

We now turn to the proof of Theorem 1.4. (A very constructive proof of the result can be found on page 125 of [6].) Suppose that Ω is a bounded domain with C^∞-smooth boundary and let $A^{-\infty}(\Omega)$ denote the space of analytic functions on Ω that are bounded by a constant times a negative power of the distance to the boundary. It is proved in [6, Chap. 28] that $A^\infty(\Omega)$ and $A^{-\infty}(\Omega)$ are mutually dual via an extension of the usual L^2 pairing. If the Bergman span is not dense in $A^\infty(\Omega)$, then there is a non-zero element g of $A^{-\infty}(\Omega)$ that is orthogonal to the Bergman span. But when g is paired with $K^m(z, a)$, the m-th derivative of g at a is obtained (see [6, p. 125]). Hence, g and all its derivatives vanish on Ω.

The same argument works if the linear span of the smaller sets $\{K(z, a) : a \in U\}$ and $\{K^m(z, a) : m = 0, 1, 2, \ldots\}$ are used in place of the Bergman span. (In fact, any set of points or points plus derivatives at points that is a set of determinacy for analytic functions can be used in the second variable a.) We remark here that the points and derivatives in the quadrature identity associated to domains of the form $\Omega = f(\Omega_1)$ that we have constructed are easily found by our construction. If the mapping f is biholomorphic and a set $\mathcal{K}_U$ is used to approximate the function 1, then the quadrature identity is of the simple form

$$\int_\Omega h\, dA = \sum_{j=1}^N c_j h(a_j),$$

where the points a_j are fixed points from the set $f(U)$. If a set $\mathcal{K}_a$ is used to approximate the function 1, then the quadrature identity is a "one point" identity of the form

$$\int_\Omega h\, dA = \sum_{k=1}^N c_k h^{(k)}(\alpha),$$

where $\alpha = f(a)$.

Next, we prove Theorem 1.2, i.e., that the polynomials z^n belong to the Bergman span associated to a bounded quadrature domain. Suppose Ω is a quadrature domain of finite area. Aharonov and Shapiro [1] proved that the boundary of Ω is piecewise real analytic and that the Schwarz function $S(z)$ extends to be continuous up to the boundary and meromorphically to Ω (with finitely many poles), and is such that $\bar{z} = S(z)$ for z in the boundary. Let h be a holomorphic function on Ω that extends continuously to the boundary. We shall now do a calculation analogous to one in [9, p. 74] (where $n = 1$) to see that

$$\iint_\Omega z^n \; \overline{h(z)} \; dA = \frac{i}{2} \iint_\Omega z^n \; \overline{h(z)} \; dz \wedge d\bar{z}$$

$$= \frac{i}{2(n+1)} \iint_\Omega \frac{\partial}{\partial z} \left(z^{n+1} \; \overline{h(z)} \right) \; dz \wedge d\bar{z} = \frac{i}{2(n+1)} \int_{b\Omega} z^{n+1} \; \overline{h(z)} \; d\bar{z}$$

$$= \frac{i}{2(n+1)} \int_{b\Omega} \overline{S(z)^{n+1}} \; \overline{h(z)} \; d\bar{z}.$$

The Residue Theorem yields that this last integral is equal to the conjugate of a fixed linear combination of values of h and finitely many of its derivatives at the points in Ω where $S(z)$ has poles. There is an element of the Bergman span that has exactly the same effect when paired with h in the L^2 inner product. Since such functions h are dense in the Bergman space, this shows that the function z^n and the element of the Bergman span must be one and the same.

The same argument can be repeated with $R(z)$ in place of z^n where $R(z)$ is a rational function without simple poles that is analytic on a neighborhood of $\overline{\Omega}$.

