RECIPES FOR CLASSICAL KERNEL FUNCTIONS ASSOCIATED TO A MULTIPLY CONNECTED DOMAIN IN THE PLANE

STEVEN R. BELL*

ABSTRACT. I have recently shown that the Bergman kernel associated to a finitely connected domain in the plane is given as an explicit rational combination of finitely many basic functions of *one complex variable*. In this paper, it is proved that all the basic functions and constants in the new formula for the Bergman kernel can be evaluated using one-dimensional integrals and simple linear algebra. In fact, all integrals used in the computations are line integrals over boundary curves; at no point is an integral with respect to area measure required. From a theoretical perspective, these results lead to an understanding of the complexity of the Bergman kernel. From a practical point of view, they give an efficient method to numerically compute the Bergman kernel.

Similar results are also proved for the Szegő kernel function, the Poisson kernel, and the classical Green's function.

1. Introduction. The Bergman kernel associated to a bounded domain in the plane is frequently expressed as an infinite sum of orthonormal holomorphic functions as $K(z,w)=\sum_{j=1}^{\infty}\varphi_j(z)\overline{\varphi_j(w)}$. This formula is useful in theoretical contexts, but of little practical use. People have tried to numerically orthonormalize a set of rational functions that span a dense subspace of the Bergman space and have taken partial sums of the infinite series in hopes of approximating the Bergman kernel of a multiply connected domain. The results of such numerical nightmares are usually disappointing. In this paper, I shall express the Bergman kernel in terms of *finitely many* elementary functions that are easy to compute. I shall also give a recipe to compute the functions and the coefficients appearing in the formula. All the terms can be computed using "one-dimensional" objects. The results I shall describe are based on a new formula for the Szegő kernel given in [4].

Suppose that Ω is a bounded finitely connected domain in the plane with C^∞ smooth boundary, i.e., that the boundary $b\Omega$ of Ω is given by finitely many non-intersecting C^∞ simple closed curves. The Bergman kernel $K(z,w)$ and the Szegő kernel $S(z,w)$ associated to such a domain are both known to extend to be in the space $C^\infty((\overline{\Omega}\times\overline{\Omega})-\mathcal{D})$ where $\mathcal{D}$ denotes the boundary diagonal $\{(z,z): z\in b\Omega\}$. Our problem is to determine a method to compute $K(z,w)$ at any given pair of points (z,w) in $(\overline{\Omega}\times\overline{\Omega})-\mathcal{D}$. We shall see that, once the *boundary values* of finitely many basic functions of one variable have been determined, the kernels become known at *all* points (z,w). Furthermore, the basic functions which comprise the kernel functions are all solutions to explicit Kerzman-Stein integral equations, and

1991 *Mathematics Subject Classification.* 32H10.

Key words and phrases. Szegő kernel, Ahlfors Map, Poisson kernel.

*Research supported by NSF grant DMS-0302512

as such, are easy to compute. All elements of the kernel functions may be computed by means of simple linear algebra and one dimensional integrals and one dimensional integral equations. At no point is it necessary to evaluate an integral with respect to two-dimensional area measure. We shall prove analogous results for the Poisson kernel and the gradient of the Green's function.

In this paper, we require our domains to have C^∞ smooth boundaries. To study the kernels on a non-smooth finitely connected domain such that no boundary component reduces to a point, one could use a classical construction to map the given domain onto a smooth domain. Such a map is built up of a sequence of Riemann mapping functions associated to simply connected domains and inversions. To compute the kernel functions on the more general domain, one could use one of the many excellent methods for computing Riemann maps of simply connected domains to build such a map and associated smooth domain and then use the map to pull back the results we describe in this paper to the original non-smooth domain.

Although the words, "numerical method," appear in this paper, this is not a paper on numerical analysis; no examples of numerical computations are given. However, the results of this paper should be interesting to numerical analysts.

2. The Ahlfors map and zeroes of the Szegő kernel. Before we can state our main theorems, we must recall some facts about the Szegő kernel function.

