THE SZEGÖ KERNEL AND PROPER HOLOMORPHIC MAPPINGS TO A HALF PLANE

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Abstract. We prove that all proper holomorphic mappings from a finitely connected domain in the plane to the right half plane can be expressed simply in terms of the Szegö kernel associated to the domain. Our decomposition also reveals the linear structure of the semi-group of all such maps and it offers a method to construct proper holomorphic maps of arbitrary mapping degree.

1. Main results

Suppose that Ω is a finitely connected domain in the plane such that no boundary component is a point. When Ω is a simply connected domain that is not equal to the whole complex plane, the Riemann map, which is a biholomorphic map to the unit disc, is a useful object. When Ω is multiply connected with connectivity $n > 1$, then proper holomorphic mappings to the unit disc can assume a similar role (see [B4, B5]). When studying proper holomorphic mappings, it is convenient to observe that such a domain Ω is biholomorphic to a bounded domain bounded by n real analytic non-intersecting Jordan curves. Since proper holomorphic maps are the main topic of this paper, we will always make this change of variables in order to be able to assume this feature about Ω. Given a point a in Ω, the Ahlfors map f_a associated to a is the solution to the extremal problem: among all holomorphic functions mapping Ω into the unit disc, f_a is the one such that $f_a'(a)$ is real and as large as possible. Ahlfors [A] proved that f_a is an n-to-one branched covering map of Ω onto the unit disc. It extends holomorphically past the boundary and maps each real analytic boundary curve of Ω one-to-one onto the unit circle. The Ahlfors map is an example of a proper holomorphic mapping of Ω to the unit disc. Garabedian [Ga] proved that the Ahlfors map is given by

$$f_a(z) = \frac{S(z,a)}{L(z,a)},$$

where $S(z,a)$ is the Szegő kernel and $L(z,a)$ is the Garabedian kernel, which is related to $S(z,a)$ via

$$\overline{S(z,a)} = \frac{1}{i} L(z,a) T(z) \quad \text{for } z \in b\Omega, \tag{1.1}$$

where $T(z)$ is the complex number of unit modulus pointing in the direction of the tangent vector at z pointing in the direction of the standard orientation.

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Proper holomorphic mappings of Ω onto the unit disc form a semi-group under the operation of multiplication. In case $\Omega \neq \mathbb{C}$ is simply connected, it is easy to generate all the proper holomorphic mappings of Ω onto the unit disc because they are given as finite Blaschke products composed with a single Riemann mapping. Proper holomorphic mappings to the *right half plane* (RHP) form a semi-group under the operation of addition. In this paper, we will reveal a method to generate all the proper holomorphic mappings of a finitely connected domain onto the RHP that will allow us to display such maps with the same ease as in the simply connected setting. (Of course the two semi-groups mentioned here are directly related via a linear fractional transformation between the unit disc and the RHP.)

We will show that quotients of the form

$$\frac{S(z,b)L(b,a)}{L(z,a)}, \tag{1.2}$$

where a and b are two distinct boundary points in one of the boundary curves of Ω, are proper holomorphic mappings of Ω onto the RHP as functions of z. When Ω is n-connected, they are n-to-one branched covering maps that extend meromorphically past the boundary of Ω. They map exactly one point on each boundary curve to the point at infinity via a simple pole. They map the rest of each boundary curve one-to-one onto the imaginary axis. The point a gets mapped to zero.

It is interesting to note that $S(z,a)/L(z,a)$ is a proper holomorphic mapping onto the unit disc if $a \in \Omega$, and $S(z,a)/L(z,b)$ is a proper holomorphic mapping to a half plane if a and b are distinct boundary points on one of the boundary curves of Ω. One could obtain interesting homotopies between these two proper mappings by letting a point a in the interior split into two and travel to two distinct points on a single boundary curve.

We will also prove that quotients like (1.2) are the building blocks of all proper holomorphic mappings from Ω to the RHP in the sense that, given a set of n or more distinct points $\{b_j\}_{j=1}^N$ in the boundary of Ω which includes at least one point from each boundary component, then there exist a point a in the boundary different from all the points b_j and positive constants c_j such that the mapping F given by

$$F(z) = \sum_{j=1}^{N} c_j \frac{S(z,b_j)L(b_j,a)}{L(z,a)}$$

is a proper holomorphic map of Ω to the RHP. The map F is an N-to-one branched cover and each b_j maps to the point at infinity. If there are m points b_j in a given boundary curve, then F maps the boundary curve m-to-one onto the imaginary axis union the point at infinity.

