JUST ANALYSIS: THE POISSON-SZEGŐ-BERGMAN KERNEL

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ABSTRACT. We derive some new connections between the Szegő kernel, the Poisson kernel, the Dirichlet-to-Neuamnn map, and the Bergman kernel in planar domains. The new formulas shed light on the complexity of the Poisson kernel in multiply connected domains.

To celebrate the legacy of Nessim Sibony

1. Preamble

The first time I met Nessim Sibony, I was part of a group of mathematicians who were trying to compliment him on a fiendishly clever and beautiful argument he had come up with in a recent paper of his. He said, in the most French way possible, "It's just analysis!" It was the most modest thing I had ever heard a mathematician say! I took him to mean that, if you shine the light of *analysis* on a problem, you are sure to find something interesting and beautiful.

In the spirit of Nessim Sibony, I shine the light of analysis on some rather old and moldy areas of mathematics. It's just complex analysis. In fact, it's just another paper on the Bergman kernel in the plane, an object that has the annoying feature that, if you think about it for any length of time, you will think that you have discovered something new and interesting about it. After shining the light of analysis on this material for awhile, I suspect that some of the founders of the subject must have known some of these things. What probably is new, however, is how all the pieces can be seen to fit together and how they extend nicely to the multiply connected setting, where they shed light on the complexity of the Poisson kernel. For that reason, I have made the paper somewhat expository and have improved and simplified some arguments that have previously appeared in order to shine the beam a little brighter.

2. Introduction

Because I have started this paper writing in the first person, I must warn the reader that, when I begin to prove theorems, I will switch, not to the royal we, but to the "we" that refers to the reader and me. I like to think that we are in this together.

Assume that Ω is a bounded simply connected domain in the plane with C^∞ smooth boundary $b\Omega$. I assume that my reader is familiar with the Bergman,

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Szegő, Garabedian, and Poisson kernels associated to Ω. We will quickly review the key properties and identities we will need; this information can be found in Bergman's book [11] or in [1], where it is given in a style and form in the same spirit as this paper.

Before we launch into a long string of definitions and properties of the classical kernel functions, we will summarize some of the easier results of this paper that do not require too much background. They fall under the heading: the Poisson kernel and the Bergman kernel.

Let a be a point in our bounded domain Ω with smooth boundary that we fix once and for all, and let $k(z,w)$ be a complex antiderivative in the z variable of the Bergman kernel $K(z,w)$ for Ω that vanishes at a. Let $T(w)$ denote the complex number of modulus one pointing in the direction of the standard orientation of the boundary at a point w in the boundary $b\Omega$ of Ω.

The first theorem relates the Poisson kernel to the antiderivative of the Bergman kernel.

Theorem 2.1. *The Poisson kernel $P(z,w)$ associated to a bounded simply connected domain in the plane with C^∞ smooth boundary Ω is given by*

$$P(z,w) = P(a,w) + Re\left[ik(z,w)\overline{T(w)}\right],$$

where a is a fixed point in Ω and $k(z,w)$ is a holomorphic antiderivative in the z variable of the Bergman kernel $K(z,w)$ associated to Ω that vanishes at $z=a$.

The second theorem shows that, in some sense, the Bergman kernel can be viewed as the kernel function for the Dirichlet-to-Neumann map, which is the mapping that takes a smooth function on the boundary of Ω to the normal derivative of its harmonic extension to Ω. This theorem does not require the domain to be simply connected.

Theorem 2.2. *Suppose Ω is a bounded finitely connected domain with C^∞ smooth boundary Ω. The Dirichlet-to-Neumann map applied to a C^∞ smooth real valued function φ on the boundary is given by the function on the boundary*

$$Re\,[h(z)T(z)]$$

where the boundary values of h are gotten from the holomorphic function

$$h(z) = \int_{w\in b\Omega} K(z,w)\varphi(w)\ d\bar{w}$$

on Ω by letting z tend to the boundary.

We note here that it is an easy matter to see that the function h in Theorem 2.2 is in $C^\infty(\overline{\Omega})$. Indeed, if we let Φ denote a C^∞ extension of φ to $\overline{\Omega}$, an application of the complex Green's theorem yields

$$h(z) = \iint_\Omega K(z,w)\frac{\partial\Phi(w)}{\partial w} dw\wedge d\bar{w},$$

and we observe that h is $-2i$ times the Bergman projection of $\frac{\partial\Phi(w)}{\partial w}$. Since the Bergman projection preserves functions in $C^\infty(\overline{\Omega})$ (see [1, p. 73] for a proof), we conclude that h is smooth. Fun fact: if the Bergman projection preserves smooth functions, so does the Poisson extension operator. The reverse is also true (see [1, p. 87]).

It feels like these theorems should have been known to the pioneers in the subject like Bergman, Schiffer, Nehari, Szegő, or Grunsky, at least in the simply connected setting, but I haven't found references to them. The theorems rose to the surface after I felt like I must be missing something obvious in my study of the Dirichlet-to-Neumann map in area quadrature domains in [5, 6]. I will give two proofs of the theorems in the simply connected case. The first follows the path I used to discover the theorems and has as a byproduct Theorem 4.1 in §4, which is a rather intriguing result by itself. The first also connects the Poisson kernel to the Szegő kernel to the Bergman kernel and generalizes nicely to the multiply connected setting, where it leads to Theorem 5.1 about the complexity of the Poisson kernel. The second is perhaps the way the theorems should have been discovered.

We turn now to summarizing the definitions, formulas, and properties we will need to state the theorems carefully and prove them.

The Szegő kernel $S(z,w)$ and the Bergman kernel $K(z,w)$ are holomorphic in z and antiholomorphic in w. They extend C^∞ smoothly to $\overline{\Omega}\times\overline{\Omega}$ minus the boundary diagonal and are Hermitian symmetric. They are nonvanishing in the simply connected case. The Garebedian kernel $L(z,w)$ is holomorphic in z and w with a singularity of the form

$$\frac{1}{2\pi(z-w)}$$

near $z=w$. It extends smoothly to $\overline{\Omega}\times\overline{\Omega}$ minus the diagonal and satisfies $L(z,w)=-L(w,z)$. The Garabedian kernel is also nonvanishing on $\Omega\times\overline{\Omega}$ minus the diagonal in $\Omega\times\Omega$, even in the finitely connected setting. Hence, when $w\in\Omega$, $1/L(z,w)$ can be viewed as a holomorphic function of z on Ω with a simple zero at the point w in Ω.

Let $A^\infty(\Omega)$ denote the space of holomorphic functions on Ω in $C^\infty(\overline{\Omega})$.

When a is a fixed point in Ω, we will use the shorthand notation

$$S_a(z) := S(z,a) \qquad \text{and} \qquad L_a(z) := L(z,a).$$

Similarly, $K_a(z) := K(z,a)$.