If Ω is a bounded n-connected domain in the plane with C^∞ smooth boundary, let γ_j, $j = 1, \dots, n$, denote the n non-intersecting C^∞ simple closed curves which define the boundary of Ω, and suppose that γ_j is parameterized in the standard sense by $z_j(t)$, $0 \le t \le 1$. We shall use the convention that γ_n denotes the *outer boundary curve* of Ω. Let $T(z)$ be the C^∞ function defined on $b\Omega$ such that $T(z)$ is the complex number representing the unit tangent vector at $z \in b\Omega$ pointing in the direction of the standard orientation. This complex unit tangent vector function is characterized by the equation $T(z_j(t)) = z_j'(t)/|z_j'(t)|$. (We remark that the notion of the "standard sense" mentioned above translates to the condition that $-iT(z)$ is a complex number pointing in the direction of the outward pointing normal vector at $z \in b\Omega$.)

To fix notation, we state that $A^\infty(\Omega)$ is the space of holomorphic functions on Ω that are in $C^\infty(\overline{\Omega})$, $L^2(\Omega)$ is the space of complex valued functions on Ω that are square integrable with respect to Lebesgue area measure dA, $L^2(b\Omega)$ is the space of complex valued functions on $b\Omega$ that are square integrable with respect to arc length measure ds, $H^2(\Omega)$ is the Bergman space of holomorphic functions on Ω that are in $L^2(\Omega)$, $H^2(b\Omega)$ is the Hardy space of functions in $L^2(b\Omega)$ that are the L^2 boundary values of holomorphic functions on Ω. The inner products associated to $L^2(\Omega)$ and $L^2(b\Omega)$ are

$$\langle u, v\rangle_\Omega = \iint_\Omega u\ \bar{v}\ dA \quad \text{and} \quad \langle u, v\rangle_{b\Omega} = \int_{b\Omega} u\ \bar{v}\ ds,$$

respectively.

For each fixed point $a \in \Omega$, the Szegő kernel $S(z, a)$, as a function of z, extends to the boundary to be a function in $A^\infty(\Omega)$. (An even stronger smoothness property is mentioned in the introduction.) Furthermore, $S(z, a)$ has exactly $(n - 1)$ zeroes in Ω (counting multiplicities) and does not vanish at any points z in the boundary of Ω. The *Garabedian kernel* $L(z, a)$ is a kernel related to the Szegő kernel via the identity

$$\frac{1}{i}L(z,a)T(z) = S(a,z) \qquad \text{for } z \in b\Omega \text{ and } a \in \Omega. \tag{2.1}$$

For fixed $a \in \Omega$, the kernel $L(z,a)$ is a holomorphic function of z on $\Omega - \{a\}$ with a simple pole at a with residue $1/(2\pi)$. Furthermore, as a function of z, $L(z,a)$ extends to the boundary and is in the space $C^\infty(\overline{\Omega} - \{a\})$. In fact, $L(z,a)$ extends to be in $C^\infty((\overline{\Omega} \times \overline{\Omega}) - \{(z,z) : z \in \overline{\Omega}\})$. Also, $L(z,a)$ is non-zero for all (z,a) in $\overline{\Omega} \times \Omega$ with $z \neq a$.

The kernel $S(z,w)$ is holomorphic in z and antiholomorphic in w on $\Omega \times \Omega$, and $L(z,w)$ is holomorphic in both variables for $z, w \in \Omega$, $z \neq w$. We shall need to know that $S(z,z)$ is real and positive for each $z \in \Omega$, and we shall need to use the basic identities $S(z,w) = \overline{S(w,z)}$ and $L(z,w) = -L(w,z)$. The Szegő kernel reproduces holomorphic functions in the sense that

$$h(a) = \langle h, S(\cdot, a)\rangle_{b\Omega}$$

for all $h \in H^2(b\Omega)$ and $a \in \Omega$.

Given a point $a \in \Omega$, the Ahlfors map f_a associated to the pair (Ω, a) is a proper holomorphic mapping of Ω onto the unit disc. It is an n-to-one mapping (counting multiplicities), it extends to be in $A^\infty(\Omega)$, and it maps each boundary curve γ_j one-to-one onto the unit circle. Furthermore, $f_a(a) = 0$, and f_a is the unique function mapping Ω into the unit disc maximizing the quantity $|f_a'(a)|$ with $f_a'(a) > 0$. The Ahlfors map is related to the Szegő kernel and Garabedian kernel via

$$f_a(z) = \frac{S(z,a)}{L(z,a)}. \tag{2.2}$$

Also, $f_a'(a) = 2\pi S(a,a) \neq 0$. Because f_a is n-to-one, f_a has n zeroes. The simple pole of $L(z,a)$ at a accounts for the simple zero of f_a at a. The other $n-1$ zeroes of f_a are given by the $(n-1)$ zeroes of $S(z,a)$ in $\Omega - \{a\}$. Let $a_1, a_2, \ldots, a_{n-1}$ denote these $n-1$ zeroes (counted with multiplicity). I proved in [3] (see also [1, page 105]) that, if a is close to one of the boundary curves, the zeroes $a_1, \ldots, a_{n-1}$ become distinct simple zeroes. It follows from this result that, for all but at most finitely many points $a \in \Omega$, $S(z,a)$ has $n-1$ distinct simple zeroes in Ω as a function of z.