We will also study maps like F when $N = n$, i.e., when there is exactly one pole on each boundary curve. We will show that such "Grunsky maps" can be used to generate all proper holomorphic maps to the RHP. In this way, we will be able to view the semi-group of proper holomorphic mappings from Ω to the RHP as something akin to a linear space.

In the last section of the paper, we show how to use these ideas to construct primitive pairs for the field of meromorphic functions on the double of a finitely connected domain.

The inspiration for this paper came from the formulas in [B-K]. In particular, the formula on page 111 makes one suspect that quotients of the form (1.2) are fundamental. If we start from this assumption, we will see that the results of [B-K] can be derived in a much more efficient and less technical manner.

Proper holomorphic mappings from a multiply connected domain to the right half plane or the unit disc have been studied extensively. Some pioneers in the subject were Bieberbach, Grunsky, Ahlfors, Garabedian, and Nehari (see [Bie, G1, G2, A, Ga, N]). Such proper mappings are intimately related to extremal problems for holomorphic functions and harmonic functions. See Heins [H1, H2, H3] and Fisher and Khavinson [F-Kh] for further examples of the relevance of these proper maps.

Although the techniques of proof used in the present paper are fairly standard, the connection between proper maps and the Szegő kernel is something new. Because the Szegő kernel is an eminently computable object, the new formulas may give rise to concrete methods to compute some important objects of conformal mapping and potential theory.

2. The Szegő kernel and its zeroes

We continue to assume that Ω is a bounded n-connected domain bounded by n non-intersecting C^∞ smooth real analytic Jordan curves. The results in this paper depend in a key way on the properties proved in [B3] of the zeroes of the Szegő kernel $S(z,a)$ in the z variable when a is a boundary point or near a boundary point. To state the properties, we will need to first review some basic properties of the Szegő kernel on domains with real analytic boundary. These facts are all proved in [B2].

The Szegő kernel $S(z,w)$ extends to be in C^∞ of $(\overline{\Omega}\times\overline{\Omega})-\{(\zeta,\zeta):\zeta\in b\Omega\}$. It extends holomorphically in z past the boundary for each w in Ω. When w is a boundary point, $S(z,w)$. extends past the boundary near every boundary point except w. It extends meromorphically past w and it has a simple pole at w. The Garabedian kernel $L(z,w)$ extends to be in C^∞ of $(\overline{\Omega}\times\overline{\Omega})-\{(\zeta,\zeta):\zeta\in\overline{\Omega}\}$. It extends holomorphically in z past the boundary for each w in Ω. When w is a boundary point, $L(z,w)$. extends past the boundary near every boundary point except w. It extends meromorphically past w and it has a simple pole with residue $1/(2\pi)$ at w. When $w\in\Omega$, $L(z,w)$ has a simple pole in the z variable at w with residue $1/(2\pi)$ at w. The Szegő kernel satisfies $S(z,w)=\overline{S(w,z)}$ and the Garabedian kernel satisfies $L(z,w)=-L(w,z)$.

For a fixed $a\in\Omega$, $S(z,a)$ has $n-1$ zeroes in the z variable, counted with multiplicities. As $a\in\Omega$ moves toward a boundary point a_0, the $n-1$ zeroes of

$S(z,a)$ in the z variable become distinct and simple and head toward distinct boundary points, one on each of the other $n-1$ boundary curves different from the curve containing a_0, and the mapping which takes a to the zero near a boundary curve is an antiholomorphic mapping which extends antiholomorphically past the boundary. In fact, it was proved in [B3] that the multivalued mapping which sends a point a in Ω to the $n-1$ zeros of $S(z,a)$ is a proper antiholomorphic correspondence. Therefore, using [B-B] (see also [M-R]), we may assert that there are disjoint neighborhoods U_j of each boundary curve γ_j with the following property. If a is a point in U_k, then the zeroes of $S(z,a)$ can be numbered a_j, $j \neq k$, and the map $a \mapsto a_j$ is an antiholomorphic one-to-one map of U_k onto U_j. Furthermore, as the point a travels once around γ_k in the standard sense, a_j travels once around γ_j in the opposite sense to the standard sense. If a is a point on the boundary, then the $n-1$ zeroes of $S(z,a)$ on the other boundary curves are simple zeroes and $S(z,a)$ has a simple pole at a.