The Green's function $G(z,w)$ associated to Ω is the harmonic function of z on $\Omega-\{w\}$ that has zero boundary values and singular part $-\ln|z-w|$ near $z=w$. The Poisson kernel $P(z,w)$ is given by

$$P(z,w)=\frac{1}{2\pi}\frac{\partial}{\partial n_w}G(z,w)$$

for $z \in \Omega$ and $w \in b\Omega$, where $\partial/\partial n_w$ represents the outward normal derivative in the w variable. The Bergman kernel is related to the Green's function via

$$K(z,w) = \frac{-2}{\pi}\frac{\partial^2 G(z,w)}{\partial z \partial \bar{w}}. \tag{2.1}$$

Another important kernel related to the Bergman kernel is the complementary kernel $\Lambda(z,w)$ given by

$$\Lambda(z,w) = \frac{-2}{\pi}\frac{\partial^2 G(z,w)}{\partial z \partial w}.$$

The Bergman kernel and its complementary kernel are related via

$$K(z,w)\overline{T(w)} = -\Lambda(z,w)T(w)$$

when $z \in \Omega$ and $w \in b\Omega$. This identity reveals that, when $a \in \Omega$, the holomorphic 1-form $K_a\, dz$ extends to the double of Ω as a meromorphic 1-form as the conjugate of $-\Lambda_a\, dz$ on the reflected side. We will let κ_a denote the extended 1-form on the double. This meromorphic 1-form has a residue free double pole at the reflection of $a \in \Omega$ in the double, since $\Lambda(z,a)$ has a residue free double pole at $z = a$. In the simply connected case, this implies that κ_a is an exact form, i.e., that $\kappa_a = dg$ where g is a meromorphic function on the double of Ω. One byproduct of our work will be that we can express g simply in terms of the Szegő and Garabedian kernels (see the remarks after Theorem 4.1). The 1-form κ_a also extends in this way to the double in the multiply connected setting and we will explore some interesting, but more complicated, analogous results in §5.

A harmonic function u on Ω that extends smoothly to the boundary can be expressed locally near a boundary point as $h+\overline{H}$, where h and H are holomorphic functions that extend smoothly to the boundary. The normal derivative of h at a boundary point w is $-ih'(w)T(w)$ and the normal derivative of $\overline{H}$ is therefore $i\,\overline{H'(w)T(w)}$. Hence,

$$\frac{\partial u}{\partial n} = -ih'(w)T(w) + i\overline{H'(w)T(w)} = -i\frac{\partial u}{\partial w}T(w) + i\frac{\partial u}{\partial \bar{w}}\overline{T(w)}.$$

We may express the normal derivative of the Green's function in this way. If we also note that

$$0 \equiv \frac{\partial G}{\partial w}T(w) + \frac{\partial G}{\partial \bar{w}}\overline{T(w)}$$

on the boundary because $G(z,w)$ is zero when $z \in \Omega$ and w is in the boundary, we obtain that that the Poisson kernel can be expressed in two ways in terms of the Green's function as follows.

$$\begin{aligned} P(z,w) &= \frac{1}{2\pi}\frac{\partial}{\partial n_w}G(z,w) \\ &= \frac{1}{2\pi}(-2i)\frac{\partial G}{\partial w}(z,w)T(w) = \frac{1}{2\pi}(2i)\frac{\partial G}{\partial \bar{w}}(z,w)\overline{T(w)}. \end{aligned} \tag{2.2}$$

If we now take the normal derivative again in the z variable and use (2.1), we obtain

$$\frac{\partial}{\partial n_z}\frac{\partial}{\partial n_w}G(z,w) = (-2i)(2i)T(z)\frac{\partial^2 G}{\partial z\partial\bar{w}}(z,w)\overline{T(w)} = -2\pi T(z)K(z,w)\overline{T(w)}$$

when z and w are in the boundary and $z \neq w$.

The Szegő kernel is related to the Garabedian kernel via

$$\overline{S(z,a)} = \frac{1}{i}L(z,a)T(z) \tag{2.3}$$

when $a \in \Omega$ and $z \in b\Omega$.

3. The Poisson kernel and the Szegő projection

We assume that Ω is a bounded *simply connected* domain with C^∞ smooth boundary. The starting point for the results of this paper is the following observation from [2] that describes how the Poisson extension operator can be described in terms of the Szegő projection.

Given a function φ in $C^\infty(b\Omega)$, its Poisson extension u is in $C^\infty(\overline{\Omega})$, and there are holomorphic functions h and H on Ω in $C^\infty(\overline{\Omega})$ such that $u = h + \overline{H}$ on Ω, and $\varphi = h + \overline{H}$ on $b\Omega$. Pick any point $a \in \Omega$ and note that we can assume that $H(a) = 0$ (after adding $\overline{H(a)}$ to h and subtracting $H(a)$ from H). Multiply $\varphi = h + \overline{H}$ by S_a and use the identity (2.3) in the form $S_a(z) = i\overline{L_a(z)T(z)}$ when $z \in b\Omega$ to obtain

$$S_a\varphi = S_a h + i\overline{L_a HT} \tag{3.1}$$

on $b\Omega$. Notice that L_aH is holomorphic on Ω because the zero of H at a cancels the pole of L_a at a. Also, the Cauchy theorem reveals that functions of the form $\overline{GT}$, where G is in $A^\infty(\Omega)$, are orthogonal to the Hardy space. Indeed, if g and G are both in $A^\infty(\Omega)$, then

$$\langle g, \overline{GT}\,\rangle = \int_\gamma g\,GT\ ds = \int_\gamma gG\ dz = 0,$$

and since $A^\infty(\Omega)$ is dense in the Hardy space, the result follows. Hence, letting P denote the Szegő projection and applying it to (3.1), we find that

$$S_a h = P(S_a\varphi) \quad \text{and so} \quad h = \frac{P(S_a\varphi)}{S_a}.$$

Take the conjugate of (3.1) and multiply by $\overline{T}$ and use (2.3) in the form $\overline{S_a(z)} = -iL_a(z)T(z)$ to obtain

$$-iL_a\overline{\varphi} = -iL_aH + \overline{S_a hT}, \tag{3.2}$$

on $b\Omega$, another orthogonal decomposition. This time we see that $-iL_aH = P(-iL_a\overline{\varphi}\,)$, and so

$$L_aH = P(L_a\,\overline{\varphi}\,) \quad \text{and} \quad H = \frac{P(L_a\,\overline{\varphi}\,)}{L_a}.$$

Consequently, the harmonic extension of φ to Ω is given by $h + \overline{H}$ where

$$h = \frac{P(S_a\varphi)}{S_a} \qquad \text{and} \qquad H = \frac{P(L_a\overline{\varphi})}{L_a}.$$

Hence, as noted in [2], the Poisson kernel is given by

$$P(z,w) = \frac{S(z,w)S(w,a)}{S(z,a)} + \frac{\overline{S(z,w)L(w,a)}}{\overline{L(z,a)}}. \tag{3.3}$$

Also noted in [2], letting $z = a$ in this formula shows that

$$P(a,w) = \frac{|S(w,a)|^2}{S(a,a)},$$

i.e., the well-known fact that the Poisson kernel is equal to the Poisson-Szegő kernel in a simply connected domain. The main results of this paper center around showing that the functions

$$\mathcal{S}(z,w,a) := \frac{S(z,w)S(w,a)}{S(z,a)} \qquad \text{and} \qquad \mathcal{L}(z,w,a) := \frac{S(z,w)L(w,a)}{L(z,a)}$$

are related to the Bergman kernel in a rather straightforward manner.

