UNIQUE CONTINUATION THEOREMS FOR THE $\bar\partial$-OPERATOR AND APPLICATIONS

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Abstract. We formulate a unique continuation principle for the inhomogeneous Cauchy-Riemann equations near a boundary point z_0 of a smooth domain in complex euclidean space. The principle implies that the Bergman projection of a function supported away from z_0 cannot vanish to infinite order at z_0 unless it vanishes identically. We prove that the principle holds in planar domains and in domains where the $\bar\partial$-Neumann problem is known to be analytic hypoelliptic. We also demonstrate the relevance of such questions to mapping problems in several complex variables. The last section of the paper deals with unique continuation properties of the Szegő projection and kernel in planar domains.

1. Introduction. The results of this paper grew out of attempts to answer some simple questions about the Bergman kernel function and the Bergman projection associated to a domain in complex euclidean space. Suppose that Ω is a bounded domain in $\mathbb{C}^n$ with $\mathbb{C}^\infty$ smooth boundary and suppose that the Bergman kernel function $K(z,w)$ associated to Ω is known to be a function in $C^\infty((\overline{\Omega}\times\overline{\Omega}) - \{(z,z) : z\in b\Omega\})$. (Kerzman's theorem [26] yields that this condition on the kernel function holds, for example, if Ω is strictly pseudoconvex, or more generally, if Ω is pseudoconvex of finite type in the sense of D'Angelo [19].) The question from which this research originates concerns the degree to which the Bergman kernel can vanish at boundary points. If $n=1$, the Bergman kernel cannot vanish at any point $(z_0,w_0)\in b\Omega\times b\Omega$, $z_0\neq w_0$ (although it must vanish at some points $(z_0,w_0)\in\Omega\times\Omega$ if Ω is multiply connected, see [38]). In several variables, it is not even known if the kernel function cannot vanish to *infinite order* at boundary points. Given a point $w_0\in\Omega$ and a point $z_0\in b\Omega$, is it possible for the holomorphic function $h(z)=K(z,w_0)$ to vanish to infinite order at z_0? More generally, given a multi-index β and two points, $z_0\in b\Omega$ and $w_0\in\overline{\Omega}$, $z_0\neq w_0$, is there a multi-index α such that

$$\frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha\partial\bar w^\beta}K(z,w)\quad\text{is non-zero at }(z_0,w_0)?$$

I will show that the answer to this question is "yes" on planar domains and in certain domains in $\mathbb{C}^n$ for which the $\bar\partial$-problem satisfies a unique continuation property. I will also explain why these questions about finite order vanishing are natural and how they can be related to the $\bar\partial$-problem and to mapping problems in several

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complex variables. In the last section of the paper, I will study analogous questions for the Szegő kernel and projection.

The symbol $A^\infty(\Omega)$ will denote the space of holomorphic functions on a domain Ω in $C^\infty(\overline{\Omega})$. Catlin [13] proved that if Ω is a bounded pseudoconvex domain with C^∞ smooth boundary, then there exist many functions in $A^\infty(\Omega)$ that vanish to infinite order at any given boundary point. Hence, infinite order vanishing of the Bergman kernel cannot be ruled out *a priori.*

2. A unique continuation problem for $\bar\partial$. If $z_0 \in \mathbb{C}^n$ and $\epsilon > 0$, let $B_\epsilon(z_0)$ denote the ball of radius ϵ about z_0. If $\alpha = \sum_{j=1}^n \alpha_j d\bar z_j$ is a $(0,1)$-form, then $\vartheta\alpha$ is a function defined via

$$\vartheta\alpha = -\sum_{j=1}^n \frac{\partial \alpha_j}{\partial z_j}.$$

The operator ϑ is the formal adjoint of the $\bar\partial$-operator. If Ω is a bounded domain in $\mathbb{C}^n$ with C^∞ smooth boundary, let $C^s_{(0,1)}(\overline{\Omega})$ denote the space of $(0,1)$-forms with coefficients in $C^s(\overline{\Omega})$, and let $\|\alpha\|_s$ denote the norm of a form $\alpha \in C^s_{(0,1)}(\overline{\Omega})$ (which is given as the supremum of the $C^s(\overline{\Omega})$ norms of the coefficients of α). Let $C^\infty_{(0,1)}(\overline{\Omega})$ denote the space of $(0,1)$-forms with coefficients in $C^\infty(\overline{\Omega})$.

We shall say that the boundary of a domain Ω is C^∞ smooth near a boundary point z_0 if there is a ball $B_\epsilon(z_0)$ such that $\Omega \cap B_\epsilon(z_0)$ is a C^∞ manifold with boundary near z_0.

Suppose that Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ and that the boundary of Ω is C^∞ smooth near a point $z_0 \in b\Omega$. Let $\epsilon > 0$ be small enough that $B_\epsilon(z_0)\cap\Omega$ is connected. We shall say that the ϑ-*Unique Continuation Property* holds at z_0 if the following condition holds.

THE ϑ-UNIQUE CONTINUATION PROPERTY. *For any $(0,1)$-form α in $C^\infty_{(0,1)}(\overline{\Omega} \cap B_\epsilon(z_0))$ whose coefficients vanish on $b\Omega \cap B_\epsilon(z_0)$, if the two conditions,*

1) *$\vartheta\alpha$ is holomorphic on $B_\epsilon(z_0)\cap\Omega$, and*
2) *$\vartheta\alpha$ vanishes to infinite order at z_0,*

hold, then $\vartheta\alpha$ must vanish identically on $B_\epsilon(z_0)\cap\Omega$.