The Garabedian kernel $L(z,w)$ is non-vanishing when z and w are distinct points in Ω. When a is a boundary point, $L(z,a)$ is non-vanishing in Ω, but it has $n-1$ simple zeroes on the boundary that coincide with the zeroes of $S(z,a)$.

The identity (1.1) is valid when z and a are distinct boundary points. In this case, the identity can be applied once with z and a as in the formula and then again with z and a interchanged in the formula to obtain the following two important identities:

$$S(a,z) = -T(z)S(z,a)\overline{T(a)} \qquad \text{and} \qquad \overline{L(z,a)} = T(z)L(z,a)T(a) \tag{2.1}$$

when a and z are distinct boundary points.

Let ω_j denote the harmonic function on Ω that is equal to one on the j-th boundary curve and equal to zero on the other boundary curves, and let F_j' denote the holomorphic function given by $2(\partial/\partial z)\omega_j$. Let $\mathcal{F}'$ denote the complex linear span of the F_j' as j runs over all the boundary curves. It is well known that $\mathcal{F}'$ is $n-1$ dimensional and that any $n-1$ of the F_j' form a basis. We will need an extended version of a result in [B3] (which itself was an extension of a result by Schiffer [Sch]). Let a be a point in Ω such that the $n-1$ zeroes $a_1, \ldots, a_{n-1}$ of $S(z,a)$ are distinct (and simple). It was proved in [B3] that both $\{S(z,a_j)L(z,a) : j = 1, \ldots n-1\}$ and $\{L(z,a_j)S(z,a) : j = 1, \ldots n-1\}$ form a basis for $\mathcal{F}'$ (see [B2, p. 80] for a proof in the spirit of this paper). We will need to know that this statement remains true when a is a boundary point of Ω. Note that the properties of the Szegő and Garabedian kernels mentioned above yield that the functions in the two spanning sets are holomorphic in a neighborhood of $\overline{\Omega}$ even when a is in the boundary because the simple poles on the boundary are exactly cancelled by simple zeroes. Since the functions converge uniformly on compact subsets of Ω in z as a tends to the boundary, the limit functions are in $\mathcal{F}'$. It is easy to see that $H_j(z) := L(z,a_j)S(z,a)$ are linearly independent because $H_j(a_j) \neq 0$, but $H_j(a_k) = 0$ if $k \neq j$. The functions $\{S(z,a_j)L(z,a)$ have the same property since the zeroes and poles of $L(z,b)$ and $S(z,b)$ occur at the same places with the same orders when b is a boundary point.

We will need the following technical lemma.

Lemma 2.1. *Suppose that $b_1, b_2, \dots, b_{n-1}$ are boundary points of Ω, one on each of $n-1$ distinct boundary curves. If $\{H_j\}_{j=1}^{n-1}$ is a basis for $\mathcal{F}'$, then*

$$\det[H_j(b_k)] \neq 0.$$

Number the boundary curves so that b_j is in γ_j and none of the points b_j fall on the n-th boundary curve. The proof of the lemma consists of remarking that the outward normal derivative of ω_j is given by

$$\frac{\partial \omega_j}{\partial n} = -iF_j' T,$$

and therefore $-iF_j'(b_k)T(b_k)$ is positive if $k = j$ and negative if $k \neq j$. Since $\sum_{j=1}^n \omega_j \equiv 1$, it follows that $\sum_{j=1}^{n-1} -iF_j'(b_k)T(b_k) = iF_n'(b_k)T(b_k)$, which is positive for each k. Grunsky proved in [G1, G2] that the determinant of such a matrix of positive and negative entries must be non-zero (see Prop. 4.1.3 on page 136 of [G2] or see Khavinson [Kh]). Hence the lemma is true if the basis is $\{F_j'\}_{j=1}^{n-1}$. But the non-vanishing of the determinant is invariant under changes of basis. The proof is complete.

3. Grunsky maps

We continue to assume that Ω is a bounded n-connected domain bounded by n non-intersecting C^∞ smooth real analytic Jordan curves. The n-to-one proper holomorphic mappings of Ω onto the RHP were called Grunsky maps in [B-K] and were expressed in terms of the Szegő and Garabedian kernels there. (We should point out here that the existence of Grunsky maps was proved by Bieberbach in [Bie].) Expressions of the form that appear in the next theorem are ubiquitous in [B-K], but their properties were not known to the authors at that time.