We will also study the Dirichlet-to-Neumann map, which is the map that takes the function φ above to the normal derivative of its Poisson extension u, which we have seen to be equal to $-ih'T + i\overline{H'T}$.

4. From Poisson kernel to Szegő kernel to Bergman kernel

We continue to assume that Ω is a bounded simply connected domain in the plane with C^∞ smooth boundary, and that all the definitions and notations of §2 are in place.

We now claim that, assuming w and a are fixed points in Ω with $w \neq a$, that

$$s(z) := \frac{d}{dz}\mathcal{S}(z,w,a) = \frac{1}{2}\frac{S(w,a)}{\overline{L(w,a)}}K(z,w).$$

We will prove this by showing that the $L^2(\Omega)$ inner product of a holomorphic function $H(z)$ in $C^\infty(\overline{\Omega})$ with $s(z)$ is equal to $cH(w)$ where $c = \frac{1}{2}\overline{S(w,a)}/L(w,a)$. Since such functions H are dense in the Bergman space, it follows that $s(z) = \bar{c}K(z,w)$. Here is the computation.

$$\langle H, s\rangle = \iint_\Omega H(z)\,\overline{s(z)}\,\left(\frac{-1}{2i}\right) dz \wedge d\bar{z} = \frac{1}{2i}\int_{b\Omega} H(z)\frac{\overline{S(z,w)S(w,a)}}{\overline{S(z,a)}}\ dz,$$

by the complex Green's identity. Note that (2.3) reveals that

$$\frac{\overline{S(z,w)S(w,a)}}{\overline{S(z,a)}} = \frac{\overline{T(z)S(z,w)S(w,a)}}{\overline{T(z)S(z,a)}} = \frac{L(z,w)\overline{S(w,a)}}{L(z,a)}$$

when z is in the boundary. Hence, continuing the calculation of the integral yields

$$= \frac{1}{2i}\overline{S(w,a)}\int_{b\Omega} H(z)\frac{L(z,w)}{L(z,a)}\ dz.$$

Note that $L(z,w)$ has a single simple pole at $z=w$ with residue $1/(2\pi)$, and $H(z)/L(z,a)$ can be identified as holomorphic on Ω with a zero at a because of the pole of $L(z,a)$ at $z=a$ and the fact that $L(z,a)$ is nonvanishing on $\overline{\Omega}-\{a\}$. Hence, the residue theorem shows that this last integral is equal to

$$\frac{1}{2i}\overline{S(w,a)}(2\pi i)\frac{1}{2\pi}\frac{H(w)}{L(w,a)},$$

and our claim is proved.

A very parallel calculation shows that, if $w \neq a$, then

$$\ell(z) := \frac{d}{dz}\mathcal{L}(z,w,a) = \frac{1}{2}\frac{\overline{L(w,a)}}{\overline{S(w,a)}}K(z,w).$$

Indeed, assuming w and a are fixed points in Ω, $w \neq a$, then the $L^2(\Omega)$ inner product of a holomorphic function $H(z)$ in $C^\infty(\overline{\Omega})$ with $\ell(z)$ is given by

$$\langle H, \ell\rangle = \frac{1}{2i}\int_{b\Omega} H(z)\frac{\overline{S(z,w)L(w,a)}}{\overline{L(z,a)}}\ dz,$$

by the complex Green's identity. Note that (2.3) reveals that

$$\frac{\overline{S(z,w)L(w,a)}}{\overline{L(z,a)}} = \frac{\overline{T(z)S(z,w)S(w,a)}}{\overline{T(z)L(z,a)}} = \frac{L(z,w)\overline{L(w,a)}}{S(z,a)}.$$

Hence, continuing the calculation of the integral yields

$$= \frac{1}{2i}\overline{L(w,a)}\int_{b\Omega} H(z)\frac{L(z,w)}{S(z,a)}\ dz,$$

and, because $L(z,w)$ has a single simple pole at $z=w$ with residue $1/(2\pi)$, and $S(z,a)$ is nonvanishing and smooth up to the boundary, the residue theorem shows that this last integral is equal to

$$\frac{1}{2i}\overline{L(w,a)}(2\pi i)\frac{1}{2\pi}\frac{H(w)}{S(w,a)},$$

and our claim is proved.

We now let w tend to the boundary and note that (2.3) shows that

$$\frac{S(w,a)}{\overline{L(w,a)}} = i\overline{T(w)}$$

when w is in the boundary. Thus,

$$\frac{d}{dz}\mathcal{S}(z,w,a) = \frac{i}{2}K(z,w)\overline{T(w)}$$

when w is in the boundary. Similarly,

$$\frac{d}{dz}\mathcal{L}(z,w,a)=\frac{i}{2}K(z,w)\overline{T(w)}.$$

Note that

$$\mathcal{L}(a,w,a)=0$$

because of the pole of $L(z,a)$ at $z=a$. Hence, letting $z=a$ in the expression

$$P(z,w)=\mathcal{S}(z,w,a)+\mathcal{L}(a,w,a)$$

for the Poisson kernel yields that, on a simply connected domain, the Poisson kernel $P(a,w)$ is equal to

$$P(a,w)=\mathcal{S}(a,w,a)=\frac{|S(a,w)|^2}{S(a,a)},$$

which, as noted earlier, is the Poisson-Szegő kernel at a. We now can take antiderivatives to obtain

$$\mathcal{S}(z,w,a)=\frac{1}{2}ik(z,w)\overline{T(w)}+P(a,w) \qquad \text{and} \qquad \mathcal{L}(z,w,a)=\frac{1}{2}ik(z,w)\overline{T(w)}.$$

Theorem 2.1 is proved. Theorem 2.2 now follows by writing out the integrals in the new Poisson formula for solving the Dirichlet problem as $h(z)+\overline{H(z)}$ and computing the normal derivative via $-ih'T+i\overline{H'T}$. Differentiating under the integral sign when z is in Ω and letting z tend to the boundary completes the proof.

Before we proceed to the multiply connected case, it is worth collecting the formulas we have derived in the statement of a theorem and pointing out an interesting consequences of one of them.