We shall use the abbreviation ϑ-UCP for ϑ-*Unique Continuation Property.* It is easy to see that the ϑ-UCP is purely local and that it does not really depend on Ω or the size of ϵ. It only depends on the germ of the hypersurface describing the boundary of Ω near z_0. I have stated the property in terms of a fixed domain Ω because I shall only apply the property in such a setting.

It may not be true that every boundary point of a smooth bounded pseudoconvex domain satisfies the ϑ-UCP, but it seems very likely to me that strictly pseudoconvex boundary points do, and maybe even pseudoconvex boundary points of finite type in the sense of D'Angelo. We shall show later that the ϑ-UCP does hold at strictly pseudoconvex boundary points of domains with real analytic boundaries by virtue of the analytic hypoellipticity of the $\bar\partial$-Neumann problem at such points.

In the case of one variable, the ϑ-UCP holds at every smooth boundary point as a direct consequence of the Schwarz reflection principle (see Jeong [25]). To see this, suppose z_0 is a point on a C^∞ smooth curve γ and φ is a function which is defined on one side of γ near z_0, which is C^∞ smooth up to γ, and which vanishes along γ. Let Ω be a small domain with C^∞ smooth boundary whose boundary coincides with γ near z_0. We may further assume that Ω lies on the side of γ on which φ is defined and that Ω is small enough that φ is in $C^\infty(\overline{\Omega})$. In one variable, the assumptions of the ϑ-UCP translate to say that if $\partial\varphi/\partial z$ is holomorphic on Ω near z_0 and vanishes to infinite order at z_0, then $\partial\varphi/\partial z \equiv 0$ on Ω near z_0. Furthermore, saying that $\partial\varphi/\partial z$ is holomorphic is simply to say that φ is harmonic. Suppose that φ is harmonic on Ω near z_0 and that $\partial\varphi/\partial z$ does vanish to infinite order at z_0. Let f denote a biholomorphic map of the upper half plane onto Ω that maps the origin to z_0. The map f extends C^∞ smoothly up to the real axis and $f'(0) \neq 0$. Consider the function $\varphi \circ f$. This function is harmonic on the upper half plane near the origin and it is C^∞ smooth up to the real axis. Since φ vanishes on γ near z_0, it follows that $\varphi \circ f$ vanishes on the real axis near the origin, and the Schwarz reflection principle yields that $\varphi \circ f$ extends to be harmonic on a neighborhood of the origin. Hence, $(\partial/\partial z)(\varphi \circ f)$ extends holomorphically to the same neighborhood of the origin. But $(\partial/\partial z)(\varphi \circ f) = f'[(\partial\varphi/\partial z) \circ f]$ and it can be seen that infinite order vanishing of $\partial\varphi/\partial z$ at z_0 implies that the function $(\partial/\partial z)(\varphi \circ f)$, which is holomorphic on a neighborhood of the origin vanishes to infinite order at the origin. Consequently, this function vanishes on a neighborhood of the origin, and it follows that $\partial\varphi/\partial z$ also vanishes in Ω near z_0. The proof is complete.

We shall see that the ϑ-UCP implies a unique continuation property for the Bergman projection in §4. First, however, we must prove an important lemma.

3. Rosay's Lemma. In this section, we shall give a proof of a theorem of Rosay which characterizes the space of smooth functions in the orthogonal complement of the Bergman space. In [36], Rosay proved his lemma in $\mathbb{C}^2$ and summarized briefly (but completely) how to generalize the result to higher dimensions. We give a detailed proof in $\mathbb{C}^n$ here because we shall need to refer to steps in the proof at points later in the paper. Also, I have needed to modify Rosay's original argument in order to adapt it to a localization scheme I use later. Let $H^2(\Omega)$ denote the Bergman space, which is the space of holomorphic functions contained in $L^2(\Omega)$.

Theorem 3.1 (Rosay's Lemma). *Suppose Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^∞ smooth boundary and suppose that u is a function in $C^\infty(\overline{\Omega})$ that is orthogonal to the Bergman space $H^2(\Omega)$. There exist functions $\alpha_j \in C^\infty(\overline{\Omega})$, $j = 1, \ldots, n$, all vanishing on $b\Omega$, such that the $(0,1)$-form $\alpha = \sum_{j=1}^n \alpha_j d\bar{z}_j$ satisfies*

$$u = \vartheta\alpha.$$

Since any function of the form $\vartheta\alpha$ where $\alpha \in C^\infty_{(0,1)}(\overline{\Omega})$ vanishes on $b\Omega$ is orthogonal to the Bergman space, Rosay's Lemma characterizes the space of smooth functions orthogonal to the Bergman space.

Before proving this result, it is worth mentioning one of its most striking consequences. Given a holomorphic function $h \in H^2(\Omega)$, only a fool would look for a function in $H^2(\Omega)$ which is equal to h near a boundary point z_0 and which is supported in a small ball centered at z_0. Rosay's Lemma, however, implies that such a localization is available in the orthogonal complement of the Bergman space. Indeed, given a function $u \in C^\infty(\overline{\Omega})$ which is orthogonal to $H^2(\Omega)$ and a point $z_0 \in b\Omega$, let χ be a C^∞ function supported in a small ball $B_\epsilon(z_0)$ which is equal to one on a neighborhood of z_0. If α is the $(0,1)$-form supplied by Rosay's Lemma, then $\tilde{u} = \vartheta(\chi\alpha)$ is a function in $C^\infty(\overline{\Omega})$ that is orthogonal to the Bergman space, that is supported in $B_\epsilon(z_0)$, and that is equal to u near z_0. (I do not know if such a localization is possible in smooth non-pseudoconvex domains.)

