A RIEMANN SURFACE ATTACHED TO DOMAINS IN THE PLANE AND COMPLEXITY IN POTENTIAL THEORY

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Abstract. We prove that if either of the Bergman or Szegő kernel functions associated to a multiply connected domain Ω in the plane is an algebraic function, then there exists a compact Riemann surface $\mathcal{R}$ such that Ω is a domain in $\mathcal{R}$ and such that a long list of classical domain functions associated to Ω extend to $\mathcal{R}$ as single valued meromorphic functions. Because the field of meromorphic functions on a compact Riemann surface is generated by just two functions, it follows that all the classical domain functions associated to Ω are rational combinations of just two functions of one variable. This result gives rise to some very interesting questions in potential theory and conformal mapping. We discuss how it may yield information about complexity in potential theory in a much more general context.

1. Introduction. On a simply connected domain in the plane, the Riemann mapping function can be used to pull back the classical kernel functions from the unit disc and questions about the complexity and algebraic properties of the kernel functions become downright transparent. I have been spurred on to answer similar questions in the multiply connected case for reasons stemming from results in several complex variables. While studying the boundary behavior of holomorphic mappings in several complex variables ([6-9]), I discovered that the Bergman projection, and consequently the Bergman kernel, transforms under holomorphic mappings which are merely *proper* (in the topological sense of the word — for example, finite Blaschke products are proper holomorphic maps of the unit disc to itself). This result leads one to suspect that it might be possible to pull back the classical kernel functions from the disc via proper holomorphic mappings. Since any finitely connected domain in the plane such that no boundary component is a point can be mapped properly onto the unit disc by an Ahlfors map, there arose the hope to find formulas for the kernel functions on such domains analogous to the formulas in the simply connected case, but with the Ahlfors map taking over the role of the Riemann map. For example, the domain $A(r) := \{z \in \mathbb{C} : |z+z^{-1}| < r\}$ is a two-connected domain with smooth real analytic boundary curves if $r > 2$. The mapping $f(z) = (1/r)(z + z^{-1})$ is a proper holomorphic map from Ω onto the unit disc that is a 2-sheeted branched covering map. If the transformation formula for the Bergman kernels under proper maps could be used to pull back the simple rational kernel functions on the disc up to $A(r)$, then it ought to follow that the kernel functions for $A(r)$ are algebraic. I have recently proved in [5] that the kernel functions associated to $A(r)$ are indeed algebraic, however, the connection between

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the kernels on these domains and the kernels on the unit disc are much more tenuous than direct pull backs. In this paper, I shall enlist the help of a Riemann surface that I construct by means of a Szegő kernel identity that I discovered in [4] to reveal further connections between kernel functions on multiply connected domains and the proper holomorphic maps of these domains onto the unit disc.

The starting point for this research is the following theorem that I proved in [5, Theorem 4.4].

Theorem 1.1. *Suppose Ω is a finitely connected domain in the plane such that no boundary component is a point. The following conditions are equivalent.*

(1) *The Bergman kernel associated to Ω is algebraic.*
(2) *The Szegő kernel associated to Ω is algebraic.*
(3) *There exists a single proper holomorphic mapping of Ω onto the unit disc which is algebraic.*
(4) *Every proper holomorphic mapping of Ω onto the unit disc is algebraic.*

This theorem will allow us to begin the construction of a Riemann surface in case one of the conditions of the theorem are met. Indeed, if the Szegő kernel is algebraic, then we shall be able to realize Ω as a subdomain of a certain compact Riemann surface to which all the kernels and mappings associated to Ω can be extended as single valued meromorphic functions. Furthermore, the complement of Ω in $\mathcal{R}$ is connected (see Theorem 3.1 and its proof in §3 for a more precise description of the Riemann surface). A key element in the construction of this Riemann surface is a formula from [4] which relates the Szegő kernel to an Ahlfors map and finitely many other functions of one complex variable. Since the field of meromorphic functions on a compact Riemann surface is generated by two elements (see Farkas and Kra [12, page 249]), we shall be able to show that the kernel functions and proper maps associated to the domain are generated by just *two functions of one complex variable.* This shall mean that the kernel functions associated to a multiply connected domain with algebraic kernel functions are just as simple as the kernel functions associated to a simply connected domain. To be precise, we prove

Theorem 1.2. *Suppose Ω is a finitely connected domain in the plane such that no boundary component is a point. If one of the conditions,*

(1) *the Bergman kernel associated to Ω is algebraic, or*
(2) *the Szegő kernel associated to Ω is algebraic, or*
(3) *there exists a single proper holomorphic mapping of Ω onto the unit disc which is algebraic,*

is met, then there exist two holomorphic functions $f(z)$ and $g(z)$ on Ω such that the Bergman kernel $K(z,w)$ and the Szegő kernel $S(z,w)$ are given as rational combinations of the four functions $f(z)$, $g(z)$, and the complex conjugates of $f(w)$ and $g(w)$. Furthermore, every proper holomorphic mapping of Ω onto the unit disk is a rational combination of f and g, and so are the domain functions F_j'.

The unit disc is the model domain of choice in the study of simply connected domains in the plane for a score of well known reasons. Besides the superlative symmetry, the kernel functions associated to the disc are simple rational functions and the classical operators of function and potential theory are extremely simple. In the multiply connected setting, the choice of the ideal model domains is not so clear cut. Candidates include the unit disc with non-overlapping circular slits removed, an annulus with circular slits removed, a plane with radial slits removed, and a unit disc with non-intersecting closed discs removed. These model domains are each useful in various contexts. I shall put forth the idea here based on Theorem 1.2 that another candidate for the class of best model domains is the set of domains of the form

$$\mathcal{A} := \left\{ z \in \mathbb{C} : |z + \sum_{k=1}^{n-1} a_k/(z - b_k)| < r \right\}$$

where r is large enough that $\mathcal{A}$ is an n-connected domain with boundary components that are smooth real analytic curves. Since $f(z) := (1/r)(z + \sum_{k=1}^{n-1} a_k/(z - b_k))$ is a proper holomorphic mapping of $\mathcal{A}$ onto the unit disc, Theorem 1.1 implies that the classical kernels and domain functions associated to $\mathcal{A}$ are algebraic functions and that every proper holomorphic mapping from $\mathcal{A}$ onto the unit disc is algebraic. Theorem 1.2 yields that the kernel functions associated to $\mathcal{A}$ are rational combinations of just two functions of one variable. I conjecture that this is as simple as the kernel functions associated to a multiply connected domain can be. In §5 I describe some open questions and avenues of future research based on these results.

2. Preliminaries. Before we can begin to prove our main results, we must review some known facts about the classical kernel functions. Many of these facts and formulas can be found in Stefan Bergman's book [10]. I have also written up most of these results in [2] in the same spirit as this paper and I include cross references here to give the interested reader access to a uniform approach to the whole subject.

To begin with, we shall assume that Ω is a bounded n-connected domain in the plane with C^∞ smooth boundary. (Later, we shall consider general n-connected domains such that no boundary component is a point.)

