SZEGŐ COORDINATES, QUADRATURE DOMAINS, AND DOUBLE QUADRATURE DOMAINS

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ABSTRACT. We define Szegő coordinates on a finitely connected smoothly bounded planar domain which effect a holomorphic change of coordinates on the domain that can be as close to the identity as desired and which convert the domain to a quadrature domain with respect to boundary arc length. When these Szegő coordinates coincide with Bergman coordinates, the result is a double quadrature domain with respect to both area and arc length. We enumerate a host of interesting and useful properties that such double quadrature domains possess, and we show that such domains are in fact dense in the realm of bounded C^∞-smooth simply connected domains.

1. INTRODUCTION

The unit disc $D_1(0)$ is the quintessential example of a double quadrature domain. Holomorphic functions h on the disc satisfy two quadrature identities:

$$\pi h(0) = \iint_{D_1(0)} h\ dA \qquad \text{and} \qquad 2\pi h(0) = \int_{bD_1(0)} h(z)\ ds,$$

whenever these integrals make sense. We will show in this paper that double quadrature domains are rather commonplace in the realm of smoothly bounded domains in the plane.

A bounded domain Ω in the complex plane is a quadrature domain with respect to area measure if there exist finitely many points $\{w_j\}_{j=1}^N$ in the domain and non-negative integers n_j such that complex numbers c_{jk} exist satisfying

$$\iint_\Omega h\ dA = \sum_{j=1}^{N} \sum_{k=0}^{n_j} c_{jk} h^{(k)}(w_j)$$

for every function h in the Bergman space $H^2(\Omega)$ of square integrable holomorphic functions on Ω. Here, dA denotes Lebesgue area measure. We shall say that such a domain is an *area quadrature domain*. Aharonov and Shapiro [1] proved that such domains have piecewise real-analytic boundaries (see also Gustafsson [13]).

A bounded domain Ω bounded by finitely many non-intersecting piecewise C^1-smooth Jordan curves is is a quadrature domain with respect to boundary

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arc length measure if there exist finitely many points $\{w_j\}_{j=1}^N$ in the domain and non-negative integers n_j such that complex numbers c_{jk} exist satisfying

$$\int_{b\Omega} h\ ds = \sum_{j=1}^{N} \sum_{k=0}^{n_j} c_{jk} h^{(k)}(w_j)$$

for every function h in the Hardy space $H^2(b\Omega)$ of holomorphic functions with L^2 boundary values. Here, ds denotes the boundary arc length measure. We shall say that such a domain is a *boundary arc length quadrature domain.* Björn Gustafsson [14] showed that such boundary arc length quadrature domains have C^∞-smooth real-analytic boundaries. (He also showed that the boundary regularity assumptions could be greatly relaxed, but we do not want to dwell on these details here.)

After some pioneering work of Shapiro and Shapiro and Ullemar [18], Gustafsson [14] studied planar quadrature domains with respect to boundary arc length from the point of view of half-order differentials and Riemann surface theory. He characterized such domains and showed that they are dense in the category of bounded domains in the plane bounded by finitely many non-intersecting Jordan curves. We shall reframe Gustafsson's results in more elementary terms and explain how to view these quadrature domains in terms of Szegő coordinates, which are the analogue of Bergman coordinates as developed in [8, 9, 10]. Because our results our most interesting in the category of C^∞-smoothly bounded domains, we shall take the royal road and restrict our attention to domains of this form.

For more information about the state of research on quadrature domains, see [12] and the volume that holds it, and [17]. We also remark that Gustafsson [15] has studied the desinity of quadrature domains problem for harmonic functions from a different perspective that has been generalized by Sakai [16] to higher dimensions.

Suppose that Ω is a bounded domain bounded by n non-intersecting C^∞-smooth Jordan curves. Let $A^\infty(\Omega)$ denote the subspace of $C^\infty(\overline{\Omega})$ consisting of holomorphic functions. Let $K(z,w)$ denote the Bergman kernel associated to Ω and let $S(z,w)$ denote the Szegő kernel. Let $K^0(z,w)$ also denote $K(z,w)$ and let $K^m(z,w) = \frac{\partial^m}{\partial \bar{w}^m} K(z,w)$. As in [10], we define the *Bergman span* associated to Ω to be the complex linear span $\mathcal{K}$ of all functions $h(z)$ of z of the form $K^m(z,a)$ as a ranges over Ω and m ranges over all non-negative integers. If U is an open subset of Ω, let $\mathcal{K}_U$ denote the complex linear span of functions of z from $\{K(z,a) : a \in U\}$, and given a point a in Ω, let $\mathcal{K}_a$ denote the complex linear span of functions of z from $\{K^m(z,a) : m = 0,1,2,\dots\}$. Similarly, define the *Szegő span* to be the complex linear span $\mathcal{S}$ of all functions $h(z)$ of z of the form $S^m(z,a)$ as a ranges over Ω and m ranges over all non-negative integers, and define $\mathcal{S}_U$ and $\mathcal{S}_a$ in the analogous way.