Theorem 3.1. *Assume that Ω is a bounded n-connected domain bounded by n non-intersecting C^∞ smooth real analytic Jordan curves. If a and b are distinct boundary points of Ω on the same boundary curve, the function of z,*

$$f(z) = \frac{S(z,b)L(b,a)}{L(z,a)},$$

is an n-to-one proper holomorphic map of Ω onto the right half plane. Among the Grunsky maps, it is determined up to multiplication by a positive constant by the conditions that it takes b and each of the $n-1$ zeroes of $L(z,a)$ to the point at infinity, and it takes a to zero.

Proof. Let $a_1, \dots, a_{n-1}$ denote the $n-1$ zeroes of $S(z,a)$ associated to the boundary point a. Note that these zeroes are the same as the zeroes of $L(z,a)$ and they fall one each on the boundary curves of Ω different from the boundary curve on which a and b reside. If z is a boundary point of Ω different from a, b, and the zeroes a_j, then the identities (2.1) can be applied to show that

$$\frac{S(z,b)L(b,a)}{L(z,a)} = \frac{T(z)S(z,b)\overline{T(b)}\,T(b)L(b,a)T(a)}{T(z)L(z,a)T(a)} = -\frac{S(b,z)\overline{L(b,a)}}{\overline{L(z,a)}},$$

i.e., that $f(z) = -\overline{f(z)}$. Hence $f(z)$ is pure imaginary when z is on the boundary. Note that f has simple poles at b and at each a_j. Therefore f extends to the double $\widehat{\Omega}$ of Ω as a meromorphic function which maps the double n-to-one (counting multiplicities) onto the extended complex plane. We will let f also denote the extension.

We will now show that f maps Ω into the RHP. Call the imaginary axis union the point at infinity the *extended imaginary axis.* Since f maps exactly one point on each boundary curve to the point at infinity, and since f maps each boundary curve into the extended imaginary axis, it follows via the Intermediate Value Theorem that f maps each boundary curve *onto* the extended imaginary axis. Since f is n-to-one, f must be a one-to-one mapping of each boundary curve onto the extended imaginary axis. Furthermore, no point inside Ω can be mapped to the imaginary axis.

It is proved in [B3] that the Szegő kernel is given by

$$S(z,w) = \frac{1}{1-\overline{g(w)}\,g(z)} \sum_{i,j=0}^{n} s_i(z)\overline{s_j(w)}, \tag{3.1}$$

where g is an Ahlfors mapping of Ω onto the unit disc and the functions $s_i(z)$ are holomorphic in a neighborhood of $\overline{\Omega}$. The Ahlfors map g extends holomorphically past the boundary of Ω, maps the boundary of Ω onto the unit circle, and maps Ω onto the unit disc as a branched n-to-one covering map. Note that, consequently, g' is non-vanishing on the boundary. Because $S(z,b)$ has a pole at b and because $S(z,z) > 0$ for $z \in \Omega$, it follows that

$$0 < \sum_{i,j=0}^{n} s_i(w)\overline{s_j(w)}$$

for w in $\overline{\Omega}$. We may restrict g to a small neighborhood in Ω of the boundary curve of Ω that contains b. Let η denote the pull back under the restriction of g of a small radial segment in the unit disc that terminates at $g(b)$. The curve η is a smooth curve in Ω that terminates at b and makes a right angle with the boundary of Ω at b. Formula (3.1) shows that $f(z)$ is real and tends to plus infinity as z tends to b along η. This forces us to conclude that f maps Ω into the RHP near b. Suppose now that there is a point in $z_0 \in \Omega$ that gets mapped to a point in the left half plane. A curve in Ω connecting w_0 to a point $b_0 \in \Omega$ near b that gets mapped to a point in the RHP would have to get mapped under f to a curve that crosses the imaginary axis. But no point in Ω can be mapped to a point in the imaginary axis. Hence no such w_0 exists and we see that f maps

Ω into the RHP. It follows that f is a proper holomorphic map of Ω onto the RHP, and indeed, f is an n-to-one branched covering map of Ω onto the RHP.