Let γ_j, $j = 1, \ldots, n$, denote the n non-intersecting C^∞ simple closed curves which define the boundary $b\Omega$ of Ω, and suppose that γ_j is parameterized in the standard sense by $z_j(t)$, $0 \le t \le 1$. We shall use the convention that γ_n denotes the *outer boundary curve* of Ω. Let $T(z)$ be the C^∞ function defined on $b\Omega$ such that $T(z)$ is the complex number representing the unit tangent vector at $z \in b\Omega$ pointing in the direction of the standard orientation (meaning that $iT(z)$ represents the *inward pointing normal vector* at $z \in b\Omega$). This complex unit tangent vector function is characterized by the equation $T(z_j(t)) = z_j'(t)/|z_j'(t)|$.

The symbol $A^\infty(\Omega)$ will denote the space of holomorphic functions on Ω that are in $C^\infty(\overline{\Omega})$. The Bergman projection is the orthogonal projection of $L^2(\Omega)$ onto its subspace consisting of holomorphic functions and the Bergman kernel $K(z, w)$ is the kernel for this projection. The Szegő projection is the orthogonal projection of $L^2(b\Omega)$ onto the Hardy Space of functions in $L^2(b\Omega)$ that represent the L^2 boundary values of holomorphic functions. The Szegő kernel $S(z, w)$ is the kernel function for the Szegő projection.

The Bergman kernel $K(z, w)$ is related to the Szegő kernel via the identity

$$K(z, w) = 4\pi S(z, w)^2 + \sum_{i,j=1}^{n-1} A_{ij} F_i'(z) \overline{F_j'(w)}, \tag{2.1}$$

where the functions $F_i'(z)$ are classical functions of potential theory described as follows ([10, page 119], or see also [2, pages 94–96]). The harmonic function ω_j which solves the Dirichlet problem on Ω with boundary data equal to one on the boundary curve γ_j and zero on γ_k if $k \neq j$ has a multivalued harmonic conjugate. The function $F_j'(z)$ is a globally defined single valued holomorphic function on Ω which is locally defined as the derivative of $\omega_j + iv$ where v is a local harmonic conjugate for ω_j. The Cauchy-Riemann equations reveal that $F_j'(z) = 2(\partial\omega_j/\partial z)$.

The Bergman and Szegő kernels are holomorphic in the first variable and anti-holomorphic in the second on $\Omega \times \Omega$ and they are hermitian, i.e., $S(w,z) = \overline{S(z,w)}$. Furthermore, the Bergman and Szegő kernels are in $C^\infty((\overline{\Omega}\times\overline{\Omega}) - \{(z,z) : z \in b\Omega\})$ as functions of (z,w) (see [2, page 100]).

We shall also need to study the Garabedian kernel $L(z,w)$, which is related to the Szegő kernel via the identity

$$\frac{1}{i} L(z,a)T(z) = S(a,z) \qquad \text{for } z \in b\Omega \text{ and } a \in \Omega. \tag{2.2}$$

For fixed $a \in \Omega$, the kernel $L(z,a)$ is a holomorphic function of z on $\Omega - \{a\}$ with a simple pole at a with residue $1/(2\pi)$. Furthermore, as a function of z, $L(z,a)$ extends to the boundary and is in the space $C^\infty(\overline{\Omega} - \{a\})$. In fact, $L(z,w)$ is in $C^\infty((\overline{\Omega} \times \overline{\Omega}) - \{(z,z) : z \in \overline{\Omega}\})$ as a function of (z,w) (see [2, page 102]). Also, $L(z,a)$ is non-zero for all (z,a) in $\overline{\Omega} \times \Omega$ with $z \neq a$ and $L(a,z) = -L(z,a)$ (see [2, page 49]).

For each point $a \in \Omega$, the function of z given by $S(z,a)$ has exactly $(n-1)$ zeroes in Ω (counting multiplicities) and does not vanish at any points z in the boundary of Ω (see [2, page 49]).

Given a point $a \in \Omega$, the Ahlfors map f_a associated to the pair (Ω, a) is a proper holomorphic mapping of Ω onto the unit disc. It is an n-to-one mapping (counting multiplicities), it extends to be in $A^\infty(\Omega)$, and it maps each boundary curve γ_j one-to-one onto the unit circle. Furthermore, $f_a(a) = 0$, and f_a is the unique function mapping Ω into the unit disc maximizing the quantity $|f_a'(a)|$ with $f_a'(a) > 0$. The Ahlfors map is related to the Szegő kernel and Garabedian kernel via (see [2, page 49])

$$f_a(z) = \frac{S(z,a)}{L(z,a)}. \tag{2.3}$$

Note that $f_a'(a) = 2\pi S(a,a) \neq 0$. Because f_a is n-to-one, f_a has n zeroes. The simple pole of $L(z,a)$ at a accounts for the simple zero of f_a at a. The other $n-1$ zeroes of f_a are given by the $(n-1)$ zeroes of $S(z,a)$ in $\Omega - \{a\}$. Let $a_1, a_2, \ldots, a_{n-1}$ denote these $n-1$ zeroes (counted with multiplicity). I proved in [3] (see also [2, page 105]) that, if a is close to one of the boundary curves, the zeroes $a_1, \ldots, a_{n-1}$ become distinct simple zeroes. It follows from this result that, for all but at most finitely many points $a \in \Omega$, $S(z,a)$ has $n-1$ distinct simple zeroes in Ω as a function of z.

Fix a point a in Ω so that the zeroes $a_1, \ldots, a_{n-1}$ of $S(z,a)$ are distinct simple zeroes. I proved in [4, Theorem 3.1] that the Szegő kernel can be expressed in terms of the $n+1$ functions of one variable, $S(z,a)$, $f_a(z)$, and $S(z,a_i)$, $i = 1, \ldots, n-1$ via the formula

$$S(z,w) = \frac{1}{1 - f_a(z)\overline{f_a(w)}} \left(c_0 S(z,a)\overline{S(w,a)} + \sum_{i,j=1}^{n-1} c_{ij} S(z,a_i)\,\overline{S(w,a_j)} \right) \tag{2.4}$$

where $f_a(z)$ denotes the Ahlfors map associated to (Ω, a), $c_0 = 1/S(a,a)$, and the coefficients c_{ij} are given as the coefficients of the inverse matrix to the matrix $[S(a_j, a_k)]$. A similar formula follows for the Garabedian kernel by noting that, if $z \in \Omega$ and $w \in b\Omega$, then $L(z,w) = -L(w,z) = -i\overline{T(w)}S(z,w)$ by identity (2.2). If we now plug (2.4) into the right hand side of this expression, distribute $i\overline{T(w)}$ through the sum, use (2.4) again, and finally use the fact that $\overline{f_a(w)} = 1/f_a(w)$ for $w \in b\Omega$, we obtain the identity

$$L(z,w) = \frac{f_a(w)}{f_a(z) - f_a(w)} \left(c_0 S(z,a) L(w,a) + \sum_{i,j=1}^{n-1} c_{ij} S(z,a_i) L(w,a_j) \right) \tag{2.5}$$

where the constants c_0 and c_{ij} are the same as the constants in (2.4). This identity extends to hold for all (z,w) in $\Omega \times \Omega$ with $z \neq w$. The two formulas (2.4) and (2.5) will be very important in what follows.