Proof of Theorem 3.1. As in Rosay [36], the proof rests firmly on Kohn's theory of the $\bar\partial$-Neumann problem with weights [30].

Let ρ be a C^∞ defining function for Ω (which means that $\Omega = \{\rho < 0\}$, $b\Omega = \{\rho = 0\}$, and $d\rho \neq 0$ on $b\Omega$). We shall use subscript z_j's and $\bar z_j$'s to denote differentiation with respect to those variables. Thus, for example, $\rho_{\bar z_j}$ is shorthand for $\partial\rho/\partial\bar z_j$.

Before we begin the proof, we must define some basic objects associated with the $\bar\partial$-problem (see Kohn [30,31] for complete details). Let P denote the Bergman projection, which is the orthogonal projection of $L^2(\Omega)$ onto the closed subspace $H^2(\Omega)$. If $t > 0$, the space $L^2_t(\Omega)$ is defined to be the Hilbert space of complex valued functions on Ω with inner product given by

$$\langle u, v\rangle_t = \int_\Omega u(z)\, v(z)\, e^{-t|z|^2}\, dV,$$

where dV denotes the standard Lebesgue measure on $\mathbb{C}^n$. If $\beta = \sum_{j=1}^n \beta_j d\bar z_j$ is a $(0,1)$-form, then ϑ_t, the formal adjoint of $\bar\partial$ with respect to the weight function $e^{-t|z|^2}$, is defined via

$$\vartheta_t\beta = e^{t|z|^2}\vartheta(e^{-t|z|^2}\beta).$$

The space $H^2(\Omega)$ can also be viewed as a closed subspace of $L^2_t(\Omega)$, and we may define the orthogonal projection P_t of $L^2_t(\Omega)$ onto $H^2(\Omega)$. This operator P_t is related to the weighted $\bar\partial$-Neumann operator N_t via Kohn's formula (see Kohn [30]),

$$P_t = I - \vartheta_t N_t \bar\partial.$$

Kohn proved that, given a positive integer s, there is a t_0 such that if $t > t_0$, the operator N_t maps $C^\infty_{(0,1)}(\overline{\Omega})$ into $C^s_{(0,1)}(\overline{\Omega})$. Kohn also proved Sobolev estimates for N_t. When Kohn's estimates are combined with the basic Sobolev Lemma estimate, we can see that there exists a positive integer M with the property that N_t maps $C^{s+M}_{(0,1)}(\overline{\Omega})$ into $C^s_{(0,1)}(\overline{\Omega})$ and N_t satisfies an estimate of the form $\|N_t\beta\|_s \le C\|\beta\|_{s+M}$ whenever t is sufficiently large. The integer M does not depend on s or t.

We shall also need to know that $(0,1)$-forms in the range of N_t satisfy the following boundary condition. If $\beta = N_t\omega$, then writing $\beta = \sum_{j=1}^n \beta_j d\bar z_j$, we have

$$\sum_{j=1}^n \beta_j \frac{\partial\rho}{\partial z_j} = 0 \qquad \text{on } b\Omega. \tag{3.1}$$

Suppose that $u \in C^\infty(\overline{\Omega})$ is orthogonal to $H^2(\Omega)$ with respect to the standard $L^2(\Omega)$ inner product. It follows that $e^{t|z|^2}u(z)$ is orthogonal to $H^2(\Omega)$ in $L^2_t(\Omega)$, and hence, that $P_t(e^{t|z|^2}u) = 0$. Therefore,

$$e^{t|z|^2}u = \vartheta_t N_t\bar\partial(e^{t|z|^2}u) = e^{t|z|^2}\vartheta\left(e^{-t|z|^2}N_t\bar\partial(e^{t|z|^2}u)\right),$$

and so

$$u = \vartheta\beta$$

where β is a $(0,1)$-form given by

$$\beta = e^{-t|z|^2} N_t \bar{\partial}(e^{t|z|^2} u).$$

Given a positive integer s, we assume that t is large enough to ensure that the coefficients of β are in $C^{s+1}(\overline{\Omega})$. It then also follows that β satisfies the boundary condition given by (3.1). Next, we use β to construct a $(0,1)$-form α in $C^s_{(0,1)}(\overline{\Omega})$ whose coefficients vanish on $b\Omega$ such that $\vartheta\alpha = \vartheta\beta = u$.

Suppose that $\{\chi_m\}_{m=0}^N$ is a C^∞ partition of unity of $\overline{\Omega}$ that is subordinate to a finite covering of $\overline{\Omega}$ consisting of small open balls $B_{r_m}(w_m)$ centered at boundary points of Ω together with the open set Ω. We assume that $\chi_0 \in C_0^\infty(\Omega)$ is the function associated with the open set Ω and that $\chi_m \in C_0^\infty(B_{r_m}(w_m))$ for $m \geq 0$. We may assume that the radii r_m are small enough that, on each ball, there is some coordinate direction z_j such that $\partial\rho/\partial z_j$ is non-vanishing on the closure of that ball.

Observe that

$$u = \sum_{m=0}^{N} \vartheta(\beta\chi_m).$$

Define $u^{(m)} = \vartheta(\beta\chi_m)$ and $\beta^{(m)} = \beta\chi_m$, and let us write $\beta^{(m)} = \sum_{j=1}^n \beta_j^{(m)} d\bar{z}_j$.

If $m = 0$, define $\alpha^{(0)} = \beta^{(0)}$. Obviously, $\vartheta\alpha^{(0)} = \vartheta\beta^{(0)}$, the coefficients of $\alpha^{(0)}$ vanish on $b\Omega$, and $\alpha^{(0)}$ is just as smooth as β.