Theorem 4.1. *If Ω is a bounded C^∞ smooth simply connected domain in the plane and a and w are points in Ω, then*

$$\frac{\partial}{\partial z}\left(\frac{S(z,w)S(w,a)}{S(z,a)}\right)=\frac{1}{2}\frac{S(w,a)}{\overline{L(w,a)}}K(z,w)$$

$$\frac{\partial}{\partial z}\left(\frac{S(z,w)L(w,a)}{L(z,a)}\right)=\frac{1}{2}\frac{L(w,a)}{\overline{S(w,a)}}K(z,w).$$

Furthermore, if w is in the boundary then,

$$\frac{\partial}{\partial z}\left(\frac{S(z,w)S(w,a)}{S(z,a)}\right)=\frac{i}{2}K(z,w)\overline{T(w)}$$

$$\frac{\partial}{\partial z}\left(\frac{S(z,w)L(w,a)}{L(z,a)}\right)=\frac{i}{2}K(z,w)\overline{T(w)}.$$

If both a and w are in the boundary and $a\neq w$, then

$$\frac{S(z,w)L(w,a)}{L(z,a)}$$

is a biholomorphic mapping of Ω onto the right half plane that sends the point a to zero and the point w to the point at infinity, and so the last formula expresses the boundary values of the Bergman kernel in terms of such an object.

The proof of Theorem 4.1 is complete except for the last statement about the quotient being a biholomorphic map to the right half plane, which is proved in [7]

We mentioned in §2 that the meromorphic 1-form κ_w is an exact meromorphic 1-form on the double of Ω. This is true even when w is a point in the boundary. The identity

$$\frac{\partial}{\partial z}\left(\frac{S(z,w)L(w,a)}{L(z,a)}\right)=\frac{i}{2}K(z,w)\overline{T(w)}$$

shows that the function $G(z)$ which is equal to $S(z,w)L(w,a)/L(z,a)$ on Ω is such that $dG=\frac{i}{2}\overline{T(w)}K_w dz$ on Ω. Identity (2.3) can be used multiple times to show that, when z and w are in the boundary, $S(z,w)L(w,a)/L(z,a)$ is equal to the conjugate of $H(z):=-S(z,w)S(w,a)/S(z,a)$. Hence, the meromorphic function $g(z)$ given by $-2iT(w)G(z)$ on Ω and $-2iT(w)$ times the conjugate of $H(z)$ on the reflected side of Ω in the double is such that $dg=\kappa_w$. Note that g has a single simple pole at the point $w\in b\Omega$ in the double.

We isolate part of this last argument as a theorem because we will use it again in the multiply connected setting.

Theorem 4.2. *Suppose Ω is a bounded C^∞ smooth finitely connected domain. If a is a point in Ω and z and w are points on the boundary, then*

$$\frac{S(z,w)L(w,a)}{L(z,a)}$$

is equal to the conjugate of

$$-\frac{S(z,w)S(w,a)}{S(z,a)},$$

and this reveals that $S(z,w)L(w,a)/L(z,a)$ extends to be a meromorphic function $G(z)$ on the double of Ω when w is a boundary point of Ω. Similarly, $S(z,w)S(w,a)/S(z,a)$ also extends to the double as a meromorphic function of z when $w\in b\Omega$. If Ω is simply connected and $w\in b\Omega$, then the meromorphic one-form κ_w is equal to dg where $g(z)=-2iT(w)G(z)$ is meromorphic on the double with a single simple pole at the point $w\in b\Omega$.

5. Poisson, Szegő, Bergman kernel connections in the multiply connected case

We now consider the case of a bounded finitely connected domain Ω with C^∞ smooth boundary. Assume that Ω is n-connected and let γ_0 denote the outer boundary curve and γ_j, $j=1,\dots n-1$, denote the $n-1$ inner boundary curves, all parameterized in the standard sense. Let ω_j denote the harmonic function on Ω that has boundary values equal to one on γ_j and equal to zero on the other boundary curves, and let $F_j' = 2(\partial\omega_j/\partial z)$ denote the associated classical holomorphic functions whose matrix of periods on the $n-1$ inner boundary curves is nonsingular. (F_j' is not the derivative of a holomorphic function.) As we did in the simply connected case, we first review the relationship between the Poisson extension operator and the Szegő projection, as described in [2].

It is a well known fact that, given a holomorphic function H on Ω, there is a holomorphic function h on Ω and complex constants c_j such that

$$h' = H + \sum_{j=1}^{n-1} c_j F_j'.$$

The constants are uniquely determined by the periods of H on the inner boundary curves and h is unique up to an additive constant. If H is in $C^\infty(\overline{\Omega})$, then so is h. Given a harmonic function u in $C^\infty(\overline{\Omega})$, $\partial u/\partial z$ is holomorphic. Hence, there is a holomorphic function g in $C^\infty(\overline{\Omega})$ such that

$$g' = \frac{\partial u}{\partial z} + \sum_{j=1}^{n-1} \tilde{c}_j F_j'.$$

Now

$$g - u - 2\sum_{j=1}^{n-1} \tilde{c}_j \omega_j$$

is an antiholomorphic function $\overline{G}$. Hence, we have shown that u can be expressed as

$$u(z) = g(z) + \overline{G(z)} + \sum_{j=1}^{n-1} c_j \omega_j(z), \tag{5.1}$$

where g and G are holomorphic functions on Ω in $C^\infty(\overline{\Omega})$ and the c_j are complex constants. As shown in [2], it is possible to pick a point a in Ω such that the $n-1$ zeroes of $S_a(z)$ in Ω are simple zeroes (see also [1], Chapter 27). We may assume that $G(a) = 0$ by absorbing its value into g. We note that, in multiply connected domains, even though the Szegő kernel has zeroes, the Garabedian kernel $L(z,a)$ is nonvanishing for $z \in \overline{\Omega} - \{a\}$, and the Szegő kernel $S(z,w)$ is nonvanishing when $z \in b\Omega$ and $w \in \Omega$.

Let φ denote the function on $b\Omega$ representing the boundary values of u and multiply (5.1) by S_a, restrict to the boundary, and use (2.3) to obtain

$$S_a\varphi = S_a g - i\overline{L_a G T} + \sum_{j=1}^{n-1} c_j S_a \omega_j.$$

Take the Szegő projection and note that $\overline{L_a G T}$ is orthogonal to the Hardy space to see that

$$g = \frac{P(S_a\varphi)}{S_a} - \sum_{j=1}^{n-1} c_j \frac{P(S_a\omega_j)}{S_a}.$$

Let $\{a_j\}_{j=1}^{n-1}$ denote the simple zeroes of S_a. Because g is holomorphic on Ω, there can be no poles at the zeroes of the $S(z,a)$, and so

$$P(S_a\varphi)(a_k) = \sum_{j=1}^{n-1} c_j P(S_a\omega_j)(a_k) \tag{5.2}$$

for $k = 1, \ldots, n-1$. We must conclude that this system of equations for c_j has a solution no matter what the function φ is. We could choose φ so that the left hand side of the system (5.2) is equal to one at a_j and equal to zero at the a_k with $k \neq j$. Since we could do this for each a_j, it follows that the $(n-1)\times(n-1)$ matrix $[A_{kj}]$ where $A_{jk} = P(S_a\omega_j)(a_k)$ is nonsingular. (Indeed, since S_a is nonvanishing on the boundary, we could choose a φ so that $S_a\varphi$ has the boundary values of any given complex polynomial. The Szegő projection of $S_a\varphi$ would be equal to that polynomial, and we could use Lagrange interpolation at the points a_k to come up with a system of equations that shows that the matrix must be nonsingular. That the matrix is nonsingular was also proved in [2], see also [1, p. 111].)