We now list two theorems that will serve as the heart of the paper. Note that if $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between finitely connected domains in the plane, then it is well known that there is a holomorphic branch of $\sqrt{f'}$ on Ω_1.

Theorem 1.1. *Suppose Ω_1 and Ω_2 are bounded domains in the plane bounded by finitely many non-intersecting C^∞-smooth Jordan curves. Suppose further that $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping. Then Ω_2 is a boundary arc length quadrature domain if and only if f' is equal to the square of an element of the Szegő span associated to Ω_1, or equivalently, if and only if $\sqrt{f'}$ is in the Szegő span.*

We shall call a biholomorphic mapping f as given in Theorem 1.1 such that $\sqrt{f'}$ is in the Szegő span a *Szegő coordinate*. This is a direct analogue of the definition of Bergman coordinate in [10] where a biholomorphic map f is called a Bergman coordinate if and only if f' is in the *Bergman* span.

The next theorem allows us to use Theorem 1.1 to approximate domains by boundary arc length quadrature domains.

Theorem 1.2. *Suppose that Ω is a bounded domain bounded by finitely many non-intersecting C^∞-smooth Jordan curves. Then the Szegő span $\mathcal{S}$ associated to Ω is dense in $A^\infty(\Omega)$. Furthermore, $\mathcal{S}_U$ is dense in $A^\infty(\Omega)$ for any open subset U of Ω, and $\mathcal{S}_a$ is dense in $A^\infty(\Omega)$ for any point $a \in \Omega$.*

Theorem 1.2 was proved in [4].

In §2, we show how to combine Theorems 1.1 and 1.2 and results of [10] to obtain the following results. The next theorem shows that boundary arc length quadrature domains are dense in the realm of bounded finitely connected smoothly bounded planar domains.

Theorem 1.3. *Suppose Ω is a bounded domain bounded by finitely many non-intersecting C^∞-smooth Jordan curves. There is a biholomorphic mapping f which is a Szegő coordinate as close to the identity in $C^\infty(\overline{\Omega})$ as we please so that $f(\Omega)$ is a boundary arc length quadrature domain.*

The proof of Theorem 1.3 was sketched in [7, p. 285-6]. We shall flesh out the argument here to make this paper more comprehensible and because we will need the fine details in order to generalize the result to double quadrature domains.

Theorem 1.4. *Suppose Ω is a bounded simply connected domain bounded by a C^∞-smooth Jordan curve. There is a biholomorphic mapping f which is both a Szegő coordinate and a Bergman coordinate as close to the identity in $C^\infty(\overline{\Omega})$ as we please so that $f(\Omega)$ is both a boundary arc length quadrature domain and an area quadrature domain, i.e., a double quadrature domain.*

Double quadrature domains have a number of remarkable properties, some of which we enumerate here.

(1) The boundary is given by finitely many non-intersecting real algebraic curves which are C^∞-smooth and real analytic. Gustafsson [13] showed that the boundary is essentially given by the zero set of an irreducible polynomial $P(z, w)$ on $\mathbb{C}^2$. He showed that the boundary is equal to $\{z : P(z, \bar{z}) = 0\}$ minus perhaps finitely many points, and he showed that $P(z, w)$ must have a a certain special form.