We now restrict our attention to a single function $u^{(m)} = \vartheta\beta^{(m)}$ with $m > 0$. We wish to construct a $(0,1)$-form $\alpha^{(m)}$ in $C^s_{(0,1)}(\overline{\Omega})$ whose coefficients vanish on $b\Omega$ such that $\vartheta\alpha^{(m)} = \vartheta\beta^{(m)}$. Notice that $\beta^{(m)}$ satisfies the boundary condition (3.1). We know that $\beta^{(m)}$ is supported in a ball $B_{r_m}(z_m)$ where $z_m \in b\Omega$ and that there is a coordinate direction z_j such that $\partial\rho/\partial z_j$ is non-vanishing on the closure of $B_{r_m}(z_m)$. For convenience, we may assume that z_1 is such a coordinate direction. We now define a $(0,1)$-form $\alpha^{(m)} = \sum_{j=1}^n \alpha_j d\bar{z}_j$ via

$$\alpha_1 = \beta_1^{(m)} + \sum_{j=2}^{n} \frac{\partial}{\partial z_j}\left(\frac{\beta_j^{(m)}\rho}{\rho_{z_1}}\right), \qquad \text{and}$$

$$\alpha_k = \beta_k^{(m)} - \frac{\partial}{\partial z_1}\left(\frac{\beta_k^{(m)}\rho}{\rho_{z_1}}\right), \qquad k = 2, 3, \ldots, n.$$

It is easy to see that α_k vanishes on $b\Omega$ for $k = 2, \ldots, n$. Furthermore, since

$$\sum_{j=2}^{n} \frac{\partial}{\partial z_j}\left(\frac{\beta_j^{(m)}\rho}{\rho_{z_1}}\right) = \frac{1}{\rho_{z_1}} \sum_{j=2}^{n} \beta_j^{(m)} \frac{\partial\rho}{\partial z_j} \qquad \text{on } b\Omega,$$

the boundary condition (3.1) implies that

$$\sum_{j=2}^{n} \frac{\partial}{\partial z_j}\left(\frac{\beta_j^{(m)}\rho}{\rho_{z_1}}\right) = -\beta_1^{(m)} \qquad \text{on } b\Omega,$$

and we see that α_1 also vanishes on $b\Omega$. A simple computation now reveals that

$$\vartheta\alpha^{(m)} = \vartheta\beta^{(m)}.$$

Furthermore, $\alpha^{(m)}$ is in $C^s_{(0,1)}(\overline{\Omega})$. The global form α that we seek is now given by $\alpha = \sum_{m=0}^N \alpha^{(m)}$.

Because the operator N_t satisfies estimates when t is sufficiently large, and because the form α constructed above is only one degree less smooth than β, we may assert that there is a positive integer M with the following property. Given a positive integer s, the procedure outlined above to obtain α from u gives rise to an operator L mapping functions in $C^\infty(\overline{\Omega})$ that are orthogonal to $H^2(\Omega)$ into $C^s_{(0,1)}(\overline{\Omega})$. Furthermore, $\alpha = Lu$ satisfies an estimate of the form $\|\alpha\|_s \leq C\|u\|_{s+M}$. We emphasize here that, although C and L depend on s and t, the integer M does not.

We shall now use a Mittag-Leffler argument to construct a $(0,1)$-form α in $C^\infty_{(0,1)}(\overline{\Omega})$ whose coefficients vanish on $b\Omega$ such that $\vartheta\alpha = u$. We shall inductively construct a sequence of $(0,1)$-forms α_s such that $\alpha_s \in C^{s+M+2}_{(0,1)}(\overline{\Omega})$, the coefficients of α_s vanish on $b\Omega$, $\vartheta\alpha_s = u$, and $\|\alpha_{s+1} - \alpha_s\|_s < 1/2^s$. The desired form α in $C^\infty_{(0,1)}(\overline{\Omega})$ will be given by

$$\alpha = \alpha_1 + \sum_{s=1}^{\infty}(\alpha_{s+1} - \alpha_s).$$

We have shown how to construct α_1. Suppose that $\alpha_1, \ldots, \alpha_s$ have been constructed satisfying the desired properties. Let $\tilde{\alpha}_{s+1}$ be a form in $C^{s+M+3}_{(0,1)}(\overline{\Omega})$ satisfying $\vartheta\tilde{\alpha}_{s+1} = u$ with coefficients that vanish on $b\Omega$. The form α_{s+1} shall be given by

$$\alpha_{s+1} = \tilde{\alpha}_{s+1} - \Phi_\epsilon(\tilde{\alpha}_{s+1} - \alpha_s) + \sigma_s,$$

where Φ_ϵ is a special smoothing operator and σ_s will be a $(0,1)$-form with small $C^s_{(0,1)}(\overline{\Omega})$ norm whose coefficients vanish on $b\Omega$ satisfying

$$\vartheta\sigma_s = \vartheta\Phi_\epsilon(\tilde{\alpha}_{s+1} - \alpha_s).$$

The smoothing operator Φ_ϵ will have the property that it maps a form $\beta \in C^{s+M+1}_{(0,1)}(\overline{\Omega})$ to a form in $C^\infty_{(0,1)}(\overline{\Omega})$ in such a way that $\|\beta - \Phi_\epsilon\beta\|_{s+M} \leq c_\epsilon\|\beta\|_{s+M+1}$ where c_ϵ tends to zero as ϵ tends to zero. Furthermore, if the coefficients of β vanish on $b\Omega$, then so do the coefficients of $\Phi_\epsilon\beta$. We shall describe how to construct such a smoothing operator Φ_ϵ at the end of this proof. Now, we shall finish the proof of the theorem, assuming that we have Φ_ϵ at our disposal.