Hence, (5.2) determines the constants c_j. Next, take the conjugate of (5.1), multiply by L_a, and repeat the process to obtain

$$L_a\overline{\varphi} = L_aG + i\overline{S_a gT} + \sum_{j=1}^{n-1} \bar{c}_j L_a\omega_j,$$

and project to see that

$$G = \frac{P(L_a\,\overline{\varphi}\,)}{L_a} - \sum_{j=1}^{n-1} \bar{c}_j \frac{P(L_a\omega_j)}{L_a}.$$

We now face the task to recognize the quotients of Szegő and Garabedian kernels appearing as kernels in these formulas as derivatives of something like we did in the simply connected case (the part of this paper that can be considered to be new). The matter is complicated by the fact that $S(z,a)$ has $n-1$ simple zeroes at $\{a_1,\ldots,a_{n-1}\}$. The easiest term to analyze is, assuming $w \neq a$,

$$\ell(z) := \frac{d}{dz}\mathcal{L}(z,w,a)$$

where

$$\mathcal{L}(z,w,a) = \frac{S(z,w)L(w,a)}{L(z,a)}.$$

Indeed, assuming w and a are fixed points in Ω, $w \neq a$, then the $L^2(\Omega)$ inner product of a holomorphic function $H(z)$ in $C^\infty(\overline{\Omega})$ with $\ell(z)$ is given by

$$\langle H, \ell\rangle = \frac{1}{2i}\int_{b\Omega} H(z)\overline{\frac{S(z,w)L(w,a)}{L(z,a)}}\ dz,$$

by the complex Green's identity. As before, the identity (2.3) reveals that

$$\frac{\overline{S(z,w)L(w,a)}}{\overline{L(z,a)}}=\frac{\overline{T(z)S(z,w)S(w,a)}}{\overline{T(z)L(z,a)}}=\frac{L(z,w)\overline{L(w,a)}}{S(z,a)}.$$

Hence, continuing the calculation of the integral yields

$$=\frac{1}{2i}\overline{L(w,a)}\int_{b\Omega}H(z)\frac{L(z,w)}{S(z,a)}\,dz,$$

and, because $L(z,w)$ has a single simple pole at $z=w$ with residue $1/(2\pi)$, and $S(z,a)$ is smooth up to the boundary and nonvanishing on the boundary, the residue theorem shows that this last integral is equal to

$$\frac{1}{2i}\overline{L(w,a)}(2\pi i)\left[\frac{1}{2\pi}\frac{H(w)}{S(w,a)}+\sum_{j=1}^{n-1}\frac{H(a_j)L(a_j,w)}{S'(a_j,a)}\right]$$

where

$$S'(a_j,a)=\frac{\partial}{\partial z}S(z,a)\bigg|_{z=a_j}$$

are nonzero constants (because the zeroes are simple). It now follows that

$$\ell(z)=\frac{1}{2}\frac{\overline{L(w,a)}}{\overline{S(w,a)}}K(z,w)+\pi L(w,a)\sum_{j=1}^{n-1}\frac{\overline{L(a_j,w)}}{\overline{S'(a_j,a)}}K(z,a_j). \tag{5.3}$$

Finally, letting w go to the boundary and using the identities (2.3) and

$$L(a_j,w)=-L(w,a_j)=-i\,\overline{S(w,a_j)T(w)}$$

when $w\in b\Omega$, we obtain that

$$\ell(z)=\frac{i}{2}K(z,w)\overline{T(w)}+i\pi\sum_{j=1}^{n-1}\frac{L(w,a)S(w,a_j)T(w)}{\overline{S'(a_j,a)}}K(z,a_j) \tag{5.4}$$

when $w\in b\Omega$ and $a\in\Omega$. The functions $L(w,a)S(w,a_j)$ appearing here are rather interesting. Let $\mathcal{F}'$ denote the $n-1$ dimensional linear span of the functions F_j', $j=1,\dots,n-1$. Schiffer [13] showed that $L(w,a)S(w,a_j)$, $j=1,\dots,n-1$, form a basis for $\mathcal{F}'$. So do the functions $S(w,a)L(w,a_j)$, $j=1,\dots,n-1$, since $F_j'T=-\overline{F_j'T}$ on the boundary and (2.3) yields that

$$L(w,a)S(w,a_j)T(w)=-\overline{S(w,a)L(w,a_j)T(w)}$$

on the boundary (see [1, p. 98] for another treatment of these facts). Notice that the zero of $S(w,a_j)$ at $w=a$ cancels the pole of $L(w,a)$ at $w=a$ and the pole of $L(w,a_j)$ at $w=a_j$ is canceled by the zero of $S(w,a)$ at a_j. We may rewrite (5.4) in the form

$$\ell(z)=\frac{i}{2}K(z,w)\overline{T(w)}-i\pi\sum_{j=1}^{n-1}\frac{\overline{S(w,a)L(w,a_j)T(w)}}{\overline{S'(a_j,a)}}K(z,a_j). \tag{5.5}$$

We define normalized basis functions u_j for $\mathcal{F}'$ via

$$u_j(w) := 2\pi \frac{S(w,a)L(w,a_j)}{S'(a_j,a)}.$$

These basis functions will show up in many of our formulas and are noteworthy because

$$u_j(a_j) = 1 \quad \text{and} \quad u_j(a_k) = 0 \text{ for } j \neq k.$$

We can now shorten (5.5) to a pleasing

$$\ell(z) = \frac{i}{2} K(z,w)\overline{T(w)} - \frac{i}{2} \sum_{j=1}^{n-1} \overline{u_j(w)T(w)} K(z,a_j). \tag{5.6}$$

An interesting consequence of the expression for ℓ in (5.3) when w is in Ω can be obtained by integrating around an inner boundary curve in the z variable and using the well known formula

$$\int_{\gamma_k} K(z,w)\ dz = -i\,\overline{F_k'(w)}.$$

Since ℓ is the complex derivative of something, we obtain

$$0 = -\frac{i}{2} \frac{L(w,a)}{\overline{S(w,a)}} \overline{F_k'(w)} - \pi i L(w,a) \sum_{j=1}^{n-1} \frac{\overline{L(a_j,w)}}{\overline{S'(a_j,a)}} \overline{F_k'(a_j)},$$

and dividing by $iL(w,a)$, multiplying by $\overline{S(w,a)}$, and taking the complex conjugate yields

$$F_k'(w) = -2\pi \sum_{j=1}^{n-1} \frac{S(w,a)L(a_j,w)}{S'(a_j,a)} F_k'(a_j).$$

Since $L(a_j,w) = -L(w,a_j)$, we obtain that

$$F_k'(w) = 2\pi \sum_{j=1}^{n-1} \frac{F_k'(a_j)}{S'(a_j,a)} S(w,a)L(w,a_j).$$

(Note that the formula is valid at the zeroes $w = a_m$ because $L(w,a_m)$ has a simple pole at a_m with residue $1/(2\pi)$ and $S(w,a)$ has simple zeroes at the a_j with $j \neq m$.) This formula confirms the result of Schiffer mentioned above that says that the functions $\{S(w,a)L(w,a_j)\}_{j=1}^{n-1}$ form a basis for $\mathcal{F}'$.