Let $\mathcal{F}'$ denote the vector space of functions given by the complex linear span of the set of functions $\{F_j'(z) : j = 1, \ldots, n-1\}$ mentioned above. It is a classical fact that $\mathcal{F}'$ is $n-1$ dimensional. It shall be important for us to relate the functions in $\mathcal{F}'$ to the Szegő and Bergman kernel functions. Notice that $S(z,a_i)L(z,a)$ is in $A^\infty(\Omega)$ because the pole of $L(z,a)$ at $z = a$ is cancelled by the zero of $S(z,a_i)$ at $z = a$. Similarly, $S(z,a)L(z,a_i)$ is in $A^\infty(\Omega)$ because the pole of $L(z,a_i)$ at $z = a_i$ is cancelled by the zero of $S(z,a)$ at $z = a_i$. A theorem due to Schiffer ([13], see also [2, page 80]) states that the set of $n-1$ functions $\{S(z,a_i)L(z,a) : i = 1, \ldots, n-1\}$ form a basis for $\mathcal{F}'$. It is also shown in [2, page 80] that the linear span of $\{S(z,a_i)L(z,a) : i = 1, \ldots, n-1\}$ is the same as the linear span of $\{L(z,a_i)S(z,a) : i = 1, \ldots, n-1\}$. Hence, formula (2.1) can be rewritten in the form

$$K(z,w) = 4\pi S(z,w)^2 + \sum_{i,j=1}^{n-1} \lambda_{ij} \mathcal{L}_i(z) \overline{\mathcal{L}_j(w)} \tag{2.6}$$

where $\mathcal{L}_i(z) = L(z,a_i)S(z,a)$.

The Bergman kernel is related to the classical Green's function via ([10, page 62], see also [2, page 131])

$$K(z,w) = -\frac{2}{\pi} \frac{\partial^2 G(z,w)}{\partial z \partial \bar{w}}.$$

Another kernel function on $\Omega \times \Omega$ that we shall study is given by

$$\Lambda(z,w) = -\frac{2}{\pi} \frac{\partial^2 G(z,w)}{\partial z \partial w}.$$

In the literature, this function is sometimes written as $L(z,w)$ with anywhere between zero and three tildes over the top. We have chosen the symbol Λ here to avoid confusion with our notation for the Garabedian kernel above. It follows from well known properties of the Green's function that $\Lambda(z,w)$ is holomorphic in z and w and is in $C^\infty(\overline{\Omega} \times \overline{\Omega} - \{(z,z) : z \in \overline{\Omega}\})$. If $a \in \Omega$, then $\Lambda(z,a)$ has a double pole at $z = a$ as a function of z and $\Lambda(z,a) = \Lambda(a,z)$ (see [2, page 134]).

The Bergman kernel is related to Λ via the identity

$$\Lambda(w,z)T(z) = -K(w,z)\overline{T(z)} \qquad \text{for } w \in \Omega \text{ and } z \in b\Omega \tag{2.7}$$

(see [2, page 135]).

The kernel $\Lambda(z,w)$ can also be expressed in terms of kernel functions associated to the boundary. By using (2.7), (2.6), and (2.2), the identity

$$\Lambda(w,z) = 4\pi L(w,z)^2 + \sum_{i,j=1}^{n-1} \lambda_{ij} L(w,a_i)S(w,a)S(z,a_j)L(z,a), \tag{2.8}$$

can be seen to hold for $z,w \in \Omega$, $z \neq w$. The coefficients λ_{ij} are the same as those appearing in (2.6). We may express the functions $S(z,a_j)L(z,a)$ in terms of the other basis $\{\mathcal{L}_j\}_{j=1}^{n-1}$ for $\mathcal{F}'$ in order to be able to rewrite formula (2.8) in the form

$$\Lambda(w,z) = 4\pi L(w,z)^2 + \sum_{i,j=1}^{n-1} \mu_{ij} \mathcal{L}_i(z)\, \mathcal{L}_j(w) \tag{2.9}$$

where $\mathcal{L}_i(z) = L(z,a_i)S(z,a)$.

We now suppose that Ω is merely an n-connected domain in the plane such that no boundary component of Ω is a point. It is well know that there is a biholomorphic mapping Φ mapping Ω one-to-one onto a bounded domain Ω_0 in the plane with real analytic boundary. The domain Ω_0 is a bounded n-connected domain with C^∞ smooth boundary whose boundary consists of n non-intersecting simple closed real analytic curves. Let subscript 0's indicate that a kernel function is associated to Ω_0. The transformation formula for the Bergman kernels under biholomorphic mappings gives

$$K(z,w) = \Phi'(z) K_0(\Phi(z),\Phi(w))\overline{\Phi'(w)}. \tag{2.10}$$

Similarly,

$$\Lambda(z,w) = \Phi'(z)\Lambda_0(\Phi(z),\Phi(w))\Phi'(w). \tag{2.11}$$

It is well known that the function Φ' has a single valued holomorphic square root on Ω (see [2, page 43]). To avoid a discussion of the meaning of the Cauchy transform and the Szegő projection in non-smooth domains, we shall opt to *define* the Szegő and Garabedian kernels associated to Ω via the natural transformation formulas,

$$S(z,w) = \sqrt{\Phi'(z)}\; S_0(\Phi(z),\Phi(w))\overline{\sqrt{\Phi'(w)}}, \tag{2.12}$$

and

$$L(z,w) = \sqrt{\Phi'(z)}\; L_0(\Phi(z),\Phi(w))\sqrt{\Phi'(w)}. \tag{2.13}$$

Finally, the Green's functions satisfy

$$G(z,w) = G_0(\Phi(z),\Phi(w)). \tag{2.14}$$

It is a routine matter to check that the transformation formulas for the kernel functions above respect all the formulas given in this section where the variables range *inside* the domain. The formulas on the boundary involving the unit complex tangent vector function $T(z)$ can be seen to be valid near boundary points of Ω that are C^1 smooth and where Φ is locally C^1 up to the boundary and Φ' is non-vanishing.