3. A formula for the Szegő kernel. The zeroes of the Szegő kernel give rise to a particularly nice basis for the Hardy space of an n-connected domain with C^∞ smooth boundary. We shall use the notation that we set up in the preceding section. We assume that $a \in \Omega$ is a fixed point in Ω that has been chosen so that the $n-1$ zeroes, $a_1, \ldots, a_{n-1}$, of $S(z,a)$ are distinct and simple. We shall let a_0 denote a and we shall use the shorthand notation $f(z)$ for the Ahlfors map $f_a(z)$.

It was shown in [4] that the set of functions $S(z,a_i)f(z)^k$, where $0 \leq i \leq n-1$ and $k \geq 0$, forms a basis for the Hardy space $H^2(b\Omega)$ and that this basis is easy to orthonormalize because it is already "nearly orthogonal." The formula in the following theorem was obtained in [4] by writing the Szegő kernel in terms of the orthogonal basis obtained from the functions listed above.

Theorem 3.1. *The Szegő kernel is given by*

$$S(z,w)=\frac{1}{1-f(z)\overline{f(w)}}\left(c_0S(z,a)\overline{S(w,a)}+\sum_{i,j=1}^{n-1}c_{ij}S(z,a_i)\,\overline{S(w,a_j)}\right)\tag{3.1}$$

for all (z,w) in Ω. Here, $f(z)$ denotes the Ahlfors map $f_a(z)$. Furthermore, the $(n-1)\times(n-1)$ matrix $\mathcal{M}=[S(a_j,a_k)]_{j=1,\ldots,n-1}^{k=1,\ldots,n-1}$ is non-singular. The coefficient c_0 in the formula is given by $c_0=1/S(a,a)$, and the coefficients c_{ij} are given as the coefficients of the inverse matrix to $\mathcal{M}$.

4. Recipes for the Szegő and Garabedian kernels. Formula (3.1) reveals that the Szegő kernel associated to an n-connected domain is composed of the $n+1$ functions, $S(z,a)$, $S(z,a_1)$, $S(z,a_2),\ldots,S(z,a_{n-1})$, and $f(z)$ (or $L(z,a)$ because $f(z)=S(z,a)/L(z,a)$). If one knows the boundary values of these $n+1$ functions, then the Szegő kernel may be evaluated at any pair of points (z,w) in $\Omega\times\Omega$ by applying the Cauchy integral formula twice, once to evaluate the functions on the right hand side of (3.1) at z, and once to evaluate the functions at w. (Actually, to compute the Garabedian kernel at an interior point, one would use the Cauchy integral formula to evaluate $L(z,a)-(2\pi)^{-1}/(z-a)$ and then add back the singular part.) In this section, we show how much effort is required to numerically compute the boundary values of the $n+1$ functions that comprise $S(z,w)$.

Kerzman and Stein [7] discovered an effective method for computing the Szegő kernel (see also [1,2,6,8,10]). They proved that the function $S_a(z)=S(z,a)$ is the solution to a Fredholm integral equation of the second kind given by

$$S_a(z)-\int_{w\in b\Omega}A(z,w)S_a(w)\;ds=\mathcal{C}_a(z),$$

where $A(z,w)$ is the Kerzman-Stein kernel and $\mathcal{C}_a(z)$ is the Cauchy kernel. To be precise,

$$A(z,w)=\frac{1}{2\pi i}\left(\frac{T(w)}{w-z}-\frac{\overline{T(z)}}{\bar{w}-\bar{z}}\right)$$

if $z,w\in b\Omega$, $z\neq w$, and $A(z,w)=0$ if $z=w$, and

$$\mathcal{C}_a(z)=\frac{1}{2\pi i}\frac{\overline{T(z)}}{\bar{a}-\bar{z}}.$$

The Kerzman-Stein kernel is skew-hermitian and, in spite of the apparent singularity at $z=w$ in the formula above, it is in $C^\infty(b\Omega\times b\Omega)$. (Kerzman and Stein discovered that the apparent singularities in the formula for $A(z,w)$ exactly cancel.) The Cauchy kernel is in $C^\infty(b\Omega)$. It follows from standard theory that this integral equation has a unique C^∞ smooth solution. (See Kerzman and Trummer [8] and [2,6] for descriptions of convenient ways to write and to solve this integral equation.)