We next note that, since $\ell(z)$ is a derivative for each fixed w in Ω, identity (5.3) yields that the periods of $\ell(z)$ must all be zero. Furthermore, since the complex linear span of $K(z,w)$ as w ranges over Ω is dense in $A^\infty(\Omega)$, it follows that the matrix of periods of the functions $K(z,a_j)$, $j = 1,\ldots,n-1$, must be nonsingular. Hence, the coefficients in front of the functions $K(z,a_j)$ in identities (5.3) and (5.4) are completely determined by the periods of the $K(z,w)$ term.

We now focus attention on equation (5.6) for fixed w in the boundary. Let $k(z,w)$ be a complex antiderivative of the right hand side of (5.6) in the z variable that vanishes at $z = a$. (We remark that $k(z,w)$ can be defined via a path integral that starts at a and ends at z.) The definition of ℓ and the fact that the pole of $L(z,a)$ in the denominator makes $\mathcal{L}(z,w,a)$ vanish at $z=a$ allows us to conclude that

$$\frac{S(z,w)L(w,a)}{L(z,a)}=k(z,w).$$

Next, we want to simplify the functions

$$H_m(z):=\int_{w\in\gamma_m}\frac{S(z,w)L(w,a)}{L(z,a)}\ ds$$

that appear in the solution to the Dirichlet problem in the form $P(L_a\omega_m)/L_a$. Notice that, by differentiating under the integral sign and using (5.6),

$$\begin{aligned}H'_m(z)&=\int_{w\in\gamma_m}\left[\frac{i}{2}K(z,w)\overline{T(w)}-\frac{i}{2}\sum_{j=1}^{n-1}\overline{u_j(w)T(w)}K(z,a_j)\right]\ ds\\&=\frac{i}{2}F'_m(z)+\sum_{j=1}^{n-1}c_{mj}K(z,a_j),\end{aligned}\tag{5.7}$$

where c_{mj} is a nonsingular matrix of periods. Since the left hand side of this equation is a derivative, the coefficients c_{mj} are completely determined by the periods of the F'_m. Define F_m to be a complex antiderivative of the right hand side of (5.7) that vanishes at $z=a$. (Once again, we remark that F_m could be defined by a path integral. We also remark that F_m is not an antiderivative of F'_m, since the periods of the later are not all zero.) Since H_m vanishes at a, again by virtue of the pole of $L(z,a)$ at $z=a$, we conclude that $F_m(z)$ is equal to $H_m(z)$.

We could continue to express all the other functions that appear as kernels in the solution to the Dirichlet problem. One way to do this would be to use Theorem 4.2. Another way would be to note that the Ahlfors map $f_a:\Omega\to D_1(0)$, which is an n-to-one branched proper holomorphic mapping from Ω onto the unit disc with $f_a(a)=0$, is given by

$$f_a(z)=\frac{S(z,a)}{L(z,a)},$$

and it follows that

$$\frac{S(z,w)S(w,a)}{S(z,a)}=\frac{S(z,w)L(w,a)}{L(z,a)}\ \frac{f_a(w)}{f_a(z)},$$

and many of the properties of the other terms in the solution to the Dirichlet problem can be read off. However, it turns out we have done enough now to be able to write out the Poisson kernel completely. The solution is further analyzed in [2] and the Poisson kernel is expressed there as

$$P(z,w) = 2\mathrm{Re}\left[\frac{S(z,w)L(w,a)}{L(z,a)} - \sum_{j=1}^{n-1}\mu_j(w)\int_{\zeta\in\gamma_j}\frac{S(z,\zeta)L(\zeta,a)}{L(z,a)}\,ds\right] + \frac{|S(w,a)|^2}{S(a,a)} + \sum_{j=1}^{n-1}(\omega_j(a)-\lambda_j(a))\mu_j(w), \tag{5.8}$$

where $\mu_j(w)$ is a *real valued* fixed linear combination

$$\mu_j(w) = \sum_{k=1}^{n-1} A_{jk}F_k'(w)T(w),$$

where the coefficients A_{jk} are determined by the system

$$\int_{\gamma_j}\mu_k\,ds = \delta_{jk}.$$

The constant $\lambda_j(a)$ is given by

$$\lambda_j(a) = \int_{\gamma_j}\frac{|S(w,a)|^2}{S(a,a)}\,ds.$$

Using our results above, we may write

$$P(z,w) = 2\mathrm{Re}\left[k(z,w) - \sum_{j=1}^{n-1}\mu_j(w)F_j(z)\right] + \frac{|S(w,a)|^2}{S(a,a)} + \sum_{j=1}^{n-1}(\omega_j(a)-\lambda_j(a))\mu_j(w),$$

As noted in [2], letting $z = a$ in the formula (5.8) for the Poisson kernel and observing that the pole of $L(z,a)$ at $z = a$ makes the top line vanish reveals that, on a multiply connected domain, the Poisson kernel and the Poisson-Szegő kernel are related via

$$P(a,w) = \frac{|S(w,a)|^2}{S(a,a)} + \sum_{j=1}^{n-1}(\omega_j(a)-\lambda_j(a))\mu_j(w).$$

I, the author, have been obsessed over the years with revealing the complexity of the Poisson kernel in finitely connected domains (see [3, 4, 5]) and how it relates to questions about the complexity of solutions to the Dirichlet problem originating in [12] and culminating in [8]. Identity (5.8) can be used to shed light on this subject. Indeed, it is proved in [2] that the Szegő kernel can be expressed via

$$S(z,w) = \frac{c_0S(z,a)\overline{S(w,a)} + \sum_{i,j=1}^{n-1}c_{ij}S(z,a_i)\,\overline{S(w,a_j)}}{(1-f_a(z)\,\overline{f_a(w)})}, \tag{5.9}$$

where $f_a(z)$ is the Ahlfors map associated to a and where the coefficients are constants. Consequently, $k(z,w) = S(z,w)L(w,a)/L(z,a)$ can be decomposed into a sum of terms involving terms like $S(z,b)/L(z,a)$, which extend meromorphically to the double of Ω as functions of z for fixed b and a. Indeed, identity (2.3) shows that

$$\frac{S(z,b)}{L(z,a)} = \frac{\overline{L(z,b)}}{\overline{S(z,a)}} \quad \text{for } z \in b\Omega,$$