3. Construction of the Riemann surface. This section shall be devoted to proving the following theorem.

Theorem 3.1. *Suppose that Ω is a finitely connected domain in the plane such that no boundary component is a point. If the Bergman or the Szegő kernel associated to Ω is algebraic, then Ω can be realized as a subdomain of a compact Riemann surface $\widetilde{\mathcal{R}}$ such that all the kernel functions $S(z,w)$, $L(z,w)$, $K(z,w)$, $\Lambda(z,w)$ extend to $\widetilde{\mathcal{R}}\times\widetilde{\mathcal{R}}$ as single valued meromorphic functions. Furthermore, the Ahlfors maps $f_a(z)$ and every proper holomorphic mapping from Ω to the unit disc extend to be single valued meromorphic functions on $\widetilde{\mathcal{R}}$. Also, the functions $F_k'(z)$, $k=1,\dots,n-1$, extend to be single valued meromorphic functions on $\widetilde{\mathcal{R}}$. Furthermore, the complement of Ω in $\widetilde{\mathcal{R}}$ is connected.*

Proof. Suppose that Ω is a finitely connected domain in the plane such that no boundary component is a point and such that the Bergman or the Szegő kernel associated to Ω is algebraic. Theorem 1.2 yields that the Ahlfors maps $f_a(z)$ associated to Ω are algebraic, and so the boundary of Ω is locally described by equations $|f_a(z)|^2=1$. Hence, the boundary of Ω consists of a finite union of real analytic curve segments with endpoints. For now, let us assume that no singular points of $f_a(z)$ occur on the boundary of Ω so that the boundary of Ω consists of n non-intersecting C^∞ smooth real analytic closed curves. (We shall consider the case where singular points of $f_a(z)$ fall on the boundary of Ω later because they shall only force modifications in the proof that distract from the main idea given now in the smooth case.) Note that the exterior of Ω contains an open set in the Riemann sphere. Hence, by making a linear fractional change of variables, we may assume that the point at infinity is not on the boundary of Ω. (Note that the transformation formulas for the kernel functions and Ahlfors maps imply that linear fractional transformations preserve algebraicity.)

We shall now use the three formulas (2.3),(2.4), and (2.5) to construct a certain Riemann surface $\mathcal{R}$ and to describe its boundary. Let us call the $n+1$ functions $S(z,a)$, $L(z,a)$, and $S(z,a_1)$, $S(z,a_2),\dots,S(z,a_{n-1})$ *core functions* and, for convenience, let us rename them $C_1(z),C_2(z),\dots C_{n+1}(z)$. Each of the $n+1$ core functions is algebraic and so we may think of them (from the viewpoint of Weierstrass) as being multivalued functions that can be analytically continued to the whole Riemann sphere. There is a finite set of points E in the extended complex plane above which one or more of the function elements associated to the core functions has an algebraic singularity. Choose a point A_0 in the outer boundary curve γ_n of Ω to act as a base point. We construct $\mathcal{R}$ by performing analytic continuation of each of the core functions *simultaneously*, starting at A_0 and moving into the exterior of Ω, paying special attention to the points in E. It is best to think of $\mathcal{R}$ as being exterior to Ω and as being attached to Ω at the outer boundary curve γ_n. Away from E, the lifting of germs along curves to a Riemann surface over the extended complex plane is routine and obvious. When we analytically continue up to a point p in E, it may happen that none of the germs of the function elements of the core functions become singular at p. In this case, we lift and analytically continue through p without incident. If, on the other hand, at least one of the elements is singular at p, then we construct a local coordinate system at the point $\tilde{p}$ above p as follows. Let us abuse notation by letting $C_1(z),C_2(z),\dots,C_{n+1}(z)$ denote the $n+1$ function elements of the core functions that are obtained as we analytically continue them up to p along a curve. Each of these elements can be viewed as a function element of a Puiseux expansion at p and so there are positive integers λ_k such that the substitution $z = p + (\zeta - p)^{\lambda_k}$ makes $C_k(\zeta)$ analytic and continuable in ζ through $\zeta = 0$. (Note that the number λ_k is equal to one if $C_k(z)$ does not have a singularity at p.) Let m be equal to the least common multiple of $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$. We can now define a local uniformizing variable ζ that is suitable for each of the function elements in the obvious manner: $z = p + (\zeta - p)^m$. This coordinate function allows us to lift all the core functions so as to be defined and single valued on a disc centered at $\zeta = 0$ and we use it to define a local chart near $\tilde{p}$.

We now define the boundary of the Riemann surface $\mathcal{R}$ as follows. The core functions $S(z, a)$, $L(z, a)$, and $S(z, a_1)$, $S(z, a_2), \ldots, S(z, a_{n-1})$ have *preferred* germs in Ω given by the values they have by virtue of the definition of the kernel functions of the domain Ω. A point $\tilde{p} \in \mathcal{R}$ over $p \in \mathbb{C}$ is in the boundary of $\mathcal{R}$ if p is in the boundary of Ω and the germs of the core functions are equal to their preferred germs on the Ω side of the projection down onto the plane. (Note that there will be sheets of $\mathcal{R}$ above Ω if it happens that one or more of the core functions continue up to $p \in b\Omega$ and are not equal to their preferred germs over Ω.)

At the moment, it is not clear if any but the outer boundary curve of Ω is involved in the formation of the boundary of $\mathcal{R}$. We shall soon see that *each* boundary curve of Ω contributes exactly one component to the boundary of $\mathcal{R}$.

Our Riemann surface $\mathcal{R}$ is clearly a finitely sheeted surface over all or part of the extended complex plane and $\mathcal{R}$ together with its boundary is compact. Formulas (2.3), (2.4), and (2.5) reveal that $S(z, w)$, $L(z, w)$, and $f_a(z)$ are well defined and single valued meromorphic functions of z on $\mathcal{R}$ when w is held fixed in Ω.

We now want to do a careful counting of the poles of $S(z, w)$ and $L(z, w)$ on $\mathcal{R}$ in the z variable when w is held fixed at a point in Ω.

Recall that $S(z, w)$ and $L(z, w)$ have no zeroes or poles as functions of z on the boundary of Ω when w is held fixed in Ω. We now want to show that the number of poles of each of these functions (counted with multiplicity) in $\mathcal{R}$ is constant independent of w as w ranges over Ω minus a certain finite set of points. We begin by studying $S(z, w)$. Formula (2.4) shows that $S(z, w)$ has two kinds of poles: roving poles that come from the first factor involving $f_a(z)$ and stationary poles that come from the sum in the second factor. We first consider the stationary poles arising from the second factor in (2.4). It will be convenient to think of $S(z, w)$ as a quotient $\mathcal{N}_S(z, w)/\mathcal{D}_S(z, w)$ where

$$\mathcal{N}_S(z, w) := S(z, a)\overline{S(w, a)} + \sum_{i,j=1}^{n-1} c_{ij} S(z, a_i)\, \overline{S(w, a_j)},$$