The Kerzman-Stein equation produces the boundary values of $S(z,a)$. The boundary values of the Garabedian kernel $L(z,a)$ can be computed via identity (2.1), and the boundary values of the Ahlfors map $f_a(z)$ can now be gotten from (2.2). The remaining functions in (3.1) can be computed via the Kerzman-Stein integral equation once the zeroes $a_1, \ldots, a_{n-1}$ have been located. Since $S(z,a)$ does not vanish on $b\Omega$, we may use the residue theorem to compute the symmetric sums

$$\sum_{j=1}^{n-1} a_j^k = \frac{1}{2\pi i} \int_{b\Omega} \frac{z^k \left(\frac{\partial}{\partial z} S(z,a)\right)}{S(z,a)} \, dz$$

for $k = 1, \ldots, n-1$. Newton's identities can now be used to compute the elementary symmetric functions of $a_1, \ldots, a_{n-1}$, and hence, the coefficients of the polynomial $\prod_{j=1}^{n-1}(\zeta - a_j)$ are determined. We have therefore shown that the problem of locating the zeroes of $S(z,a)$ is equivalent to computing $n-1$ line integrals and finding the roots of a polynomial of degree $n-1$.

An interesting consequence of the results of this section is that, once the boundary values of the $n+1$ basic functions have been computed, it becomes as easy to evaluate Szegő projections at a point $z \in \Omega$ as it is to find Cauchy transforms at that point.

We remark here that it was shown in [4] that the Garabedian kernel can be expressed in terms of finitely many functions as

$$L(z,w) = \frac{f_a(w)}{f_a(z) - f_a(w)} \left(c_0 S(z,a) L(w,a) + \sum_{i,j=1}^{n-1} \bar{c}_{ij} S(z,a_i) L(w,a_j) \right) \tag{4.1}$$

for $z, w \in \Omega$, $z \neq w$. Note that the constants c_0 and c_{ij} are the same as the constants in (3.1).

5. A recipe for the Bergman kernel. In this section, we shall give a procedure for computing the Bergman kernel. The Bergman kernel $K(z,w)$ is related to the Szegő kernel via the identity

$$K(z,w) = 4\pi S(z,w)^2 + \sum_{i,j=1}^{n-1} k_{ij} F_i'(z) \overline{F_j'(w)},$$

where the functions $F_i'(z)$ are classical functions of potential theory described as follows. The harmonic function ω_j which solves the Dirichlet problem on Ω with boundary data equal to one on the boundary curve γ_j and zero on γ_k if $k \neq j$ has a multivalued harmonic conjugate. The function $F_j'(z)$ is a globally defined single valued holomorphic function on Ω which is locally defined as the derivative of $\omega_j + iv$ where v is a local harmonic conjugate for ω_j. The Cauchy-Riemann equations reveal that $F_j'(z) = 2(\partial \omega_j / \partial z)$.

Let $\mathcal{F}'$ denote the vector space of functions given by the complex linear span of the set of functions $\{F_j'(z) : j = 1, \ldots, n-1\}$. It is a classical fact that $\mathcal{F}'$ is $n-1$ dimensional. Notice that $S(z,a_i)L(z,a)$ is in $A^\infty(\Omega)$ because the pole of $L(z,a)$ at $z = a$ is cancelled by the zero of $S(z,a_i)$ at $z = a$. A theorem due to Schiffer [9] (see also [1,3]) states that the $n-1$ functions $S(z,a_i)L(z,a)$, $i = 1, \ldots, n-1$ form a basis for $\mathcal{F}'$. We may now write

$$K(z,w) = 4\pi S(z,w)^2 + \sum_{i,j=1}^{n-1} \mu_{ij} S(z,a_i) L(z,a) \overline{S(w,a_j) L(w,a)}, \tag{5.1}$$

It is shown in [1, page 80] that the linear span of $\{S(z,a_i)L(z,a) : i = 1, \dots, n-1\}$ is the same as the linear span of $\{L(z,a_i)S(z,a) : i = 1, \dots, n-1\}$. Hence, formula (5.1) can be rewritten in the form

$$K(z,w) = 4\pi S(z,w)^2 + \sum_{i,j=1}^{n-1} \lambda_{ij} L(z,a_i)S(z,a)\,\overline{L(w,a_j)S(w,a)}. \tag{5.2}$$

This last formula, together with (3.1), gives us an obvious strategy for computing the Bergman kernel.