The uniqueness clause in the theorem follows from the fact (see Grunsky [G2]) that any other proper holomorphic mapping to the RHP with the same poles as f above must be given by $cf(z) + iC$ where c and C are real constants and $c > 0$. □

We next turn to the problem of constructing a Grunsky map that is a proper holomorphic mapping to the RHP with prescribed poles, one per boundary curve.

Given n boundary points $b_1, \ldots, b_n$, one on each of n boundary curves, let b_n denote the point on the outer boundary and let a denote a point on the outer boundary curve different from b_n such that none of the $n-1$ zeroes of $L(z,a)$ coincide with a b_j. (Since the zero map, which maps the point a in the boundary to the zero a_j on the j-th boundary curve, is a homeomorphism of the two boundary curves, it is possible to choose such a point a.)

We will construct a mapping $F(z)$ given by

$$F(z) = \sum_{j=1}^{n} c_j \frac{S(z,b_j)L(b_j,a)}{L(z,a)} = \frac{1}{L(z,a)} \sum_{j=1}^{n} c_j S(z,b_j)L(b_j,a), \tag{3.2}$$

where, after fixing c_n to be a positive constant, the coefficients c_j will be uniquely determined positive constants such that

$$\sum_{j=1}^{n} c_j S(z,b_j)L(b_j,a) = 0$$

when z is a zero of $L(z,a)$, i.e., so that

$$\sum_{j=1}^{n-1} c_j S(a_i,b_j)L(b_j,a) = -c_n S(a_i,b_n)L(b_n,a),$$

for $i = 1, \ldots, n-1$. The first of identities (2.1) allow us to rewrite this system as

$$\sum_{j=1}^{n-1} c_j T(b_j)S(b_j,a_i)L(b_j,a) = -c_n T(b_n)S(b_n,a_i)L(b_n,a) \tag{3.3}$$

for $i = 1, \ldots, n-1$. Now, since the functions $S(z,a_i)L(z,a)$, $i = 1, \ldots, n-1$, form a basis for $\mathcal{F}'$, Lemma 2.1 yields that the determinant of the coefficient matrix for the system (3.3) is non-zero. Hence the complex numbers c_j exist (partly because the $T(b_k)$ are non-zero, too).

We next show that the coefficients c_j must be real. Identities (2.1) reveal that

$$T(z)S(z,a_i)L(z,a) = -\overline{T(z)S(z,a_i)L(z,a)},$$

when $z \in b\Omega$. This shows that the matrix of coefficients and the right hand side of (3.3) are pure imaginary, and it follows that the c_j are real valued.

Next, we will show that each c_j must be non-zero. The same argument used in the proof of Theorem 3.1 shows that F maps each boundary curve of Ω into the extended imaginary axis. Hence, F extends to the double of Ω as a meromorphic function. We will let F also denote the extension of F to the double. Notice that F has at most n poles. It has one pole on the outer boundary (because $c_n > 0$), and at most one on each of the other boundary curves. If F has a pole on a boundary curve, then the Intermediate Value Theorem can be used to see that F maps that boundary curve *onto* the extended imaginary axis. If F has a total of $m < n$ poles on the boundary, then F is an m-to-one mapping of the double of Ω onto the extended complex plane (counting multiplicities). We must now conclude that each boundary curve of Ω that contains a pole of F gets mapped one-to-one onto the extended imaginary axis. If there is a boundary curve that does not contain a pole of F (i.e., if $m < n$), then we encounter points on the boundary that map to the imaginary axis and yield the existence of points in the extended complex plane that have more than m pre-images under F. This contradiction implies that m must be equal to n, and we conclude that each c_j must be non-zero.

Finally, we must show that each c_j must be positive. The same argument used in the proof of Theorem 3.1 shows that F maps Ω into the RHP near b_j if c_j is positive and into the left half plane if c_j is negative. We know that F maps Ω into the RHP near b_n because $c_n > 0$. If one of the coefficients c_j is negative, then we may connect a point near b_j that gets mapped to a point in the left half plane to a point near b_n that gets mapped to a point in the right half plane by a curve in Ω. The Intermediate Value Theorem yields a point along the curve that must map to a point in the imaginary axis, and this would make F more than n-to-one in places, in violation of the result proved in the previous paragraph.

We note that there was nothing special about choosing the coefficient c_n associated with the outer boundary to be a fixed positive number. The system (3.3) is such that, if one of the coefficients c_j is set to be a fixed positive real number, then all the other coefficients are uniquely determined and must also be positive real numbers. We collect these results in the following theorem.