(2) The complex polynomials belong to the Bergman span.
(3) The complex polynomials belong to the Szegő span.
(4) The complex unit tangent vector function $T(z)$ is a rational function of z and $\bar{z}$ for z in the boundary (which of course is equivalent to being a rational function of Re z and Im z). The function $T(z)$ is also the restriction to the boundary of a meromorphic function on the double without poles on the boundary. This means that $T(z)$ can be extended to the domain as either a meromorphic function with no poles on the boundary, or an anti-meromorphic function with no poles on the boundary.
(5) The Schwarz function $S(z)$ exists, extends to be analytic on a neighborhood of the boundary, extends to be meromorphic on the domain without poles on the boundary, and is algebraic. Because $S(z) = \bar{z}$ on the boundary, both z and $S(z)$ extend meromorphically to the double of the domain. Gustafsson noted that all the meromorphic functions on the double are generated by the extensions of z and $S(z)$ to the double.
(6) Both the Bergman kernel $K(z, w)$ and the Szegő kernel $S(z, w)$ are rational functions of z, $S(z)$, $\bar{w}$, and $\overline{S(w)}$. Hence, they both extend to the double cross the double as functions which are meromorphic in z and anti-meromorphic in w. If the domain is multiply connected, the two kernels are rational functions of $f_1(z)$, $f_2(z)$, $\overline{f_1(w)}$, and $\overline{f_2(w)}$, where f_1 and f_2 are two Ahlfors maps associated to two generic points in the domain. In the simply connected case, the kernels are even more elementary (see below).
(7) Both the Bergman and Szegő kernels are rational functions of z, $\bar{z}$, w, and $\bar{w}$ when z and w are restricted to the boundary.
(8) The Kerzman-Stein kernel $A(z, w)$ is a rational function of z, $\bar{z}$, w, and $\bar{w}$ for z and w on the boundary.

If the domain is a simply connected double quadrature domain, then even more can be said. Let f denote a Riemann map associated to a point in the domain.

(1) Since f extends to the double, it is a rational function of z and $S(z)$, and it is a rational function of z and $\bar{z}$ on the boundary. It is also algebraic. Since the extension of f to the double generates the meromorphic functions on the double, it also follows that z and $S(z)$ are rational functions of $f(z)$. (This is another way of seeing that f has a rational inverse, a well know fact about simply connected quadrature domains with respect to area.)
(2) The Bergman kernel $K(z, w)$ and the Szegő kernel $S(z, w)$ are rational functions of $f(z)$ and $\overline{f(w)}$.
(3) The Poisson kernel $p(z, w)$ is the real part of a function that is rational in z and $S(z)$, and w and $\bar{w}$ (for z in the domain and w on the boundary). It is also the real part of a rational function of $f(z)$ and w and $\bar{w}$. It is even the real part of a rational function of $f(z)$ and $f(w)$.

We shall explain these results in §5. In §6, we shall explain a connection between the Schwarz function, double quadrature domains, and proper holomorphic mappings onto the unit disc.

2. Density of quadrature domains in the simply connected case

In this section, we give a simple proof that boundary arc length quadrature domains are dense in the realm of bounded C^∞-smoothly bounded simply connected domains, and then we go on to prove that even double quadrature domains are dense. (We do not know an example of a multiply connected double quadrature domain. Perhaps they do not exist.)

Suppose that Ω is a bounded simply connected domain in the plane bounded by a C^∞-smooth Jordan curve. We may now construct a boundary arc length quadrature domain close to Ω as follows. By Theorem 1.2, we may suppose that h is a function in the Szegő span that is close to the constant function 1 in $A^\infty(\Omega)$. (Of course, we take h close enough to 1 so as to be non-vanishing.) Let H be a holomorphic square root of h on Ω, and let f be a complex antiderivative of H, where we choose the constant of integration so that $f(z)$ is close to the identity in $A^\infty(\Omega)$. By choosing h close enough to 1 in $A^\infty(\Omega)$, we may guarantee that f is close enough to the identity that f is one-to-one on Ω and $f(\Omega)$ is as C^∞-close as we desire to Ω.

Now according to Theorem 1.1, $f(\Omega)$ is a boundary arc length quadrature domain that is C^∞ close to Ω.

We can repeat the above argument using $\mathcal{S}_a$ for a point a in Ω instead of the complete Szegő span. We obtain a biholomorphic mapping f to an arc length quadrature domain close to Ω satisfying an identity,

$$f' = \left(\sum_{m=0}^{N} c_m S^m(z,a) \right)^2.$$

We want to see that f' is also in the Bergman span so that $f(\Omega)$ is also an area quadrature domain via Theorem 1.3 of [10].