The difficulty of computing the functions appearing in (5.2) has been discussed. We now describe a method for computing the coefficients λ_{ij}. We shall write $K_w(z)$ in place of $K(z,w)$ and $S_w(z)$ in place of $S(z,w)$ to emphasize that we are thinking of w as being fixed and we are viewing these kernels as functions of z. Let us also write $\mathcal{L}_i(z) = L(z,a_i)S(z,a)$. Thus, formula (5.2) may be rewritten as

$$K_w - 4\pi S_w^2 = \sum_{i,j=1}^{n-1} \lambda_{ij} \overline{\mathcal{L}_j(w)}\,\mathcal{L}_i. \tag{5.3}$$

The complement of $\overline{\Omega}$ in $\mathbb{C}$ is the union of domains D_j, $j = 1, \dots, n$, where the boundary of D_j is described by the boundary curve γ_j. Recall that γ_n denotes the outer boundary curve of Ω. For $j = 1, \dots, n-1$, pick a point b_j in D_j. We now consider the effect of integrating (5.3) against the function $1/(z-b_k)$. Notice that

$$\langle (z-b_k)^{-1}, K_w\rangle_\Omega = \frac{1}{w-b_k}$$

because the Bergman kernel reproduces holomorphic functions. Since $1/(z-b_k) = (\partial/\partial z)\ln|z-b_k|^2$, we may use the complex Green's identity to compute

$$\langle (z-b_k)^{-1}, S_w^2\rangle_\Omega = \iint_{z\in\Omega} (\partial/\partial z)\ln|z-b_k|^2\ \overline{S_w(z)^2}\ (\frac{i}{2}dz\wedge d\bar z) =$$
$$i\int_{z\in b\Omega} \ln|z-b_k|\ \overline{S_w(z)^2}\ d\bar z.$$

Define numbers $A_{ik} = \langle (z-b_k)^{-1}, \mathcal{L}_i\rangle_\Omega$. We may use the complex Green's identity again to obtain

$$A_{ik} = i\int_{z\in b\Omega} \ln|z-b_k|\ \overline{\mathcal{L}_i(z)}\ d\bar z.$$

We now collect the integrals above as dictated by (5.3), and we set $w = a_m$, $m = 1, \dots, n-1$ to obtain the system,

$$\frac{1}{a_m - b_k} - 4\pi i\int_{z\in b\Omega} \ln|z-b_k|\ S(a_m,z)^2\ d\bar z = \sum_{i,j=1}^{n-1} \lambda_{ij} A_{ik}\overline{\mathcal{L}_j(a_m)}.$$

To show that this system determines the numbers λ_{ij}, we need only check that the matrices given by $\mathbb{A} = [A_{ik}]$ and $\mathbb{L} = [\mathcal{L}_j(a_m)]$ are invertible. That $\mathbb{L}$ is invertible is obvious because

$$L(w,a_j)S(w,a) = \begin{cases} 0, & \text{if } w = a_m,\ m \neq j \\ \frac{1}{2\pi}\frac{\partial}{\partial z}S(a_j,a), & \text{if } w = a_j, \end{cases}$$

and $(\partial/\partial z)S(a_j, a) \neq 0$ because a has been chosen so that the zeroes of $S(z, a)$ are simple zeroes. To show that $\mathbb{A}$ is invertible, we shall need to use an argument from [3]. If $G = \sum_{k=1}^{n-1} c_k F_k'$, then $G = 2(\partial/\partial z)\omega$ where $\omega = \left(\sum_{k=1}^{n-1} c_k \omega_k\right)$. It is proved in [3, page 12] that the constants c_k are given by the integral

$$c_k = -\frac{1}{2\pi i} \int_{z \in b\Omega} \ln|z - b_k| G(z) dz,$$

where b_k is the fixed point chosen from D_k. Notice that c_k is the value of ω on γ_k. Suppose $\mathbb{A}$ is not invertible. Then there would exist constants σ_i, not all zero, such that

$$\sum_{i=1}^{n-1} A_{ik} \bar{\sigma}_i = 0$$

for each k, and the complex conjugate of this equality yields that

$$\int_{z \in b\Omega} \ln|z - b_k| \left(\sum_{i=1}^{n-1} \sigma_i \mathcal{L}_i(z) \right) \, dz = 0. \tag{5.4}$$