Theorem 3.2. *Assume that Ω is a bounded n-connected domain bounded by n non-intersecting C^∞ smooth real analytic Jordan curves. Given n boundary points b_j, one on each boundary curve, it is possible to choose a point a in the boundary different from all the b_j so that a Grunsky map associated to the points b_j is given by*

$$F(z) = \sum_{j=1}^{n} c_j \frac{S(z, b_j)L(b_j, a)}{L(z, a)}$$

where, after fixing one of the coefficients, say c_k, to be a positive constant, the other coefficients c_j are positive constants that are uniquely determined by the system

$$0 = \sum_{j=1}^{n} c_j \; iS(a_i, b_j)L(b_j, a),$$

for $i = 1, \ldots, n-1$. The coefficients $iS(a_i, b_j)L(b_j, a)$ are real numbers. Notice that $F(a) = 0$. The set of all Grunsky maps with poles at the b_j is given by $cF + iC$ where c and C are real constants and $c > 0$.

We now turn to the problem of generating *all* the proper holomorphic mappings from our given domain Ω to the RHP. The work of Heins [H1, H2, H3] and Fisher and Khavinson [F-Kh] lead one to suspect that Grunsky maps are the building blocks for such mappings and that the solution will depend on linear equations. It is clear that real linear combinations of Grunsky maps using only positive coefficients are proper holomorphic mappings to the RHP. Here, we show that the semi-group of proper holomorphic mappings of Ω onto the RHP is *precisely* the set of real linear combinations of Grunsky maps using only positive coefficients. Furthermore, given a proper holomorphic mapping F to the RHP with poles at points $\{b_j\}_{j=1}^N$ in the boundary of Ω (which, we have shown, must include at least one point from each boundary component), then there exist a point a in the boundary different from all the points b_j and positive constants A_j such that the mapping F is given by

$$F(z) = \sum_{j=1}^{N} A_j \frac{S(z, b_j)L(b_j, a)}{L(z, a)}. \tag{3.4}$$

Suppose F is a proper holomorphic mapping to the RHP. It was shown in [B-K] that F can be expressed as a real linear combination of Grunsky maps where the Grunsky maps in the sum only have poles at poles of F. We may select a point a on the boundary which is different from all the b_j and such that none of the zeroes $a_1, \ldots, a_{n-1}$ of $S(z, a)$ coincide with any of the b_j. We have shown that Grunsky maps with poles amongst the set of b_j are given by sums like (3.2). When the sum is combined, an expression like (3.4) is obtained. It now follows that all of the coefficients in the sum must be positive because we have seen that $S(z, b_j)L(b_j, a)/L(z, a)$ maps Ω into the RHP near b_j,

Suppose now that we are given $N > n$ boundary points $\{b_j\}_{j=1}^N$ that include at least one point from each boundary curve. It is easy to see that there exists a proper holomorphic map to the RHP with only simple poles at each b_j. Indeed, label the b_j so that the first n of them fall one per boundary curve. We have shown that the system

$$0 = \sum_{j=1}^{n} A_j S(a_i, b_j)L(b_j, a)$$

has a unique solution when $A_n = 1$ with all the other A_j being positive. If we set A_k to be small enough positive numbers for $k = n+1, \ldots, N$, and continue to set $A_n = 1$, then the system,

$$0 = \sum_{j=1}^{N} A_j S(a_i, b_j)L(b_j, a)$$

for $i = 1, \ldots, n-1$, will have a unique solution where the A_j are still positive for $j = 1, \ldots, n-1$. Now

$$F(z) = \sum_{j=1}^{N} A_j \frac{S(z, b_j)L(b_j, a)}{L(z, a)}$$

is the desired proper map.

The set of all proper holomorphic mappings to the RHP with simple poles only at the points b_j are the mappings of the form

$$F(z) = \sum_{j=1}^{N} A_j \frac{S(z, b_j)L(b_j, a)}{L(z, a)},$$

where each A_j is positive and

$$0 = \sum_{j=1}^{N} A_j S(a_i, b_j)L(b_j, a)$$

for $i = 1, \ldots, n-1$. Hence, it is a standard problem in linear algebra to find all such things. The set of all such vectors (A_j) in the positive N-tant is given by convex combinations of extremal rays, where the extremal rays are the solutions to the system with as many of the coefficients equal to zero as possible, and the rest positive. We may repeat the geometric argument that we used in our construction of Grunsky maps with prescribed poles to see that it is impossible to have a solution with fewer than n positive A_j, and that there must be at least one A_j associated to a b_j on each of the n boundary curves. Hence, the extremal ray solutions correspond to Grunsky maps. (This is reminiscent of the argument used by M. Heins in [H2] to determine the building blocks for the linear space of positive harmonic functions on a Riemann surface.)