Let $L(z,w)$ denote the Garabedian kernel associated to Ω and let $L^m(z,w) = (\partial/\partial w)^m L(z,w)$. Let $\Lambda(z,w)$ denote the Schiffer kernel function, which is meromorphic in z and w with a double pole at $z = w$ and which is related to the Bergman kernel $K(z,w)$ of Ω via

$$K(w,z)\overline{T(z)} = -\Lambda(w,z)T(z)$$

when $w \in \Omega$ and $z \in b\Omega$. Here $T(z)$ denotes the complex unit tangent vector function pointing in the direction of the standard orientation. The basic properties of these kernel functions are described in [5]. We also define $\Lambda^m(z,w) = (\partial/\partial w)^m \Lambda(z,w)$. Let b_j denote complex coefficients that will be determined later. The relationship

$$S(w,z) = \frac{1}{i} L(z,w) T(z)$$

for $w \in \Omega$ and $z \in b\Omega$ together with the relationship between $K(z,w)$ and $\Lambda(z,w)$ allow us to state that $H(z)T(z) = -\overline{G(z)T(z)}$ for z in $b\Omega$, where

$$H(z) = \left[\left(\sum_{m=0}^{N} c_m S^m(z,a) \right)^2 - \left(\sum_{j=0}^{M} b_j K^j(z,a) \right) \right]$$

and

$$G(z) = \left[\left(\sum_{m=0}^{N} \bar{c}_m L^m(z,a) \right)^2 - \left(\sum_{j=0}^{M} \bar{b}_j \Lambda^j(z,a) \right) \right].$$

It can be deduced that $\left(\sum_{j=0}^{N} \bar{c}_j L^j(z,a)\right)^2$ is residue free since

$$\int_{b\Omega} \left(\sum_{m=0}^{N} \bar{c}_m L^m(z,a) \right)^2 \, dz = \int_{b\Omega} \left(\sum_{m=0}^{N} c_m \overline{S^m(z,a)} \right)^2 \, d\bar{z},$$

and this last integral is zero by Cauchy's Theorem. Since $\Lambda(z,a)$ has principal part $(-1/\pi)(z-a)^{-2}$ at $z=a$, it follows that a positive integer M and constants b_j can be chosen so that $G(z)$ has a removable singularity at a. Now both $H(z)$ and $G(z)$ are holomorphic on Ω and extend smoothly to the boundary in such a way that $H(z)T(z) = -\overline{G(z)T(z)}$ on the boundary. We now claim that this can only be true if both H and G are identically zero. Indeed, functions of the form $\overline{G(z)T(z)}$ are orthogonal to holomorphic functions in the Hardy space and functions of the form $H(z)T(z)$ are orthogonal to anti-holomorphic functions. Thus, $H(z)T(z)$ is orthogonal to holomorphic functions and anti-holomorphic functions. But every C^∞-smooth function on $b\Omega$ is the restriction to the boundary of a harmonic function in $C^\infty(\overline{\Omega})$, and every such harmonic function can be written as $h + \overline{h}$ where $h \in A^\infty(\Omega)$. Since smooth functions are dense in $L^2(b\Omega)$, it follows that $H \equiv 0$ and $G \equiv 0$. We can now state that

$$\left(\sum_{m=0}^{N} c_m S^m(z,a) \right)^2 = \left(\sum_{j=0}^{M} b_j K^j(z,a) \right),$$

and this shows that f' is in the Bergman span. Hence f is both a Szegő and a Bergman coordinate and $f(\Omega)$ is a double quadrature domain.

We remark here that, after finding the proof above, we discovered that Avci [2] showed in his unpublished Stanford PhD thesis using other methods that an arc length quadrature domain that satisfies a one point quadrature identity would also have to be an area quadrature domain. We include our proof here because it shows the close and explicit connection between Szegő and Bergman coordinates.

3. Density of boundary arc length quadrature domains in the multiply connected case

In this section, we show how Gustafsson's argument using half-order differentials and a special Runge Theorem on Riemann surfaces [14, p. 77-78] can be modified along the lines of the previous section to prove C^∞ density of boundary arc length quadrature domains in the category of smooth finitely connected domains.

Let Ω be a bounded domain bounded by n non-intersecting C^∞-smooth Jordan curves. There is a function h in the Szegő span that is as close to one in $C^\infty(\overline{\Omega})$ as we desire. Gustafsson [14, p. 78] constructed rational functions $R_j(z)$, $j = 1, \ldots, n-1$ with special orthogonality conditions as follows. Let γ_j, $j = 1, \ldots, n$ denote the boundary curves of Ω where γ_n denotes the outer boundary. Choose a point z_j inside γ_j, $j = 1, \ldots, n-1$ (i.e., choose one point from each of the holes in Ω). The rational functions are given by