Let $G = \sum_{i=1}^{n-1} \sigma_i \mathcal{L}_i$. Since G is in the linear span of $\{F_j'\}_{j=1}^{n-1}$, condition (5.4) and the fact from [3] imply that $G = 2(\partial/\partial z)\omega$ where ω is a harmonic function on Ω that vanishes on each boundary curve of Ω, i.e., that $G \equiv 0$. Now each σ_k must be zero because the functions $\mathcal{L}_i$ are linearly independent. This contradiction yields that the matrix $\mathbb{A}$ must be non-singular and the proof is finished.

The method described above for computing the Bergman kernel presumes that the boundary of the domain is smooth. If a finitely connected domain does not have smooth boundary, and if none of its boundary components are points, it can be mapped conformally onto a domain whose boundary is smooth (via a sequence of Riemann maps of simply connected domains and inversions). The transformation formula for the Bergman kernels under biholomorphic mappings can then be used to determine the functions appearing in the formula of the following theorem.

Theorem 5.1. *Suppose Ω is a finitely connected domain such that no boundary component of Ω is a point. Let $f(z)$ denote an Ahlfors map of Ω onto the unit disc. The Bergman kernel $K(z, w)$ associated to Ω is a function of the form*

$$K(z, w) = \frac{1}{(1 - f(z)\overline{f(w)})^2} \left(\sum_{j,k=1}^{n(n+1)/2} C_{jk} H_j(z) \overline{H_k(w)} \right) + \sum_{i,j=1}^{n-1} \lambda_{ij} G_i(z) \overline{G_j(w)}$$

where the functions H_j and G_j are functions of one variable in the Bergman space.

7. The Poisson kernel. I showed in [3] how the Szegő projection can be used to solve the Dirichlet problem, and I showed in [4] how this method of solving the Dirichlet problem gives rise to a formula for the Poisson kernel of a bounded domain with smooth boundary in terms of the Szegő kernel.

Let Ω denote an n-connected domain with C^∞ smooth boundary. We shall use the same notation for describing Ω that we used above, and as above, we select a point $a \in \Omega$ such that the zeroes $a_1, \ldots, a_{n-1}$ of $S(z, a)$ are all distinct and simple. Let $S_a(z) = S(z, a)$ and $L_a(z) = L(z, a)$.

The Szegő projection P associated to Ω is the orthogonal projection of $L^2(b\Omega)$ onto the Hardy space $H^2(b\Omega)$. The Szegő kernel is the kernel for the Szegő projection in the sense that, given a function $u \in L^2(b\Omega)$, the projection Pu is identified with a holomorphic function $h = Pu$ defined on Ω whose L^2 boundary values are equal to Pu, and

$$(Pu)(z) = \int_{w \in b\Omega} S(z, w)\, u(w)\; ds.$$

The Szegő projection maps $C^\infty(b\Omega)$ into $C^\infty(\overline{\Omega})$ (see [1,5] for proofs of these basic facts).

Recall that the set of functions $\{L(z, a_k)S(z, a)\}_{k=1}^{n-1}$ spans the same linear space as the set of functions $\{F'_k\}_{k=1}^{n-1}$. Define an $(n-1) \times (n-1)$ matrix of periods via

$$\mathcal{A}_{jk} = -i \int_{\gamma_j} L(z, a_k)S(z, a)\, dz,$$

for $j = 1, \ldots, n-1$. Because the matrix of periods of F'_k is non-singular, so is $[\mathcal{A}_{jk}]$. Let $[B_{jk}]$ denote the inverse of $[\mathcal{A}_{jk}]$. The following theorem was proved in [4].