Let $\omega = \Phi_\epsilon(\tilde{\alpha}_{s+1} - \alpha_s)$ and let $v = \vartheta\omega$. Note that v is in $C^\infty(\overline{\Omega})$ and that v is orthogonal to $H^2(\Omega)$ because ω vanishes on $b\Omega$. Notice that, since $\vartheta(\tilde{\alpha}_{s+1} - \alpha_s) = u - u = 0$, it follows that

$$v = \vartheta\left[\Phi_\epsilon(\tilde{\alpha}_{s+1} - \alpha_s) - (\tilde{\alpha}_{s+1} - \alpha_s)\right].$$

Hence, by taking ϵ small, the norm $\|v\|_{s+M}$ can be made small. We may now find a form σ_s which is in $C^{s+M+3}_{(0,1)}$ whose coefficients vanish on $b\Omega$ such that $\vartheta\sigma_s = v$.

Furthermore, by taking ϵ small, and by using the solution operators mentioned above, we may guarantee that the norm $\|\sigma_s\|_s$ is as small as we please. We now define

$$\alpha_{s+1} = \tilde{\alpha}_{s+1} - \Phi_\epsilon(\tilde{\alpha}_{s+1} - \alpha_s) + \sigma_s.$$

Since $\tilde{\alpha}_{s+1} - \alpha_s - \Phi_\epsilon(\tilde{\alpha}_{s+1} - \alpha_s) + \sigma_s$ has been constructed to be a form in $C^{s+M+3}_{(0,1)}$ with small s-norm whose coefficients vanish on $b\Omega$, the proof is complete.

Finally, we must show how to construct the smoothing operator Φ_ϵ. The construction hinges on the following simple one real variable argument. Suppose that $f(x)$ is a function in $C^{s+1}(\mathbb{R})$ that vanishes at the origin. Notice that

$$f(x) = \int_0^x f'(t)\,dt = x\int_0^1 f'(tx)\,dt.$$

Let θ_ϵ be an approximation of the identity in $C_0^\infty(\mathbb{R})$ and define

$$(\Phi_\epsilon f)(x) = x\int_0^1 (\theta_\epsilon * f)'(tx)\,dt. \tag{3.2}$$

It is easy to see that $(\Phi_\epsilon f)$ is a function in $C^\infty(\mathbb{R})$ that vanishes at the origin and that the s-norm of $f - \Phi_\epsilon f$ on a compact ball is bounded by a constant c_ϵ times the $(s+1)$-norm of f where $c_\epsilon \to 0$ as $\epsilon \to 0$. To construct the operator Φ_ϵ on Ω, we may use a partition of unity to reduce our problem to creating an operator Φ_ϵ that acts on forms that are supported on a small ball $B_r(z_0)$ where $z_0 \in b\Omega$. We may further use a C^∞ change of variables in order to be able to assume that $z_0 = 0$ and that the boundary of Ω is equal to the real hyperplane Im $z_1 = 0$ near z_0. It will also suffice to construct on operator that maps *functions* vanishing on $b\Omega$ to the same kind of functions. Finally, the operator given by (3.2), using $x = \text{Im } z_n$ and allowing the other variables Re z_1 and $z_2, \dots, z_n$ to be carried along as parameters, satisfies the conditions we require. The proof is finished.

We shall also need the following local version of Rosay's Lemma.

Theorem 3.2 (Local Rosay Lemma). *Suppose Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ and that the boundary of Ω is C^∞ smooth near a boundary point z_0 which is of finite type in the sense of D'Angelo. If u is a function in $L^2(\Omega)$ which is C^∞ smooth up to $b\Omega$ near z_0 and which is orthogonal to the Bergman space $H^2(\Omega)$, then there exist an $\epsilon > 0$ and functions $\alpha_j \in C^\infty(\overline{\Omega} \cap B_\epsilon(z_0))$, $j = 1, \dots, n$, which vanish on $b\Omega$ near z_0 such that the $(0,1)$-form $\alpha = \sum_{j=1}^n \alpha_j d\bar{z}_j$ satisfies*

$$u = \vartheta\alpha \qquad \text{on } \Omega \cap B_\epsilon(z_0).$$

Consequently, there exists a function $\tilde{u} \in C^\infty(\overline{\Omega})$ which is supported in $\Omega \cap B_\epsilon(z_0)$ such that $\tilde{u} = u$ near z_0 and $\tilde{u} \perp H^2(\Omega)$.

The proof of this local version is a direct application of the subelliptic estimates of the $\bar\partial$-Neumann problem at points of finite type proved by Catlin [14-16]. The Bergman projection is given by

$$P = I - \vartheta N \bar\partial,$$

where N is the (unweighted) $\bar{\partial}$-Neumann operator. Catlin [14, p. 164] showed that, even in a non-smooth pseudoconvex domain, the Neumann operator exists and the operators N, $\vartheta N\bar{\partial}$, and $N\bar{\partial}$ are bounded in L^2 norms. If u is orthogonal to $H^2(\Omega)$, then $Pu = 0$ and so

$$u = \vartheta N\bar{\partial}u.$$

If u is C^∞ smooth up to the boundary near z_0, then the subelliptic estimates for the $\bar{\partial}$-Neumann problem near points of finite type yield that the coefficients of the $(0,1)$-form $\beta = N\bar{\partial}u$ are C^∞ smooth up to the boundary near z_0. We have produced a form β satisfying $u = \vartheta\beta$ where β satisfies the boundary condition (3.1) near z_0. Now the same procedure that we used in the proof of Theorem 3.1 can be used to obtain a $(0,1)$-form α from β which vanishes on $b\Omega$ near z_0.