$$R_j(z) = \frac{1}{2\pi i(z - z_j)} + (z - z_j)Q_j(z),$$

where the Q_j are any polynomials satisfying

$$Q_j(z_k) = -\frac{1}{2\pi i(z_k - z_j)^2}$$

for $k \neq j$ (making $R_j(z_k) = 0$ for $k \neq j$). The functions R_j have the property that

$$\int_{\gamma_k} R_j \; dz = \delta_{kj} \quad \text{and} \int_{\gamma_k} R_j R_m \; dz = 0$$

for $j, k, m = 1, \ldots n-1$. Since each rational function R_j is in $A^\infty(\Omega)$, we may approximate it in $A^\infty(\Omega)$ by a function r_j in the Szegő span. Notice that it easy to find coefficients A_j such that the periods of $(h - \sum_{j=1}^{n-1} A_j R_j)^2$ vanish because the system is

$$\int_{\gamma_k} h^2 \; dz + 2\sum_{j=1}^{n-1} A_j \int_{\gamma_k} hR_j \; dz = 0$$

for $k = 1, \ldots, n-1$. Since h is close to one, the coefficient matrix

$$\int_{\gamma_k} hR_j \; dz$$

is close to the identity and $\int_{\gamma_k} h^2 \, dz$ is close to zero. Thus, the A_j are uniquely determined and are close to zero. If we modify the problem of making the periods vanish by replacing the R_j by our functions r_j in the Szegő span, then the system becomes

$$\int_{\gamma_k} h^2 \; dz + 2\sum_{j=1}^{n-1} A_j \int_{\gamma_k} hr_j \; dz + \sum_{j,m=1}^{n-1} A_j A_m \int_{\gamma_k} r_j r_m \; dz = 0,$$

and since the coefficients in the linear part are close to the identity matrix and the coefficients of the quadratic terms are small, the implicit function theorem yields that the A_j are uniquely determined and are close to the small solution to the system with R_j in place of r_j. Notice that by taking h to be sufficiently close to one, the coefficients A_j can be made small enough that $h - \sum_{j-1}^{n-1} A_j r_j$ is as close to one as desired. Thus, this modification of Gustafsson's argument shows that there is a function $F = h - \sum_{j-1}^{n-1} A_j r_j$ in the Szegő span which is close to one whose square F^2 has vanishing periods. Now let f be an antiderivative of F^2. Since F^2 is close to one, we may choose the constant of integration so that f is close to the identity. If h is close enough to one, then f will be close enough to the identity to make f biholomorphic on Ω and Theorem 1.1 yields that f maps Ω to a boundary arc length quadrature domain that is C^∞ close to Ω.

4. Proofs of the main theorems

Suppose that $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between bounded domains in the plane bounded by finitely many non-intersecting C^∞-smooth Jordan curves. Let $F = f^{-1}$. It is well known that a holomorphic square root of f' exists on Ω_1 and a square root of F' exists on Ω_2. Since f' and F' extend C^∞-smoothly to the respective boundaries and are non-vanishing there, the same is true of their square roots. Notice that the change of variables formula yields that

$$\int_{b\Omega_1} |f'|(u \circ f)\; ds = \int_{b\Omega_2} u\; ds$$

when u is in the Hardy space $H^2(b\Omega_2)$. Let

$$\langle u, v\rangle_j = \int_{b\Omega_j} u\,\bar{v}\; ds$$

denote the Hardy space inner product on Ω_j, $j = 1, 2$. The last identity can be written

$$\langle \sqrt{f'}(u \circ f), \sqrt{f'}\rangle_1 = \langle u, 1\rangle_2.$$

It is easy to verify that if $u \in L^2(b\Omega_2)$ and $v \in L^2(b\Omega_1)$, then

$$\langle \sqrt{f'}(u \circ f), v\rangle_1 = \langle u, \sqrt{F'}(v \circ F)\rangle_2.$$

To prove the first half of Theorem 1.1, assume that $\sqrt{f'}$ is in the Szegő span associated to Ω_1. If h is an element of the Hardy space $H^2(b\Omega_2)$, then

$$\int_{b\Omega_2} h\,ds = \int_{b\Omega_1} |f'|(h \circ f)\; ds = \langle \sqrt{f'}(h \circ f), \sqrt{f'}\rangle_1,$$

and if $\sqrt{f'}$ is in the Szegő span, then this last inner product yields a finite linear combination of values of $\sqrt{f'}(h \circ f)$ and its derivatives at finitely many points in Ω_1, which reduces to a finite linear combination of values of h and its derivatives at finitely many points in Ω_2. This shows that Ω_2 is a boundary arc length quadrature domain.