and

$$\mathcal{D}_S(z, w) := 1 - f_a(z)\overline{f_a(w)}.$$

If w is a fixed point in Ω, then the stationary poles of $S(z, w)$ in the z variable on $\mathcal{R}$ occur at poles of $\mathcal{N}_S(z, w)$. We now consider whether it is possible for poles of the various terms in this sum to cancel one another. Let $\mathcal{P}$ denote the set of points in $\mathcal{R}$ where one or more of the function elements associated to the *core* functions $S(z, a)$, $S(z, a_1)$, $S(z, a_2), \ldots, S(z, a_{n-1})$ has a pole, and suppose z_0 is a point in $\mathcal{P}$. (If z_0 is also in the set E of singular points, we use the local uniformizing variable used in the construction of $\mathcal{R}$ as the underlying coordinate variable.) Let N be the order of the pole at z_0 of highest order among $S(z,a)$, $S(z,a_1)$, $S(z,a_2),\ldots,S(z,a_{n-1})$. We claim that, for generic w in Ω, $\mathcal{N}_S(z,w)$ has a pole of order N at z_0 in the z variable. Indeed, the coefficient of order N in the principal parts in the z variable of $\mathcal{N}_S(z,w)$ is given by a function of w of the form

$$H(w) := b_0 S(w,a) + \sum_{i,j=1}^{n-1} c_{ij} b_i \, S(w,a_j)$$

where not all the constants $b_0, b_1, \ldots, b_{n-1}$ are zero. We shall now prove that $H(w)$ can only vanish at finitely many w in $\overline{\Omega}$. Let $\beta_j = \sum_{i=1}^{n-1} c_{ij} b_i$. Because $[c_{ij}]$ is a non-singular matrix, $(b_1,\ldots,b_{n-1}) = 0 \in \mathbb{C}^{n-1}$ if and only if $(\beta_1,\ldots,\beta_{n-1}) = 0 \in \mathbb{C}^{n-1}$. If the function $H(w)$, which is holomorphic on a neighborhood of $\overline{\Omega}$, were to vanish at an infinite number of points in $\overline{\Omega}$, then it would be identically zero on $\overline{\Omega}$, and by integrating $\overline{H(w)}$ over the boundary against a generic polynomial $p(w)$, the reproducing property of the Szegő kernel would yield that

$$0 = \overline{b_0}\, p(a) + \sum_{j=1}^{n-1} \overline{\beta_j}\, p(a_j).$$

For this to hold true for *any* polynomial requires that $b_0 = 0$ and $(\beta_1,\ldots,\beta_{n-1}) = 0$. Hence, all the constants $b_0, b_1, \ldots, b_{n-1}$ are zero, and this contradiction shows that there are at most finitely many points w in $\overline{\Omega}$ where $\mathcal{N}_S(z,w)$ does not have a pole of order N at z_0. Let us call this finite set Q. We shall augment Q by finitely many points as necessary as we proceed in order to keep the zeroes and poles of $S(z,w)$ and $L(z,w)$ from co-mingling as w ranges over $\Omega - Q$. We now add the points to Q that we would get by repeating the argument above for z_0 for each of the finitely many points in $\mathcal{R}$ where one or more of the core functions has a pole.

Notice that we may now state that the zeroes of $\mathcal{N}_S(z,w)$ in the z variable stay away from the poles of $\mathcal{N}_S(z,w)$ in the z variable as long as w stays in $\overline{\Omega} - Q$.

The same reasoning that we used at one of the poles z_0 above shows that, given a point z in $\mathcal{R}$, there are at most finitely many points $w \in \overline{\Omega}$ such that $\mathcal{N}_S(z,w) = 0$.

Next, we treat the stationary poles of the denominator term in the expression $\mathcal{D}_S(z,w) = 1 - f_a(z)\overline{f_a(w)}$ for $S(z,w)$. Let us now add the points $w = a, a_1, \ldots, a_{n-1}$ to Q in case they are not already there so that $f_a(w)$ is non-vanishing on $\overline{\Omega} - Q$. Hence, when $w \in \overline{\Omega} - Q$, the poles of $\mathcal{D}_S(z,w)$ in the z variable occur at the poles of $f_a(z)$, and thus, are fixed. If one of these points falls in $\mathcal{P}$, it will simply reduce the order of or cancel one of the fixed poles of $\mathcal{N}_S(z,w)$.

For $w \in \overline{\Omega} - Q$, the zeroes of $\mathcal{D}_S(z,w)$ occur where $f_a(z) = 1/\overline{f_a(w)}$. Since $f_a(z)$ takes on only finitely many values at the points where the core functions have poles, we may add finitely many points to Q, if necessary, to ensure that the zeroes of $\mathcal{D}_S(z,w)$ (where $f_a(z) = 1/\overline{f_a(w)}$) do not occur at points in $\mathcal{P}$ when w is in $\overline{\Omega} - Q$.

Finally, near a generic point (z_0,w_0) in $(\mathcal{R} - \mathcal{P}) \times (\Omega - Q)$, we may express $\mathcal{N}_S(z,w)$ and $\mathcal{D}_S(z,w)$ as products of irreducible Weierstrass polynomials normalized in the z variable and we may factor out common factors. Since the discriminant of such a generalized polynomial in z is a holomorphic function of w and since the the resultant of two such polynomials is also a holomorphic function of w, elementary arguments similar to those in [11, page 7] reveal that, away from a finite set of points w near w_0, the number of poles of $\mathcal{N}_S(z,w)/\mathcal{D}_S(z,w)$ near z_0 is constant independent of w. Because $\mathcal{N}_S(z,w)$ and $\mathcal{D}_S(z,w)$ have no zeroes or poles near $b\Omega \times b\Omega$, we may use a finite cover to be able to assert that there is finite set $Q \subset \Omega$ and a positive integer P_S such that $S(z,w)$ has P_S poles as a function of z in $\mathcal{R}$ when $w \in \Omega - Q$. (Because the core functions are holomorphic and non-vanishing on the boundary of $\mathcal{R}$, the argument principle applied to a triangulation of the closure of $\mathcal{R}$ shows that the number of zeroes minus the number of poles of $S(z,w)$ in the z variable is constant as w ranges over Ω. We have chosen Q so that the zeroes and poles of $S(z,w)$ in the z variable do not interact as w ranges over $\Omega - Q$.)

Analogous reasoning can be applied to $L(z,w)$ using (2.5). Define

$$\mathcal{N}_L(z,w) := c_0 S(z,a)L(w,a) + \sum_{i,j=1}^{n-1} c_{ij} S(z,a_i)L(w,a_j).$$