Now that we have set up the notation for the $\bar{\partial}$-Neumann problem, we can explain an important example where the ϑ-UCP can be seen to hold at certain strictly pseudoconvex boundary points of pseudoconvex domains. Suppose Ω is a bounded pseudoconvex domain and that z_0 is a strictly pseudoconvex boundary point such that $b\Omega$ is a real analytic hypersurface near z_0. Suppose α is a $(0,1)$-form on Ω whose coefficients are C^∞ smooth up to $b\Omega$ near z_0 and vanish there. Suppose further that α satisfies properties (1) and (2) in the statement of the ϑ-UCP. By replacing α by $\chi\alpha$ where χ is a C^∞ cut off function supported near z_0 that is equal to one on a neighborhood of z_0, we may assume that α is globally C^∞ smooth, that $\vartheta\alpha$ is in $L^2(\Omega)$, that α vanishes on $b\Omega$, and that $\vartheta\alpha$ is holomorphic near z_0. Under these conditions, it follows that $\vartheta\alpha$ is orthogonal to $H^2(\Omega)$. Thus, $P(\vartheta\alpha) \equiv 0$. But $P(\vartheta\alpha) = \vartheta\alpha - \vartheta N\bar{\partial}(\vartheta\alpha)$, and hence, $\vartheta\alpha = \vartheta N\bar{\partial}(\vartheta\alpha)$ on Ω. Since $\vartheta\alpha$ is holomorphic near z_0, it follows that $\bar{\partial}\vartheta\alpha$ is zero near z_0 and the analytic hypoellipticity of the $\bar{\partial}$-Neumann problem at strictly pseudoconvex boundary points (see [39-41]) implies that $\vartheta N\bar{\partial}(\vartheta\alpha)$ extends to be real analytic on a neighborhood of z_0. Therefore, $\vartheta\alpha$ extends to be holomorphic on a neighborhood of z_0. Consequently, infinite order vanishing of $\vartheta\alpha$ at z_0 implies vanishing on a full neighborhood of z_0 and the ϑ-UCP property is seen to hold at z_0.

The ϑ-UCP also holds at weakly pseudoconvex boundary points where the boundary is real analytic whenever the $\bar{\partial}$-problem is known to be locally analytic hypoelliptic there (see Derridj and Tartakoff [20] for examples of such boundary points). Recent work of Christ and Geller [18] shows that local analytic hypoellipticity can fail at certain weakly pseudoconvex boundary points, even when they are of finite type. This is one of the reasons that, although I am reasonably confident that the ϑ-UCP holds at strictly pseudoconvex boundary points, I have doubts about the truth of the ϑ-UCP at general weakly pseudoconvex boundary points of finite type.

4. Unique continuation for the Bergman projection. We are now in a position to see how the ϑ-UCP relates to the Bergman projection and kernel. A bounded domain Ω in $\mathbb{C}^n$ with C^∞ smooth boundary is said to satisfy *Condition R* if its Bergman projection preserves the space $C^\infty(\overline{\Omega})$.

Theorem 4.1. *Suppose that Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^∞ smooth boundary that satisfies Condition R and suppose that z_0 is a boundary point of Ω which satisfies the ϑ-UCP. Suppose $\varphi \in C_0^\infty(\Omega)$. If $P\varphi$ vanishes to infinite order at z_0, then $P\varphi$ must vanish identically on Ω.*

Proof. Given $\varphi \in C_0^\infty(\Omega)$, let $h = P\varphi$. Since $h - \varphi$ is a function in $C^\infty(\overline{\Omega})$ that is orthogonal to $H^2(\Omega)$, Rosay's lemma (Theorem 3.1) yields a form $\alpha \in C^\infty_{(0,1)}(\overline{\Omega})$ whose coefficients vanish on $b\Omega$ such that

$$h - \varphi = \vartheta\alpha.$$

Finally, $\vartheta\alpha$ is equal to the holomorphic function h near z_0, and the ϑ-UCP at z_0 implies the conclusion of the theorem.

Remark. Actually, the assumption that $\varphi \in C^\infty_0(\Omega)$ in Theorem 4.1 can be replaced with the weaker assumption that $\varphi \in L^2(\Omega)$ has compact support. Indeed, if $\varphi \in L^2(\Omega)$ has compact support $\mathcal{K}$ in Ω, we may construct a function $\psi \in C^\infty_0(\Omega)$ such that $P\psi = P\varphi$ as follows. Let δ denote the distance from $\mathcal{K}$ to $b\Omega$ and let

$$\theta_\delta(z) = \delta^{-2n}\theta(z/\delta)$$

where θ is a function in $C^\infty_0(B_1(0))$ that is radially symmetric about the origin with $\int \theta\, dV = 1$. Let $\psi = \theta_\delta * \varphi$. A straightforward application of Fubini's theorem and the averaging property of holomorphic functions yields that

$$\int_\Omega \varphi\overline{h}\; dV = \int_\Omega \psi\overline{h}\; dV$$

for all functions h that are holomorphic on Ω. Hence $P\varphi = P\psi$. Later, we shall need to take this argument one step further. If $\mathcal{K}$ is a compact subset of Ω and $d\mu$ is a complex finite Borel measure on $\mathcal{K}$, then we may define the Bergman projection of $d\mu$ via

$$(P\, d\mu)(z) = \int_{w\in\mathcal{K}} K(z,w)\, d\mu.$$

We may argue as above to see that the function $\psi = \theta_\delta * d\mu$ is a function in $C^\infty_0(\Omega)$ satisfying $P\psi = P\, d\mu$.