We may now summarize what we have accomplished in the following theorem.

Theorem 3.3. *The set of all proper holomorphic mappings from Ω to the RHP is given by the set of positive linear combinations of Grunsky maps.*

4. Constructing primitive pairs via the Szegő kernel

We conclude this paper by demonstrating the ease with which one can write down a primitive pair for the space of meromorphic functions on the double of a finitely connected domain with smooth boundary using the ideas above. A proper holomorphic mapping from such a domain to the RHP extends to the double via the Reflection Principle. We will explain how to find two proper maps to the RHP whose extensions to the double form a primitive pair. (A primitive pair of a compact Riemann surface is a pair of functions such that any meromorphic function of the surface is a rational combination of the two functions.)

We start with the absolute easiest way to construct a pair. Suppose that F_1 is a Grunsky map. Pick one point b_j from each boundary curve such that the values $F_1(b_j)$ are distinct complex numbers in the finite complex plane. (This is easy because each boundary curve gets mapped one-to-one onto the extended imaginary axis.) Now let $F_2(z)$ be the Grunsky map associated to the set $\{b_j\}$. Since F_1 separates the points in $F_2^{-1}(\infty)$, the extensions of F_1 and F_2 form a primitive pair (see Farkas and Kra [F-K]).

Now we describe a somewhat harder way to get a pair, but the pair is easier to write down. Suppose that Ω is a bounded domain bounded by n non-intersecting real analytic curves. Pick two points, a and b, in one of the boundary curves γ_k, and let $a_1, \dots, a_{n-1}$ denote the zeroes of $L(z, a)$ (which fall one in each of the other boundary curves). We know that $S(z, b)/L(z, a)$ is a proper holomorphic mapping of Ω onto the RHP. We will now show that there is a third point α in γ_k (different from a and b) so that $F_1(z) = S(z, b)/L(z, a)$ and $F_2(z) = S(z, b)/L(z, \alpha)$ extend to the double and form a primitive pair. We know that F_1 has simple poles at b and each of the a_j (and no other poles). To complete the proof, we need to prove that F_2 separates the points $a_1, \dots, a_{n-1}$ with values in the finite complex plane. The equation $F_2(a_j) = F_2(a_i)$ is equivalent to $0 = S(a_j, b)L(a_i, \alpha) - S(a_i, b)L(a_j, \alpha)$, or

$$0 = c_{ij} L(\alpha, a_i) + L(\alpha, a_j), \tag{4.1}$$

where c_{ij} is a non-zero constant. (We have used the fact that $L(z, w) = -L(w, z)$ here.) We now claim that equation (4.1) can hold for at most finitely many points α in γ_k. Indeed, if the meromorphic function of α on the right hand side of (4.1) were identically zero on γ_k, then it would be identically zero in α on Ω. It would then follow from identity (1.1) that

$$0 = \bar{c}_{ij} S(a_i, \alpha) + S(a_j, \alpha) \tag{4.2}$$

for all α in Ω. It follows from Lemma 7.3 of [B1] that the complex linear combinations of functions of z of the form $S(z, \alpha)$ as α runs over Ω can be used to approximate any polynomial uniformly on $\overline{\Omega}$. Hence the function on the right hand side of (4.2) cannot be identically zero in α. Since it extends holomorphically past the boundary curve γ_k in α, it can have at most finitely many zeroes on γ_k. By avoiding all the finite sets that arise in this way for $i, j = 1, \dots, n-1$ with $i \neq j$, we obtain an open dense set of α on γ_k such that F_2 separates points as desired.

The Szegő kernel is eminently computable via the Kerzman-Stein-Trummer method (see [K-S, K-T]). Hence, these observations may give rise to a concrete way to calculate primitive pairs. Primitive pairs are the building blocks of many of the basic kernel functions of complex analysis (see [B4]).

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