We shall now show that, after adding finitely many points to Q, if necessary, $\mathcal{N}_L(z,w)$ has poles in the z variable on $\mathcal{R}$ *at exactly the same points as* $\mathcal{N}_S(z,w)$ *with exactly the same order* when $w \in \overline{\Omega} - Q$. As above, suppose one or more of the *core* functions $S(z,a)$, $S(z,a_1)$, $S(z,a_2), \ldots, S(z,a_{n-1})$ has a pole at z_0 in $\mathcal{R}$. Let N be the order of the pole at z_0 of highest order among $S(z,a)$, $S(z,a_1)$, $S(z,a_2), \ldots, S(z,a_{n-1})$. We claim that, for generic w in Ω, $\mathcal{N}_L(z,w)$ has a pole of order N at z_0 in the z variable. Indeed, if some of the top order terms in the poles in z of the core functions involved in the definition of $\mathcal{N}_L(z,w)$ cancel each other, a linear relation of the form

$$0 = b_0 L(w,a) + \sum_{i,j=1}^{n-1} c_{ij} b_i \, L(w,a_j)$$

would hold where not all the constants $b_0, b_1, \ldots, b_{n-1}$ are zero. We shall now prove that such a linear relation can only hold for at most finitely many w in $\overline{\Omega}$. Let $\beta_j = \sum_{i=1}^{n-1} c_{ij} b_i$. Because $[c_{ij}]$ is a non-singular matrix, $(b_1, \ldots, b_{n-1}) = 0 \in \mathbb{C}^{n-1}$ if and only if $(\beta_1, \ldots, \beta_{n-1}) = 0 \in \mathbb{C}^{n-1}$. If the function

$$H(w) := b_0 L(w,a) + \sum_{j=1}^{n-1} \beta_j \, L(w,a_j),$$

which extends holomorphically across the boundary of Ω, were to vanish at an infinite number of points in $\overline{\Omega}$, then it would be identically zero on $\overline{\Omega}$, and by integrating $H(w)T(w)$ over the boundary against a generic polynomial $p(w)$, we can use (2.2) and the reproducing property of the Szegő kernel to see that

$$0 = b_0 \, p(a) + \sum_{j=1}^{n-1} \beta_j \, p(a_j),$$

which implies that $b_0 = 0$ and $(\beta_1, \ldots, \beta_{n-1}) = 0$. Hence, all the constants $b_0, b_1, \ldots, b_{n-1}$ are zero, and this contradiction shows that there are at most finitely many points w in $\overline{\Omega}$ where $\mathcal{N}_L(z,w)$ does not have a pole of order N at z_0. Let us add these points, if they are not already there, to Q.

Define

$$g_L(z,w) := \frac{f_a(w)}{f_a(z) - f_a(w)}.$$

For $w \in \overline{\Omega} - Q$, the zeroes of $g_L(z,w)$ in the z variable occur at poles of $f_a(z)$ and the poles of $g_L(z,w)$ occur where $f_a(z) = f_a(w)$. Since $f_a(z)$ takes on only finitely many values at the points where the core functions have poles, we may add finitely many points to Q, if necessary, to ensure that the poles of $g_L(z,w)$ (where $f_a(z) = f_a(w)$) are not cancelled by zeroes of $\mathcal{N}_L(z,w)$ for w in $\overline{\Omega} - Q$.

Away from points in $\mathcal{P} \times Q$, we may argue as we did in the study of $S(z,w)$ above to see that there is a positive integer P_L which is equal to the number of poles of $L(z,w)$ as a function of z in $\mathcal{R}$ when $w \in \Omega - Q$.

We shall now deduce relationships between P_S and P_L based on formula (2.2) and the fact that, for $w \in \Omega$ near the boundary, the simple pole of $L(z,w)$ in the z variable at $z = w$ corresponds to a simple pole of $S(z,w)$ in the z variable at the point obtained from w by Schwarz Reflection of w across the boundary (see [4, page 1351]) and that near the boundary, these are the only poles of $S(z,w)$ and $L(z,w)$. Because we shall need to modify the proof of this fact when the boundary is not globally smooth, we shall give a quick proof of it here. Since $b\Omega$ is given by real analytic curves, there exists an *antiholomorphic reflection function* $R(z)$ with the properties that $R(z)$ is defined and is antiholomorphic on a neighborhood $\mathcal{O}$ of $b\Omega$, $R(z_0) = z_0$ when $z_0 \in b\Omega$, $(\partial/\partial\bar{z})R(z)$ is non-vanishing on $\mathcal{O}$, and $R(z)$ maps $\mathcal{O} \cap \Omega$ one-to-one onto $\mathcal{O} - \overline{\Omega}$. Let w be a point in Ω that is close to the boundary and let A be a fixed point in Ω. By (2.2), we have $-i\, L(z,A)T(z) = S(A,z)$ and $-i\, L(z,w)T(z) = S(w,z)$ for $z \in b\Omega$. Divide the second of these identities by the first and use the fact that $R(z) = z$ on $b\Omega$ to obtain

$$\frac{S(w,R(z))}{S(A,R(z))} = \frac{L(z,w)}{L(z,A)} \qquad \text{for } z \in b\Omega.$$

The function on the left hand side of this identity is holomorphic in z on a neighborhood of $b\Omega$; so is the function on the right hand side. Since these functions agree on $b\Omega$, they must be equal on a neighborhood of $b\Omega$. We may assume that $\mathcal{O}$ is small enough that $S(z,A)$ and $L(z,A)$ have no poles or zeroes in $\mathcal{O}$. We may now read off from the identity that the simple pole of $L(z,w)$ in the z variable at $z = w$ corresponds to a simple pole of $S(z,w)$ in the z variable at $w = R(z)$. It is also apparent that there are no other poles of $S(z,w)$ and $L(z,w)$ near the boundary, and if w_0 is a point in the boundary of Ω, then $S(z,w_0)$ and $L(z,w_0)$ both have a simple pole at w_0 and no other poles on the boundary of Ω.

Let us consider what happens to the poles of $S(z,w)$ and $L(z,w)$ as w approaches a point w_0 in the outer boundary curve γ_n minus Q from the inside of Ω. When w is actually in the boundary of Ω and $z \in \Omega$, we may rewrite (2.2) as

$$\text{(3.1)} \qquad \frac{1}{i} L(w,z)T(w) = S(z,w).$$

This identity extends to hold for all z in $\mathcal{R}$ and hence it follows that $L(z,w_0)$ and $S(z,w_0)$ have exactly the same poles as functions of z on $\mathcal{R}$ when w_0 is a fixed point in the boundary of Ω (recall that $L(w,z) = -L(z,w)$). As w approaches w_0

from the inside of Ω, the fact about the pole of $L(z,w)$ at w corresponding to a pole of $S(z,w)$ at the reflection of w on the outside of Ω shows that

$$P_S = P_L + 1. \tag{3.2}$$

We are now in a position to prove that each of the boundary curves of Ω is involved in the formation of the boundary of $\mathcal{R}$ and so, by attaching Ω to $\mathcal{R}$ along their mutual boundary, we will obtain the compact Riemann surface $\widetilde{\mathcal{R}}$ mentioned in Theorem 3.1. Indeed, if γ_k is a boundary curve of Ω that is not reached in the construction of $\mathcal{R}$, then by letting $w \in \Omega - Q$ approach a fixed point w_0 in $\gamma_k - Q$, we see that

$$P_S = P_L,$$

and this contradicts (3.2).

It is now clear from the definition of $\mathcal{R}$, that by attaching Ω to the boundary of $\mathcal{R}$ in the obvious way, we obtain a *compact* Riemann surface $\widetilde{\mathcal{R}}$ on which the functions $S(z,w)$, $L(z,w)$, and $f_a(z)$ are well defined and single valued meromorphic functions of z when w is held fixed in Ω. Finally, the relationships between the kernel functions and other functions of potential theory described in §2 complete the proof of Theorem 3.1 in case the boundary of Ω is smooth.