Finally, we remark that when Theorems 1.2 and 1.3 are combined, it is easy to see that in the setting of Theorem 1.3, when f' is in the Bergman span associated to Ω_1 and Ω_2 is a quadrature domain, every function of the form $f' f^n$ for $n = 0, 1, 2, \ldots$ must also be in the Bergman span. (This result follows from the transformation formula for the Bergman kernels under proper holomorphic mappings, see [6, p. 68]). This reveals that $f = (f'f)/f'$ is a quotient of elements in the Bergman span, and consequently, in the case where f is biholomorphic, f can be seen to be a Bergman coordinate function as defined in [10].

4. The case of several complex variables.

It would seem natural to call a bounded domain Ω in $\mathbb{C}^n$ a quadrature domain if the integral of an L^2 holomorphic function over the domain with respect to Lebesgue volume measure (on $\mathbb{R}^{2n}$) is equal to a fixed finite linear combination of values and derivatives of the function at a fixed finite set of points in the domain. Let $K(z, w)$ denote the Bergman kernel associated to Ω, and for a multi-index α of length n, let $K^\alpha(z, w) = \frac{\partial^{|\alpha|}}{\partial \bar{w}^\alpha} K(z, w)$. Let $K^\nu(z, w)$ also denote the Bergman kernel when $\nu = (0, 0, \ldots, 0)$ is the zero multi-index. Define the Bergman span associated to Ω to be the complex linear span of the functions $K^\alpha(z, a)$ as a ranges over Ω and α ranges over all multi-indices of length n. With these definitions in place, the proof of Theorem 1.3 goes over line for line with the complex Jacobian $u = \det \left[\partial f_i / \partial z_j \right]$ in place of the the simple derivatives.

Theorem 4.1. *Suppose Ω_1 and Ω_2 are bounded domains in $\mathbb{C}^n$. Suppose further that $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping. Then Ω_2 is a quadrature domain if and only if u belongs to the Bergman span associated to Ω_1.*

The density lemma for the Bergman span was first proved in smooth bounded domains that satisfy Condition R in [3], and so it seems distinctly possible to prove the density of quadrature domains in some subclass of the category of smooth domains that satisfy Condition R. To do so, one would need to be able to show that given a function $U(z)$ in the Bergman span that is close to 1 in $A^\infty(\Omega_1)$, one can find a biholomorphic mapping f that is close to the identity such that the Jacobian of f is *equal* to U. This problem is easy to solve on certain domains. On the unit ball, for example, one can take f to be $(z_1, \dots, z_{n-1}, \mu(z))$ where $\mu(z) = \int_0^{z_n} U(z_1, \dots, z_{n-1}, \tau)\, d\tau$.

It is clear that quadrature domains in several variables do not have similar strong properties to their one variable cousins. They do not have real algebraic boundaries, or even real analytic boundaries, in general, since it is easy to construct circular domains with smooth non-real-analytic boundaries. I also doubt that they will have the same utility when it comes to Bergman coordinates (see [10]). A more interesting class of domains might be the Quadrature domains (with a capital "Q") such that the polynomials belong to the Bergman span. Complete circular domains that contain the origin have this property (see [4, 7]). In fact the polynomials are given as the linear span of $K(z, 0)$ and $K^\alpha(z, 0)$ as α ranges over all multi-indices. In this sense, one might call complete circular domains that contain the origin *one-point Quadrature domains*. (The only classical one-point quadrature domains in the plane are discs.) If Ω_2 is a complete circular domain that contains the origin and $f : \Omega_1 \to \Omega_2$ is a biholomorphic polynomial mapping with unit Jacobian and polynomial inverse, then Ω_1 is also a one point Quadrature domain in this generalized sense. I suspect that these are the only one point Quadrature domains in several variables. Understanding these objects might have a bearing on the famous Jacobian Conjecture.

Another category of domains in several complex variables that might prove to be interesting is the class of domains such that the intersection of every one-dimensional complex line through a fixed point is a quadrature domain. I like to call this generalization of circular domains "quadular domains." Since quadrature domains are so dense, maybe it is possible to approximate quite general domains by quadular domains, and maybe proper holomorphic mappings between quadular domains have similar nice properties to mappings between circular domains (see [4, 7]).

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