To prove the converse, suppose that Ω_2 is a boundary arc length quadrature domain. Then, given any $g \in H^2(b\Omega_1)$,

$$\langle g, \sqrt{f'}\rangle_1 = \langle \sqrt{F'}(g \circ F), 1\rangle_2,$$

and since Ω_2 is a boundary arc length quadrature domain, this last inner product yields a finite linear combination of values of $\sqrt{F'}(g \circ F)$ and its derivatives at finitely many points in Ω_2, which reduces to a finite linear combination of values of g and its derivatives at finitely many points in Ω_1. Thus, $\sqrt{f'}$ has the same effect as an element of the Szegő span when paired against a function g in the Hardy space. It follows that $\sqrt{f'}$ must in fact be equal to that element in the Szegő span.

5. Properties of double quadrature domains

Suppose Ω is a double quadrature domain in the plane. Because Ω is an area quadrature domain, it has a boundary given by an algebraic curve, and because it is a boundary arc length quadrature domain, that curve consists of C^∞ smooth real analytic curves (see [1, 13, 14]). Because Ω is an area quadrature domain, the complex polynomials belong to the Bergman span (see [10]). Also, the Schwarz function $S(z)$ exists and is meromorphic on Ω, extends analytically past the boundary, and satisfies $S(z) = \bar{z}$ on the boundary (see [1, 13]). This identity shows that the two functions z and $S(z)$ extend meromorphically to the double of Ω. Gustafsson [13] noted that the extensions form a primitive pair for the field of meromorphic functions on the double, i.e., all the meromorphic functions on the double are given as rational combinations of the two. It is shown in [7] (see also [8]) that the Bergman kernel $K(z, w)$ associated to an area quadrature domain is a rational function of z, $S(z)$, and $\bar{w}$, and $\overline{S(w)}$. Consequently, $K(z, w)$ is a rational function of z, $\bar{z}$, w, and $\bar{w}$ when z and w are restricted to the boundary. It was also shown in [8, 7] that the Bergman kernel is a rational combination of two Ahlfors maps associated to two generic points in the domain. Aharonov and Shapiro [1] showed that the Ahlfors maps are algebraic.

Because Ω is an arc length quadrature domain, there is a meromorphic function $H(z)$ on Ω which extends analytically past the boundary and such that $\overline{H(z)}$ is equal to the complex unit tangent vector function $T(z)$ on the boundary (see [14, 2]). The identity relating the Szegő and Garabedian kernels can be used with this fact to see that both the Szegő and the Garabedian kernels extend to the double in the first variable when the second variable is held fixed. Indeed, the identity

$$\overline{S(z, a)} = \frac{1}{i} L(z, a) T(z) \qquad \text{for } z \in b\Omega$$

shows that $L(z, a) = i\overline{S(z, a)}/\overline{H(z)}$ on the boundary, and this shows that $L(z, a)$ extends meromorphically to the double in z. Similarly, $S(z, a)$ extends. These identities also reveal that $T(z)$ is equal to a function $G(z)$ on the boundary where $G(z)$ is meromorphic on Ω and extends analytically past the boundary. Hence, $T(z)$ is the restriction to the boundary of a meromorphic function that has no singularities on the boundary of Ω. These arguments are reversible and we may state the following theorem.

Theorem 5.1. *A bounded domain with a piecewise C^1 smooth boundary is a boundary arc length quadrature domain if and only if the Szegő kernel $S(z, w)$ associated to the domain extends meromorphically to the double as a function of z for each fixed w in the domain. This is the case if and only if the complex* *unit tangent vector function is the restriction to the boundary of a meromorphic function that has no singularities on the boundary of* Ω*.*

The same theorem holds with the Garabedian kernel in place of the Szegő kernel.

If Ω is a double quadrature domain, then we have shown that $T(z)$ is the restriction to the boundary of a meromorphic function on the double, and we also know that such functions are generated by z and $S(z)$. Hence, it follows that $T(z)$ is the restriction of a rational combination of z and $S(z)$ to the boundary, i.e., $T(z)$ is a rational function of z and $\bar{z}$.

When Theorem 5.1 is combined with results from [6], it follows that the Szegő kernel $S(z,w)$ associated to a double quadrature domain is a rational combination of z, $S(z)$, and $\bar{w}$, and $\overline{S(w)}$. Consequently, $S(z,w)$ is a rational function of z, $\bar{z}$, w, and $\bar{w}$ when z and w are restricted to the boundary.