Theorem 3.2 can be used to prove a local version of Theorem 4.1. In the next theorem, the domain Ω is not assumed to be globally C^∞ smooth.

Theorem 4.2. *Suppose Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ and that the boundary of Ω is C^∞ smooth near a boundary point z_0 which is of finite type in the sense of D'Angelo. Suppose further that the ϑ-UCP holds at z_0. Given $\varphi \in L^2(\Omega)$ which vanishes on $B_\epsilon(z_0) \cap \Omega$ for some $\epsilon > 0$, if $P\varphi$ vanishes to infinite order at z_0, then $P\varphi$ must vanish identically on Ω.*

Proof. Let $h = P\varphi$. The subelliptic estimate for the $\bar\partial$-problem at z_0 implies that h is C^∞ smooth up to the boundary near z_0. Since $h - \varphi$ is a function in $L^2(\Omega)$ and in $C^\infty(\overline{\Omega} \cap B_\epsilon(z_0))$ that is orthogonal to $H^2(\Omega)$, Theorem 3.2 yields a form $\alpha \in C^\infty_{(0,1)}(\overline{\Omega} \cap B_\epsilon(z_0))$ whose coefficients vanish on $b\Omega$ such that $h - \varphi = \vartheta\alpha$ near z_0. The conclusion of the theorem follows from the ϑ-UCP at z_0 and the fact that $\vartheta\alpha = h$ near z_0.

Because the Bergman kernel function associated to a bounded domain Ω is equal to the Bergman projection of a function in $C^\infty_0(\Omega)$, the two theorems above yield information about the finite order vanishing of the kernel function at boundary points. To be precise, given a point w_0 in a bounded pseudoconvex domain Ω with C^∞ smooth boundary that satisfies Condition R, let δ denote the distance from

w_0 to $b\Omega$ and let $\theta_{w_0}(z) = \delta^{-2n}\theta((z - w_0)/\delta)$ where, as before, θ is a function in $C_0^\infty(B_1(0))$ that is radially symmetric about the origin such that $\int \theta\, dV = 1$. The Bergman kernel $K(z, w)$ satisfies

$$K(z, w_0) = (P\theta_{w_0})(z).$$

Given a multi-index β, let

$$\theta_{w_0}^\beta = (-1)^{|\beta|} \frac{\partial^{|\beta|}}{\partial \bar{z}^\beta} \theta_{w_0}(z).$$

The Bergman kernel also satisfies

$$\frac{\partial^{|\beta|}}{\partial \bar{w}^\beta} K(z, w_0) = (P\theta_{w_0}^\beta)(z).$$

The following theorem follows from these facts together with Theorem 4.1.

Theorem 4.3. *Suppose that Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^∞ smooth boundary that satisfies Condition R and suppose that z_0 is a boundary point of Ω which satisfies the ϑ-UCP. Given a multi-index β and a point $w_0 \in \Omega$, there exists a multi-index α such that*

$$\frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \partial \bar{w}^\beta} K(z_0, w_0) \neq 0.$$

The next theorem allows both z_0 and w_0 to be in the boundary and does not assume that the domain is globally C^∞ smooth.

Theorem 4.4. *Suppose Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ and that the boundary of Ω is C^∞ smooth near a boundary point z_0 which is of finite type in the sense of D'Angelo. Suppose further that z_0 is a boundary point of Ω which satisfies the ϑ-UCP. If $w_0 \in \overline{\Omega}$ is such that, either*

a) *$w_0 \in \Omega$, or*
b) *$w_0 \in b\Omega$, $w_0 \neq z_0$, the boundary of Ω is C^∞ smooth near w_0, and w_0 is a point of finite type in the sense of D'Angelo,*

then, given a multi-index β, there exists a multi-index α such that

$$\frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \partial \bar{w}^\beta} K(z_0, w_0) \neq 0.$$

Proof. We shall prove the theorem in case w_0 is a boundary point. The proof in case $w_0 \in \Omega$ is similar to the proof of Theorem 4.3 and we leave it to the reader. Assume that z_0 and w_0 are boundary points, that the boundary is C^∞ smooth and of finite type near these points, and that $z_0 \neq w_0$. It is proved in [3,12] that the Bergman kernel $K_\Omega(z, w)$ for Ω extends C^∞ smoothly to $b\Omega \times b\Omega$ near (z_0, w_0). (Actually, this fact can also be deduced easily from the decomposition of the Bergman kernel that we are about to describe.) It is possible to construct (see [3]) a small pseudoconvex domain D with C^∞ smooth boundary which is of finite type in the sense of D'Angelo such that

1) $D \subset \Omega$,
2) $B_\epsilon(w_0) \cap D = B_\epsilon(w_0) \cap \Omega$ for some $\epsilon > 0$, and
3) $z_0 \notin \overline{D}$.