Theorem 7.1. *The Poisson extension $\mathcal{E}u$ of u to Ω is given by an integral*

$$(\mathcal{E}u)(z) = \int_{w \in b\Omega} p(z, w)\, u(w)\; ds,$$

where $p(z, w)$ is the Poisson kernel and is given by

$$p(z, w) = 2Re \left[\frac{S(z, w)L(w, a)}{L(z, a)} - \sum_{j=1}^{n-1} H_j(z)\mu_j(w) \right] + \frac{|S(w, a)|^2}{S(a, a)} + \sum_{j=1}^{n-1} (\omega_j(z) - \lambda_j)\mu_j(w). \tag{7.1}$$

where the $H_j(z)$ are holomorphic functions in $A^\infty(\Omega)$ given by

$$H_j(z) = \int_{\zeta \in \gamma_j} \frac{S(z, \zeta)L(\zeta, a)}{L(z, a)}\; ds,$$

the λ_j are constants given by

$$\lambda_j = \int_{\zeta \in \gamma_j} \frac{|S(\zeta, a)|^2}{S(a, a)}\; ds,$$

and where the $\mu_j(w)$ are real valued C^∞ function on $b\Omega$ given by

$$\mu_j(w) = \sum_{k=1}^{n-1} B_{jk} S(a_k, w)S(w, a).$$

It is also shown in [4] that the functions $\mu_j(w)$ are linearly independent functions that do not depend on a, but we shall not need these facts here.

Actually, it was not shown in [4] that the functions H_j are in in $A^\infty(\Omega)$, so we shall prove this here. The function $1/L(z,a)$ has a simple zero at a due to the pole of $L(z,a)$ at a. Furthermore, because $L(z,a)$ is non-vanishing on $\overline{\Omega}-\{a\}$ and is C^∞ smooth up to the boundary, it follows that $1/L(z,a)$ is in $A^\infty(\Omega)$. The formula defining H_j shows that

$$H_j(z)=\frac{1}{L(z,a)}(P\chi_j)(z)$$

where $\chi_j(\zeta)$ is a function in $C^\infty(b\Omega)$ that is equal to zero on γ_k if $k\neq j$, and equal to the C^∞ function $L(\zeta,a)$ for $\zeta\in\gamma_j$. Since the Szegő projection preserves functions in $C^\infty(b\Omega)$, we see that H_j is in $A^\infty(\Omega)$. We also remark here that the task of evaluating $H_j(z)$ at a point z in Ω is no more expensive than evaluating $S(z,a)$ because the boundary values of all the basic functions are presumed to be known, and it is just as easy to plug these functions into the formula defining H_j as it would be to use the Cauchy integral formula to evaluate $S(z,a)$.

The appearance of the term $\omega_j(z)$ in the formula above might seem disappointing because

$$\omega_j(z)=\frac{1}{2\pi i}\iint_{w\in\Omega}\frac{F_j'(w)}{w-z}\,dw\wedge d\bar{w}$$

and it looks like we will be forced to compute some double integrals if we want to compute the Poisson kernel. (I gave a method to compute F_j' in [3, page 12].) However, it is possible to compute the functions $\omega_j(z)$ without having to compute a single double integral. Indeed, if we apply the Poisson integral formula using (7.1) to the functions

$$\ln|w-b_k|,\qquad k=1,\dots,n-1,$$

where b_k is a fixed point in the domain D_k (defined to be the inside of γ_k, see §5), we obtain

$$\ln|z-b_k|=2\mathrm{Re}\ [h_k(z)]+\sum_{j=1}^{n-1}c_{jk}\omega_j(z)$$

where the functions $h_k(z)$ are explicit holomorphic functions and the c_{jk} are constants. Since no non-trivial linear combination of the functions $\ln|z-b_k|$ can be expressed as the real part of a holomorphic function on Ω, it follows that the matrix $[c_{jk}]$ is non-singular, and hence, that $\omega_j(z)$ can be expressed as the real part of an explicit holomorphic function plus a linear combination of the $\ln|z-b_k|$. It follows that the Poisson kernel, like the Bergman and Szegő kernel, can be computed by forming simple combinations of functions of one complex variable, and without using double integrals.