and this shows how to define the extension to the reflected side of Ω in the double. Similarly, terms like $S(w,b)/S(w,a)$ extend meromorphically to the double because they are equal to the conjugate of $L(w,b)/L(w,a)$ on the boundary. It is shown in [4] that the field of meromorphic functions on the double of Ω is generated by any one given Ahlfors map $f_a(z)$ and another Ahlfors map $f_b(z)$ associated to a point $b \neq a$ in Ω. Hence, when we divide identity (5.9) by $L(z,a)\overline{S(w,a)}$ and distribute the denominator through the sum, we may conclude that $S(z,w)/L(z,a)$ is a rational combination of $f_a(z)$, $f_b(z)$, $\overline{f_a(w)}$, $\overline{f_b(w)}$, and $\overline{S(w,a)}$. Since the Ahlfors map $f_a(w)$ is equal to $S(w,a)/L(w,a)$, it follows that $L(w,a) = S(w,a)/f_a(w)$, and we conclude that the principal term $S(z,w)L(w,a)/L(z,a)$ of the Poisson kernel is a rational combination of the same basic functions, $f_a(z)$, $f_b(z)$, $\overline{f_a(w)}$, $\overline{f_b(w)}$, and $\overline{S(w,a)}$. We next claim that $F_j'(w)/S(w,a)^2$ extends meromorphically to the double. Indeed,

$$F_j'(w)T(w) = -\overline{F_j'(w)T(w)} \quad \text{on } b\Omega,$$

and (2.3) yields that

$$S(w,a)^2T(w) = -\overline{L(w,a)^2T(w)} \quad \text{on } b\Omega.$$

Dividing the two equations yields that $F_j'(w)/S(w,a)^2$ is equal to the conjugate of $F_j'(w)/L(w,a)^2$ on the boundary, and this defines the claimed extension. We now conclude that $F_j'(w)$ is a rational combination of $f_a(w)$ and $f_b(w)$ times $S(w,a)^2$. Finally, since $L(w,a) = S(w,a)/f_a(w)$ and (2.3) yields that $T(w) = i\overline{S(w,a)}/L(w,a) = f_a(w)\overline{S(w,a)}/S(w,a)$ on $b\Omega$, it follows that $T(w)$ can also be expressed simply in terms of $f_a(w)$ and $S(w,a)$. We may now assert the following theorem.

Theorem 5.1. *Suppose Ω is a bounded finitely connected domain with C^∞ smooth boundary. There are two points a and b in Ω such that the Poisson kernel $P(z,w)$ associated to Ω is a rational combination of the two Alfors maps $f_a(z)$, $f_b(z)$, $f_a(w)$, $f_b(w)$, the Szegő kernel $S(w,a)$ associated to the point a, the functions $F_m(z)$, $m = 1, \dots, n-1$, and conjugates of all these functions.*

It is interesting that all the functions in this list are holomorphic functions of one variable and that the functions $F_m(z)$ occur as linear factors only in terms that are in $C^\infty(\overline{\Omega} \times b\Omega)$.

Given a bounded finitely connected domain with smooth boundary, there is a biholomorphic mapping to a nearby double quadrature domain Ω that is as C^∞ close to the identity map as desired as shown in [10]. It is also proved there that, on a smooth double quadrature domain, $S(z,a)$ extends meromorphically to the double of Ω, and consequently, $S(z,a)$ too is a rational combinations of the two

Ahlfors maps f_a and f_b that generate the field of meromorphic functions on the double. Hence, in this case, the Poisson kernel is a rational combination of just the functions $f_a(z)$, $f_b(z)$, $f_a(w)$, $f_b(w)$, the functions $F_m(z)$, $m = 1, \ldots, n-1$, and all their conjugates. This confirms the feeling that double quadrature domains should be thought of as replacements for the unit disc in the multiply connected setting via a "Riemann mapping theorem for multiply connected domains" that says smooth finitely connected domains are biholomorphic to nearby double quadrature domains.

One might have expected the harmonic measure functions ω_m to be required someplace in the formula. We will show momentarily that the ω_m can be thought of as being concealed in the functions F_m. We remark that, because the Poisson kernel is the normal derivative of the Green's function, we could further analyze the formulas to deduce that $\frac{\partial}{\partial w}G(z,w)$ is a rational combination of exactly the same functions.

It is traditional to call the $n-1$ periods along the inner boundary curves γ_j in the double the *alpha periods* and the periods around the handles the *beta periods*. The beta periods are traditionally described by taking cuts out of the domain from points on the inner curves connecting them to the outer boundary curve in such a way that the domain minus the nonintersecting cuts is simply connected. The β_j period is then defined as an integral that starts from the point in γ_j and follows the cut to the outer boundary, then travels back to the starting point along minus the cut curve in the reflected copy of Ω in the double. The $n-1$ alpha and the $n-1$ beta periods form a homology basis for the double. The identities $F_m'T = -\overline{F_m'T}$ and $K_aT = -\overline{\Lambda_aT}$ that hold on the boundary reveal that the holomorpic one-forms $F_m'dz$ extend to be holomorphic one-forms on the double and the holomorphic one-form K_adz extends to be a meromorphic one-form on the double with a residue free double pole at a in the reflected side of Ω in the double. The holomorphic one-form $H_m'dz$ where H_m' is given via (5.7) is such that it extends to the double as a meromorphic one-form whose alpha periods are zero, and whose beta periods are zero, except for one. Indeed, Schiffer and Spencer [14] showed that the beta periods of the extensions of K_adz are zero. The beta periods of $F_m'dz$ are zero, except for the one β_m that goes from γ_m to the outer boundary and comes back along the same path in the reflected side. The period of that one is 2. Because the poles of the extension of $H_m'dz$ are residue free, it follows that it is possible to analytically continue a germ of a holomorphic antiderivative $F_m = H_m$ of H_m' in Ω to the double as a mutivalued meromorphic function that jumps by 2 every time that β_m is traversed, and no jumps along the other curves in the homology basis. The function given by ω_m in Ω and $-\omega_m$ on the reflected side of Ω in the double is a harmonic function on the double minus the curve γ_m that has this same jumping behavior, and hence F_m can be viewed as contributing the same type of behavior. It is interesting to note that $F_m - \omega_m$ is a complex valued harmonic function on Ω that extends harmonically to the double minus the zeroes of $S(z,a)$ in the reflected side, where it has pole-like singularities.

The functions appearing in (5.6) are rather interesting from the point of view of the last paragraph. When $w \in b\Omega$, the 1-form

$$\left[\frac{i}{2}K(z,w)\overline{T(w)} - \frac{i}{2}\sum_{j=1}^{n-1}\overline{u_j(w)T(w)}K(z,a_j)\right]\ dz$$

extends to the double as the conjugate of

$$-\left[\frac{i}{2}\Lambda(z,w)\overline{T(w)} - \frac{i}{2}\sum_{j=1}^{n-1}\overline{u_j(w)T(w)}\Lambda(z,a_j)\right]\ dz$$

on the reflected side of Ω in the double. Its alpha and beta periods are all zero and its poles are all double residue free poles. Hence, it is equal to dg for some meromorphic function g on the double with simple poles at w and the reflections of the points a_j in the double. We have shown that

$$g(z) = \frac{S(z,w)L(w,a)}{L(z,a)}$$

on the Ω side. Theorem 4.2 defines g on the reflected side. If both w and a are allowed to be on the boundary, then these identities extend by continuity. It is shown in [7] that, in this case, such a function $g(z)$ on Ω is a proper n-to-one branched mapping of Ω onto the right half plane. It is interesting to note that any proper holomorphic map from Ω to the right half plane can be expressed as a positive linear combination of such maps plus a pure imaginary constant, and so the derivatives of proper holomorphic mappings of Ω to the right half plane can be expressed in terms of the Bergman kernel.