To finish the proof of Theorem 3.1, we must allow the possibility that singular points of the core functions might fall on the boundary of Ω. We now take a moment to describe the boundary of Ω and the way the Ahlfors map behaves there when f_a has algebraic singularities on the boundary of Ω. (Keep in mind the example where Ω is equal to the extended complex plane minus n non-intersecting x's and the boundary behavior of the extension of the Ahlfors map up to one of the x's can be visualized by thinking of the unit circle as if it had been stretched around the x and "shrink wrapped" down to it. Extending the Ahlfors map up to the x would set up a map from the "perimeter" of the x to the circle that is one-to-one on the ends of each stroke of an x, 1-to-2 on the legs of an x, and 1-to-4 at the center of an x.)

Because f_a is algebraic and because f_a maps $b\Omega$ into the unit circle, the boundary of Ω is given by n connected components that are each a union of smooth real analytic curve segments with endpoints. Indeed, given a point z_0 in the boundary of Ω, there are only three possible configurations of the boundary of Ω at z_0. The easiest case occurs when we may choose a small disc $D_\epsilon(z_0)$ so that in this disc, the boundary of Ω is a single real analytic curve that enters the disc at one point on the circle $|\zeta - z_0| = \epsilon$ and exits at another and such that the real analytic curve divides the disc into two connected parts, one inside Ω and the other exterior to Ω. In this case, f_a extends holomorphically past the boundary of Ω along the real analytic curve in $D_\epsilon(z_0)$ by reflection, and by shrinking ϵ, we may assume that f_a is holomorphic on $D_\epsilon(z_0)$. Another possibility is that we may choose $D_\epsilon(z_0)$ in the same way, except that the real analytic curve divides the disc into two connected parts that are *both inside* Ω. In this case, f_a extends holomorphically past the boundary of Ω via analytic continuation from either side and we should think of the real analytic curve as being a "slit" as is commonly done when constructing a Riemann surface by "cutting and pasting." Note that, in this case, the continuations of $f_a(z)$ along the two outward pointing normals (that point in opposite directions) must yield function elements on the "outside" of Ω that are different from f_a on the inside of Ω because the extensions of the Ahlfors map to the two different sides must map the real analytic curve one-to-one onto two different and non-overlapping arcs on the unit circle. Finally, the third possibility is that z_0 is a "center of a star" in the sense that we may choose a small disc $D_\epsilon(z_0)$ so that in this disc, the boundary component of Ω that contains z_0 consists of one or more real analytic curves $\nu_1, \nu_2, \ldots, \nu_q$ which originate at z_0 and which radiate outward from z_0 and intersect the boundary of $D_\epsilon(z_0)$ in q distinct points, and such that the only point common to more than one of the curves ν_j in the closure of $D_\epsilon(z_0)$ is z_0. Indeed, because the Ahlfors map $f_a(z_0 + \zeta^k)$ is holomorphic in ζ near $\zeta = 0$ for some positive integer k, and because f_a maps the boundary of Ω into the unit circle, these curves can be easily parametrized. Since real analytic curves can only intersect in finitely many points, it is possible to choose ϵ small enough so that z_0 is the only point common to more than one curve in the closure of $D_\epsilon(z_0)$. Now the q curves ν_j divide $D_\epsilon(z_0)$ into q sectors and, by choosing ϵ small enough, each sector is either inside Ω or outside Ω. We now see that f_a extends continuously up to the curves ν_j from sectors that are inside Ω and that analytic continuation may be performed past the curves to the "outside" of Ω, thinking of the curves as "slits." Furthermore, f_a extends continuously up to the star point z_0 from the inside of sectors that are inside Ω. (Note that it will happen that f_a extends with different values on different sides of the curves ν_j and that f_a might very well have several different values at z_0 as that point is approached from different sectors.)

Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ denote the connected components of the boundary of Ω. Let E_b denote the (finite) set of points which is the union of the set of boundary points of Ω where one or more of the core functions on $\overline{\Omega}$ are singular and the set of singular points of the boundary of Ω, i.e., points where one or more of the real analytic curves that comprise the boundary cross and/or terminate.

Choose a point A_0 on the boundary component γ_n that is on a smooth part of one of the smooth real analytic curves that comprise γ_n. This point A_0 will act as the base point as we construct $\mathcal{R}$ by performing simultaneous analytic continuation of the core functions into the "exterior" of Ω as we did above in the smooth case, except that now we might have to think of certain boundary curves as being "slits." The first difference from the construction in the smooth case above might occur as we attempt to exit the domain at A_0. It might happen that both sides of the smooth boundary curve are inside Ω. If this is the case, choose one side of the curve at A_0 to be the *inside* of Ω and analytically continue the core functions as we pass over A_0 in the direction of the outward pointing normal there. Once we have passed out of Ω along this "slit," we may proceed exactly as we did in the smooth case to construct the Riemann surface $\mathcal{R}$.

We must be more careful when we define the boundary of $\mathcal{R}$. As before, the core functions $S(z,a)$, $L(z,a)$, and $S(z,a_1)$, $S(z,a_2), \ldots, S(z,a_{n-1})$ have *preferred* germs over Ω given by the values they have by virtue of the definition of the kernel functions on Ω. We continue to think of a point $\tilde{p}$ over $p \in \mathbb{C}$ to be in the boundary of $\mathcal{R}$ if p is in the boundary of Ω and the germs of the core functions are equal to their preferred germs on the Ω side of the projection down onto the plane. If p is not in E_b and is a point on a smooth part of a smooth real analytic curve comprising the boundary that is a "one sided" boundary curve, then this works exactly as in the smooth case above. If p is not in E_b and is a point on a smooth part of a smooth real analytic curve comprising the boundary that is a "two sided" boundary curve, then we think of the boundary of $\mathcal{R}$ as being attached to Ω along the proper side of the curve-slit in the obvious way.

We next consider the boundary of $\mathcal{R}$ and the problem of analytically continuing the core functions at a point z_0 in E_b. We may define a local uniformizing variable ζ as we did above that makes each of the core functions single valued near z_0. Choose a small disc $D_\epsilon(z_0)$ as we did above so that in this disc, the boundary component of Ω that contains z_0 consists of real analytic curves $\nu_1, \nu_2, \ldots, \nu_q$ which originate at z_0 and which radiate outward from z_0 and intersect the boundary of $D_\epsilon(z_0)$ in exactly one point each, and such that the only point common to more than one of the curves ν_j is z_0. Now it is an easy matter to see how to "glue" the boundary of $\mathcal{R}$ to Ω, thinking of the curves ν_j as slits along which the boundary of $\mathcal{R}$ is to be attached to the boundary of Ω if all the function elements of the core functions agree with their preferred germs at points that lie above Ω on the "inside" side of the slit. Note that it may very well happen that the center point z_0 will be included (by continuity) in the boundary of $\mathcal{R}$ as the boundary is approached from more than one sectors that are "exterior" to Ω, and hence, it might happen that z_0 has *more than one point above it* in the boundary of $\mathcal{R}$.