If we set $z=a$ in (7.1), we obtain the formula,

$$p(a,w)=\frac{|S(w,a)|^2}{S(a,a)}+\sum_{j=1}^{n-1}(\omega_j(a)-\lambda_j(a))\mu_j(w). \tag{7.2}$$

where $\lambda_j(a)$ is now a function defined via

$$\lambda_j(a)=\int_{\zeta\in\gamma_j}\frac{|S(\zeta,a)|^2}{S(a,a)}\,ds.$$

This formula relates the Poisson kernel to the Poisson-Szegő kernel $|S(w,a)|^2/S(a,a)$ in a multiply connected domain. (These two kernels are equal in simply connected domains.) We have shown that (7.2) is valid when $a \in \Omega$ is a point where the $n-1$ zeroes of $S(z,a)$ as a function of z are all simple zeroes. However, it is clear that (7.2) is valid for all $a \in \Omega$ because the functions in it are all continuous, $S(a,a) > 0$, and the set of points a where the zeroes of $S(z,a)$ are not simple is finite. Formula (7.2) looks like it offers a better approach for computing the Poisson kernel than (7.1), however, the functions $\lambda_j(a)$ have the undesirable feature that they are given by a quotient

$$\frac{1}{S(a,a)} \int_{\zeta \in \gamma_j} |S(\zeta,a)|^2 \, ds$$

where both $S(a,a)$ and the integral tend to infinity as a approaches γ_j in such a way that the quotient is C^∞ smooth up to γ_j.

We remark that it is shown in [4] that the functions $\lambda_j(a)$ are non-harmonic functions of a in $C^\infty(\overline{\Omega})$ that have the same boundary behavior as the harmonic measure functions ω_j, i.e., $\lambda_j(a) \to 1$ as $a \to \gamma_j$ and $\lambda_j(a) \to 0$ as $a \to \gamma_k$, $k \neq j$.

8. The Green's function. The gradient of the Green's function is closely related to the Poisson kernel. This connection was explored in [4], and in light of (7.1), the following theorem was proved.

Theorem 8.1. *Suppose that Ω is a bounded n-connected domain with C^∞ smooth boundary. The gradient of the Green's function associated to Ω can be read off from the formula*

$$\frac{\partial G}{\partial \bar{w}}(z,w) = \pi \left(\frac{S(z,w)\,\overline{L(w,z)}}{S(z,z)} - i \sum_{j=1}^{n-1} (\omega_j(z) - \lambda_j(z))\overline{g_j(w)} \right)$$

for all $z, w \in \Omega$, $z \neq w$ where where $g_j(w) = -i\sum_{k=1}^{n-1} \overline{B_{jk}} S(w,a_k)L(w,a)$ and

$$\lambda_j(z) = \int_{\zeta \in \gamma_j} \frac{|S(\zeta,z)|^2}{S(z,z)} \, ds.$$

Given points z and a in Ω, $z \neq a$, let σ_z be a curve in in $\overline{\Omega}$ that starts at a point ζ_0 on the boundary, ends at z, and avoids a. The Green's function could be computed by using the formula in (8.1) and

$$G(z,a) = 2\mathrm{Re} \left(\int_{\sigma_z} \frac{\partial G}{\partial \bar{w}}(w,a) \, d\bar{w} \right),$$

but I cannot recommend this method as I do the others.

I close by mentioning that I would like to know that the Green's function is finitely complex in the same sense that the other kernels are, but I have not been able to get my hands as directly on the Green's function as I have with the other kernels.

References

1. S. Bell, *The Cauchy transform, potential theory, and conformal mapping*, CRC Press, Boca Raton, 1992.
2. ———, *Numerical computation of the Ahlfors map of a multiply connected planar domain*, J. Math. Anal. and Appl. **120** (1986), 211–217.
3. ———, *The Szegő projection and the classical objects of potential theory in the plane*, Duke Math. J. **64** (1991), 1–26.
4. ———, *Complexity of the classical kernels of potential theory*, to appear..
5. S. Bergman, *The kernel function and conformal mapping*, Math. Surveys 5, AMS, Providence, 1950.
6. P. Henrici, *Applied and computational complex analysis,* Vol. 3, John Wiley, New York, 1986.
7. N. Kerzman and E. M. Stein, *The Cauchy kernel, the Szegő kernel, and the Riemann mapping function*, Math. Ann. **236** (1978), 85–93.
8. N. Kerzman and M. Trummer, *Numerical conformal mapping via the Szegő kernel*, Numerical conformal mapping, L. N. Trefethen, ed., North Holland, Amsterdam, 1986, pp. 111–123.
9. M. Schiffer, *Various types of orthogonalization*, Duke Math. J. **17** (1950), 329–366.
10. M. Trummer, *An efficient implementation of a conformal mapping method based on the Szegő kernel*, SIAM J. of Numer. Anal. **23** (1986), 853–872.

Mathematics Department, Purdue University, West Lafayette, IN 47907 USA
E-mail address: `bell@math.purdue.edu`