It will be interesting to see if any of these new relationships between the kernel functions shed light on the proposed alternate proof of Hejhal's theorem about the relationship between the Bergman and Szegő kernels in multiply connected domains in [9].

6. Alternative ways to think about Theorems 2.1 and 2.2

Let Ω denote a bounded finitely connected domain with C^∞ smooth boundary. We will first give a more direct proof of Theorem 2.2.

Suppose φ is a C^∞ smooth real valued function on the boundary. Let u be the harmonic extension of φ to Ω. The normal derivative of u is

$$-i\frac{\partial u}{\partial z}T(z) + i\frac{\partial u}{\partial \bar{z}}\overline{T(z)}.$$

For z in Ω, $u(z)$ is given by the Poisson integral, which by (2.2), can be written it two ways:

$$u(z) = \int_{w\in b\Omega} \frac{i}{\pi}\frac{\partial G(z,w)}{\partial \bar{w}}\varphi(w)\overline{T(w)}\ ds = \int_{w\in b\Omega} \frac{-i}{\pi}\frac{\partial G(z,w)}{\partial w}\varphi(w)T(w)\ ds.$$

We will use the first way to write $\partial u/\partial z$ and the second way to write $\partial u/\partial \bar{z}$ to see that the normal derivative of u is equal to $-ihT + i\overline{HT}$ where the boundary values of the holomorphic functions h and H are gotten from

$$h(z) = \int_{w \in b\Omega} \frac{i}{\pi} \frac{\partial^2 G(z,w)}{\partial z \partial \bar{w}} \varphi(w) \overline{T(w)}\ ds$$

and

$$\overline{H(z)} = \int_{w \in b\Omega} \frac{-i}{\pi} \frac{\partial^2 G(z,w)}{\partial \bar{z} \partial w} \varphi(w) T(w)\ ds.$$

Because the conjugate of $\partial^2 G/\partial \bar{z} \partial w$ is equal to $\partial^2 G/\partial z \partial \bar{w}$, we see that $H = h$, and so $\frac{\partial u}{\partial n}$ is equal to two times the real part of $-ihT$, and now (2.1) yields that the normal derivative of u is given by $(\operatorname{Re} g(z)T(z))$ where the boundary values of g are gotten from the expression

$$g(z) = \int_{w \in b\Omega} K(z,w)\varphi(w)\overline{T(w)}\ ds.$$

Since $d\bar{w} = \overline{T(w)}\, ds$, the proof is complete.

We now assume that Ω is a bounded simply connected domain with C^∞ smooth boundary. To prove Theorem 2.1, we note that we showed at the end of §4 that the normal derivative of

$$U(z) := \int_{w \in b\Omega} \left(\frac{1}{2} \left[ik(z,w)\overline{T(w)} \right] + \frac{1}{2} \left[-i\overline{k(z,w)}T(w) \right] \right) \varphi(w) ds_w$$

is given by the integral in Theorem 2.2. Hence, the Poisson extension u of φ and U get mapped to the same function via the Dirichlet-to-Neumann map. Therefore, they differ by a constant. Plugging in $z = a$ reveals that the constant is equal to $u(a)$, which is given by the Poisson integral formula at a. This completes the proof of Theorem 2.1, using only Theorem 2.2.

Another way to deduce Theorem 2.1 is to note that

$$\frac{\partial}{\partial z} \left(\frac{1}{2} ik_w(z)\overline{T(w)} - P(z,w) \right) = \frac{1}{2} iK_w(z)\overline{T(w)} - \frac{i}{\pi} \frac{\partial^2 G}{\partial z \partial \bar{w}} \overline{T(w)} \equiv 0.$$

Hence, $\overline{H(z)} := \frac{1}{2} ik_w(z)T(w) - P(z,w)$ is an antiholomorphic function. Taking $\partial/\partial \bar{z}$ and using the second way to relate the Poisson kernel to the Green's function in (2.2), noting that the conjugate of $\partial^2 G/(\partial \bar{z} \partial w)$ is equal to $\partial^2 G/(\partial z \partial \bar{w})$, yields that $\overline{H'(z)}$ is equal to the conjugate of $-\frac{i}{2} K_w(z)\overline{T(w)}$, and so

$$H'(z) = -\frac{i}{2} K_w(z)\overline{T(w)},$$

and therefore $H(z)$ differs from $-\frac{i}{2} k_w(z)\overline{T(w)}$ by a constant, which can be shown is equal to $P(a,w)$ by plugging in $z = a$. Finally, the definition of H reveals that

$$\overline{-\frac{1}{2} ik_w(z)T(w)} - P(a,w) = \frac{1}{2} ik_w(z)T(w) - P(z,w),$$

and the theorem follows.

These simpler, more direct, proofs might convince the reader that the main theorems are not terribly interesting. This is further reinforced by noting that the transformation formula for the Bergman kernel under biholomorphic mappings reveals that, in the simply connected setting,

$$k(z,w) = \frac{\overline{f'(w)}}{\pi\overline{f(w)}(1 - f(z)\,\overline{f(w)})} - \frac{\overline{f'(w)}}{\pi\overline{f(w)}}$$

where $f(z)$ is a Riemann map to the unit disc with $f(a) = 0$ and w is a point in $\overline{\Omega}$. Since the Green's function is

$$-\mathrm{Ln}\left|\frac{f(z) - f(w)}{1 - f(z)\,\overline{f(w)}}\right|,$$

it would seem that many of the formulas we have derived for the Poisson kernel could be deduced by expressing it as the normal derivative of the Green's function in this form. This is where our "Just analysis" might be seen as just pointless analysis. However, this shortcut bypasses all the fascinating connections between the Poisson, Szegő, and Bergman kernels, and how they generalize to the multiply connected setting. Whereas it is rather easy to see that the Poisson kernel is given as an explicit rational combination of $f(z)$, $f(w)$, $f'(w)$, and $T(w)$ in the simply connected setting, it seems to require something like the work of §5 to understand the complexity of the Poisson kernel in terms of Ahlfors maps in the multiply connected setting.

7. Postlude

It is interesting to note that, whereas Nessim Sibony's statement, "It's just analysis!," was the most modest thing I have ever heard a mathematician say, a very similar sounding statement by another noteworthy mathematician in reference to the solution to the $\bar{\partial}$-Neumann problem, "It's just integration by parts!," was the most deliciously arrogant!

I was young when I heard Nessim Sibony say "It's just analysis!" Now that I have witnessed his life's work and his prodigious contributions to geometric analysis and this venerable journal, I wonder if maybe he meant that the best analysis is *geometric* analysis. Otherwise it's just analysis.

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