The set Q can be chosen exactly as in the smooth case so that the number of poles P_S and P_L are well defined for $w \in \overline{\Omega} - Q$.

At this moment, it is not clear if any but one piece of the boundary component γ_n near A_0 where we started the analytic continuation is involved in the formation of the boundary of $\mathcal{R}$. We shall soon see that each "sided" segment in the boundary is contained in the boundary of $\mathcal{R}$. Let ν denote the smooth real analytic curve that contains the point A_0 from which we started the construction of $\mathcal{R}$ and let D_+ denote the "half" of $D_\epsilon(A_0)$ that we chose as the "inside" of Ω which is bounded by ν and let D_- denote the "half" of $D_\epsilon(A_0)$ that represents the exterior of Ω (and the interior of $\mathcal{R}$). Identity (3.1) extends to hold on $\mathcal{R}$ as before and hence it follows that $L(z, w_0)$ and $S(z, w_0)$ have exactly the same poles as functions of z on $\mathcal{R}$ when w_0 is a fixed point in ν. As w approaches w_0 from the D_+ side of Ω, the fact that the pole of $L(z, w)$ at w corresponds to a pole of $S(z, w)$ at the reflection of w on the D_- side of the "exterior" of Ω shows that

$$P_S = P_L + 1. \tag{3.3}$$

We may now think of Ω as being attached to $\mathcal{R}$ along ν and we may analytically continue the core functions into Ω using their preferred germs. Pick another point w_1 on any other smooth part of a real analytic arc ν_1 in $b\Omega - E_b$. If w_1 is not part of the boundary of $\mathcal{R}$, then by letting w approach w_1 from the inside of Ω, the same reasoning we used in the smooth case shows that $P_S = P_L$, and this contradicts 3.3. Because of our ability to analytically continue the core functions by their preferred values in Ω, this shows that all the smooth parts of the boundaries together with their appropriate "sides" are all attached to $\mathcal{R}$. The finite number of points in E_b can now be filled in by continuity to complete the picture.

It is now evident that by attaching Ω to $\mathcal{R}$, we have constructed a compact Riemann surface $\widetilde{\mathcal{R}}$ on which $S(z, w)$, $L(z, w)$, and $f_a(z)$ are well defined and single valued meromorphic functions of z when w is held fixed in Ω. To finish the proof of Theorem 3.1 we need only note that identities (2.4) and (2.5) now show that $S(z, w)$ and $L(z, w)$ extend to $\widetilde{\mathcal{R}} \times \widetilde{\mathcal{R}}$ as single valued functions and (2.6) and (2.8) reveal the same for $K(z, w)$ and $\Lambda(z, w)$. The fact that the functions F_j' extend follows from the fact mentioned in §2 that F_j' is a linear combination of functions of the form $S(z, a)L(z, a_i)$.

The next short section deals with the simply connected case, $n = 1$.

4. The simply connected case. Let a be a fixed point in a simply connected domain $\Omega \neq \mathbb{C}$ and let $f_a(z)$ denote the Riemann mapping function mapping Ω one-to-one onto the unit disc $D_1(0)$ with $f_a(a) = 0$ and $f_a'(a) > 0$. The Szegő kernel $S(z, w)$ associated to Ω may be expressed as

$$S(z,w) = \frac{c\, S(z,a)\overline{S(w,a)}}{1 - f_a(z)\overline{f_a(w)}},$$

where $c = 1/S(a,a)$. This formula is analogous to (2.4) and it shows that if the Szegő kernel is algebraic, then Riemann mappings are algebraic. Conversely, if a Riemann map is algebraic, the transformation formula for the Szegő kernel (2.12) shows that $S(z, w)$ is algebraic. Hence, Riemann mappings are algebraic if and only if the Szegő kernel is algebraic. Furthermore, the Szegő kernel is a rational combination of the two functions of one variable, $f_a(z)$ and $S(z, a)$. In a simply connected domain, the Bergman kernel is related to the Szegő kernel very simply via

$$K(z,w) = 4\pi S(z,w)^2.$$

This formula shows that the Bergman kernel is algebraic if and only if the Szegő kernel is algebraic and that the Bergman kernel is a rational combination of the same two functions of one variable, $f_a(z)$ and $S(z, a)$.

5. Consequences of the existence of the Riemann surface and open questions. Suppose Ω is a finitely connected domain in the plane such that no boundary component is a point and suppose that one of the conditions of Theorem 1.2 is met. Suppose $f : \Omega \to D_1(0)$ is a proper holomorphic mapping (such as an Ahlfors map). We know that f extends to the Riemann surface $\widetilde{\mathcal{R}}$ as a single valued meromorphic function. Suppose the order of f on $\widetilde{\mathcal{R}}$ is m. Choose a point λ with $|\lambda| > 1$ so that $f^{-1}(\lambda)$ consists of m distinct points in $\widetilde{\mathcal{R}}$. We may construct a meromorphic function g on $\widetilde{\mathcal{R}}$ as in Farkas and Kra [12, page 248-249] which is holomorphic on $\Omega \subset \widetilde{\mathcal{R}}$ such that f and g form a primitive pair for the field of meromorphic functions on $\widetilde{\mathcal{R}}$. (This means that any meromorphic function on the double of Ω can be written as a rational combination of these two functions.) Now the kernel identities of §2 reveal that $K(z, w)$ and $S(z, w)$ are rational combinations of $f(z)$, $g(z)$ and conjugates of $f(w)$ and $g(w)$. Furthermore, all proper holomorphic maps of Ω onto the unit disc are rational combinations of f and g and so are the functions F_j'.

A fascinating problem that remains is to determine if the function $g(z)$ can be taken to be something explicit and well known. For example, can g be taken to be a second proper holomorphic map of Ω onto the unit disc? To show that such a thing is possible, one would have to find a proper holomorphic map g that separates the m points in $f^{-1}(\lambda)$ (see [1, page 321-324]). Since proper holomorphic maps from Ω to the disc extend to be meromorphic on the double of Ω (see [5]), the consequences mentioned above would imply that all the kernels and domain functions associated to Ω extend to the *double* of Ω as single valued meromorphic functions. Another interesting possibility is that maybe $g(z)$ could be taken to be $S(z, b)$ or $K(z, b)$ for some choice of b in Ω.

Perhaps the most interesting problem that presents itself in light of Theorem 1.2 is the question as to whether every n-connected domain in the plane such that no boundary component is a point can be mapped conformally onto a domain of the form

$$\mathcal{A} := \left\{ z \in \mathbb{C} : |z + \sum_{k=1}^{n-1} a_k/(z - b_k)| < r \right\}$$

for some choice of the parameters a_k, b_k, and r. If such a thing were possible, then the transformation formula for the Bergman kernels under conformal maps would allow the information obtained by means of the Riemann surface attached to $\mathcal{A}$ to be pulled back to a general domain.

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