The Kerzman-Stein kernel is given by

$$A(z,w) = \frac{1}{2\pi i}\left(\frac{T(w)}{w-z} - \frac{\overline{T(z)}}{\bar{w}-\bar{z}}\right)$$

for z and w in the boundary of Ω, and it follows that the Kerzman-Stein kernel associated to a double quadrature domain is a rational function of z, $\bar{z}$, w, and $\bar{w}$.

We shall now show that the complex polynomials are in the Szegő span associated to a double quadrature domain. Let $H(z)$ denote the meromorphic function that is equal to $\overline{T(z)}$ on the boundary. Suppose $h \in A^\infty(\Omega)$. Since $\bar{z} = S(z)$ on the boundary, we may write

$$\int_{b\Omega} h\,\bar{z}^n\,ds = \int_{b\Omega} h(z)S(z)^n\,\overline{T(z)}\,dz = \int_{b\Omega} h(z)S(z)^n H(z)\,dz,$$

and this last integral is equal to a finite linear combination of values of h and its derivatives at finitely many points in Ω by the Residue Theorem. Hence, z^n has the same effect when paired with h that a certain element in the Szegő span would have. Since $A^\infty(\Omega)$ is dense in the Hardy space, it follows that z^n is equal to the element in the Szegő span.

The converse of this last result is true, namely, that if the complex polynomials are in the Szegő span, then the domain is a double quadrature domain. To see this, note that $z = z/1$. If both z and 1 are in the Szegő span, then z is a quotient of elements in the Szegő span, and as such, it extends to the double as a meromorphic function (since the relationship between the Szegő and the Garabedian kernels reveals that quotients of functions in the Szegő span would be equal to quotients of conjugates of functions in the "Garabedian span" on the boundary). The condition that z extends to the double is the classic condition that is equivalent to the domain being an area quadrature domain. It is easy to see that a smooth domain is a boundary arc length quadrature domain if and only if the constant function 1 is in the Szegő span. Hence, we have proved the following theorem.

Theorem 5.2. *A bounded domain with a piecewise C^1 smooth boundary is a double quadrature domain if and only if the complex polynomials belong to the Szegő span associated to the domain.*

Theorem 5.2 is an appealing analogue of Theorem 1.2 from [10] which states that a bounded domain is an area quadrature domain if and only if the complex polynomials belong to the Bergman span.

The Poisson kernel of a smooth simply connected domain is given by

$$p(z,w) = 2\text{Re}\,\left(\frac{S(z,w)S(w,a)}{S(a,a)}\right) - \frac{|S(w,a)|^2}{S(a,a)},$$

where $z \in \Omega$, $w \in b\Omega$, and a is a fixed point in Ω (see [3, p. 1367]). Hence, when Ω is a simply connected double quadrature domain, $p(z,w)$ is the real part of a function that is rational in z and $S(z)$, and w and $\bar{w}$ (just like in the unit disc, where $S(z) = 1/z$).

The rest of the properties mentioned in §1 involving the Riemann map f follow from the remarks above and the fact that f extends meromorphically to the double and the extension generates the field of meromorphic functions on the double, i.e., every meromorphic function on the double is a rational function of the extension of the Riemann map. Hence, for example, on a quadrature domain with respect to area, the functions z and $S(z)$ extend to the double, and so they are rational functions of $f(z)$. Since, $f(w) = 1/\overline{f(w)}$ on the boundary, the restriction to the boundary of a rational function of $f(w)$ and $\overline{f(w)}$ is equal to a rational function of $f(w)$ alone. Hence, it follows from the remarks above about the Poisson kernel is the real part of a rational function of $f(z)$ and $f(w)$.

6. The Schwarz function and proper holomorphic mappings to the disc

We conclude this paper by noting a relationship between proper holomorphic mappings of a domain to the unit disc and the existence of the Schwarz function, which is the hallmark of a bounded quadrature domain with respect to area. We go on to reveal a further relationship between proper holomorphic mappings and the extension of the unit tangent function to the double, which characterizes a bounded quadrature domain with respect to area.

Assume that Ω is a (bounded) area quadrature domain. For the moment, assume that zero does not belong to Ω. As remarked earlier, the Schwarz function $S(z)$ exists and is meromorphic in Ω. Let $\{a_j\}_{j=1}^N$ denote the poles of $S(z)$ in Ω, and suppose a_j is a pole of $S(z)$ of order n_j. Let f_j denote the Ahlfors map associated to a_j, which is a proper holomorphic mapping of Ω onto the unit disc such that a_j is a simple zero of f_j. (The Ahlfors map associated to a point a in Ω is the unique solution to the extremal problem to maximize $h'(a)$ under the conditions that h maps Ω into the unit disc and $h'(a)$ is real. It maps a bounded n-connected domain onto the unit disc as an n-to-one branched covering map.) We note here that, since the boundary of Ω is piecewise smooth, proper holomorphic mappings to the unit disc extend continuously to the boundary. Conversely, a holomorphic mapping to the unit disc that is continuous up to the boundary is proper if and only if it is non-constant and it maps the boundary to the unit circle. Notice that, since $|f_j(z)| = 1$ and $\bar{z} = S(z)$ on the boundary, the function

$$\Phi(z) = \left(\prod_{j=1}^{N} f_j(z)^{n_j}\right) \frac{S(z)}{z}$$

has removable singularities at each a_j, is continuous up to the boundary, and has unit modulus on the boundary. If Ω is multiply connected, then Φ must have a zero in Ω since the Schwarz function must have a zero in the domain (see [2]). Hence Φ is a proper holomorphic mapping of Ω to the unit disc. It follows that $S(z)$ is z times the quotient of two proper holomorphic mappings of Ω to the unit disc. If the point zero were in the domain, we could add the Ahlfors map f_0 to the product and obtain the same result. If Ω is simply connected, then we can define

$$\Psi(z) = \left(\prod_{j=1}^{N} f_j(z)^{n_j}\right) \left(\prod_{k=1}^{M} F_k(z)^{-m_k}\right) \frac{S(z)}{z},$$

where f_j is the Riemann map associated to a point a_j where $S(z)$ has a pole of order n_j and F_k is the Riemann map associated to a point b_k where $S(z)$ has a zero of order m_k. In this case, Ψ has constant modulus one on the boundary and no zeros in Ω. Hence it is a unimodular constant and we deduce that $S(z)$ is equal to a unimodular constant times z times a quotient of products of Riemann maps. Since proper maps are all given by unimodular constants times products of Riemann maps in this setting, the same result follows, i.e., that $S(z)$ is z times the quotient of two proper holomorphic mappings of Ω to the unit disc.

If the domain Ω is bounded by finitely many non-intersecting Jordan curves, a simple converse follows. Indeed, since a proper holomorphic mapping f to the unit disc would be continuous up to the boundary and have unit modulus on the boundary in this case, the identity $f(z) = 1/\overline{f(z)}$, which holds on the boundary, reveals that f extends to the double. If $\bar{z}$ is equal to z times the quotient of two proper holomorphic mappings to the unit disc on the boundary, then z is seen to extend to the double, and it follows that the domain is an area quadrature domain. The theorem with its converse reads as follows.

Theorem 6.1. *A bounded domain in the plane bounded by finitely many non-intersecting Jordan curves is an area quadrature domain if and only if, on the boundary, the function $\bar{z}/z$ is equal to the boundary values of a quotient of proper holomorphic mappings of the domain onto the unit disc.*

Note that if $\bar{z}/z = f_1/f_2$ on the boundary, then the Schwarz function is given by $S(z) = zf_1(z)/f_2(z)$.

If a domain has piecewise C^1 smooth boundary, it is a boundary arc length quadrature domain if and only if the unimodular function $T(z)$ extends to the double as a meromorphic function. Similar reasoning to the argument above with $T(z)$ in place of $S(z)/z$ yields the following result.

Theorem 6.2. *A finitely connected bounded domain with piecewise C^1 smooth boundary is a boundary arc length quadrature domain if and only if the complex unit tangent vector function $T(z)$ is equal to a quotient of proper holomorphic mappings of the domain onto the unit disc.*

Since an area quadrature domain has piecewise C^1 smooth boundary, we may also state this last theorem.

Theorem 6.3. *A bounded domain in the plane bounded by finitely many non-intersecting Jordan curves is a double quadrature domain if and only if, on the boundary, the function $\bar{z}/z$ is equal to the boundary values of a quotient of proper holomorphic mappings of the domain onto the unit disc, and the complex unit tangent vector function $T(z)$ is also equal to the boundary values of a quotient of proper holomorphic mappings of the domain onto the unit disc.*

Finally, we remark that the semi-group of all proper holomorphic mappings of a finitely connected domain to the unit disc has been described completely in [11].

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