Let $K_\Omega(z,w)$ denote the Bergman kernel associated to Ω and let $K_D(z,w)$ denote the Bergman kernel associated to D. Since D is of finite type, the $\bar\partial$-Neumann problem on D is subelliptic and Kerzman's theorem yields that $K_D(z,w)$ is in $C^\infty((\overline{D}\times\overline{D})-\{(z,z): z\in bD\})$. If β is a multi-index, define

$$K_D^\beta(z,w)=\frac{\partial^{|\beta|}}{\partial\bar w^\beta}K_D(z,w),$$

and define $K_\Omega^\beta(z,w)$ similarly. If $w\in D$, define

$$\Phi_w^\beta(z)=\begin{cases} K_D^\beta(z,w), & \text{if } z\in D\\ 0, & \text{if } z\in\Omega-D.\end{cases}$$

We now claim that, if $w\in D$ and $z\in\Omega$, then

$$K_\Omega^\beta(z,w)=(P\Phi_w^\beta)(z).$$

To see this, note that, given $h\in H^2(\Omega)$, we may write

$$\int_\Omega h\,\overline{\Phi_w^\beta}\,dV=\int_D h(z)\,\overline{K_D^\beta(z,w)}\,dV=\frac{\partial^{|\beta|}}{\partial w^\beta}h(w).$$

Since $K_\Omega^\beta(z,w)$ is a holomorphic function of z in $H^2(\Omega)$ that has the same effect when paired with $h\in H^2(\Omega)$, it follows that $K_\Omega^\beta(z,w)=(P\Phi_w^\beta)(z)$. Let χ be a function in $C_0^\infty(B_\epsilon(w_0))$ that is equal to one on a neighborhood of w_0. We may now write

$$K_\Omega^\beta(\cdot,w)=P(\chi\Phi_w^\beta)+P((1-\chi)\Phi_w^\beta).$$

If the point $w\in D$ is allowed to approach w_0, the smoothness property of the Bergman kernel on D (Kerzman's theorem) implies that $(1-\chi)\Phi_w^\beta$ tends in $L^2(\Omega)$ to the function $(1-\chi)\Phi_{w_0}^\beta\in L^2(\Omega)$. Furthermore, even though $\chi\Phi_w^\beta$ does not converge to a function in $L^2(\Omega)$, the form $\bar\partial(\chi\Phi_w^\beta)$ tends in $C^\infty_{(0,1)}(\overline{\Omega})$ to $\bar\partial(\chi\Phi_{w_0}^\beta)$, which, by Kerzman's theorem, is also in $C^\infty_{(0,1)}(\overline{\Omega})$. Recall that the Bergman projection can be written $P=I-\vartheta N\bar\partial$. Since $\chi\Phi_w^\beta$ and $(1-\chi)\Phi_w^\beta$ are supported away from z_0, it follows that, for z near z_0 and w near w_0, we have

$$K_\Omega^\beta(z,w)=-\vartheta N[\bar\partial(\chi\Phi_w^\beta)]-\vartheta N\bar\partial[(1-\chi)\Phi_w^\beta], \tag{4.1}$$

and that as $w\to w_0$, the functions in this decomposition all converge very nicely to yield the decomposition,

$$K_\Omega^\beta(z,w_0)=-\vartheta N[\bar\partial(\chi\Phi_{w_0}^\beta)](z)-\left(\vartheta N\bar\partial[(1-\chi)\Phi_{w_0}^\beta]\right)(z),$$

which is valid for z near z_0. Both functions on the right hand side of this decomposition are in $L^2(\Omega)$ and are orthogonal to $H^2(\Omega)$. Furthermore, the subelliptic estimate for the $\bar\partial$-Neumann problem at z_0 implies that both functions also extend C^∞ smoothly up to $b\Omega$ near z_0. Hence, Theorem 3.2 implies that $K_\Omega^\beta(z,w_0)=\vartheta\alpha$ for a $(0,1)$-form α whose coefficients extend C^∞ smoothly up to $b\Omega$ near z_0 and vanish on $b\Omega$ near z_0. Finally, the ϑ-UCP implies that $K_\Omega^\beta(z, w_0)$ cannot vanish to infinite order as a function of z at z_0, and the proof is complete.

We remark that, in the plane, all bounded domains with C^∞ smooth boundary satisfy Condition R and all C^∞ smooth boundary points of bounded domains satisfy the ϑ-UCP. Therefore, Theorems 4.1–4.4 hold in the plane without all the unsightly extra hypotheses.

The non-vanishing property of the Bergman kernel described in Theorem 4.4 is exactly what is needed to set up *local Bergman-Ligocka* coordinates of the type used in [10] and [5] to study the boundary behavior of biholomorphic maps. In particular, the following theorem would follow if the ϑ-UCP were known to hold at strictly pseudoconvex boundary points (see Klingenberg [27] and [5]).

Theorem 4.5. *Suppose Ω is a bounded weakly pseudoconvex domain in $\mathbb{C}^n$ with C^∞ smooth boundary that is of finite type in the sense of D'Angelo, and suppose that f_j is a sequence of automorphisms of Ω that converge to a holomorphic map $f : \Omega \to \overline{\Omega}$. If the ϑ-UCP holds at a strictly pseudoconvex boundary point z_0 of Ω, then there is an $\epsilon > 0$ such that the limit map f is in $C^\infty(B_\epsilon(z_0) \cap \overline{\Omega})$ and f_j converges to f in this space.*

We remark that it is a standard fact that the limit map f in Theorem 4.5 must either be an automorphism of Ω or a constant mapping $f \equiv w_0$ where w_0 is a weakly pseudoconvex boundary point of Ω, and hence it is an easy part of the theorem to prove that f is in $C^\infty(\overline{\Omega})$. The hard part of the proof is to see that the sequence of automorphisms converge to f in $C^\infty(B_\epsilon(z_0) \cap \overline{\Omega})$.

5. Bergman kernel density theorems. In a pseudoconvex domain, the ϑ-UCP is closely related to a density property of the Bergman kernel. In what follows, we shall refer to the linear span of a set of functions in a somewhat abbreviated fashion. For example, we shall mention the complex linear span of the set of functions $\{K(z, w) : w \in \Omega\}$, and by this we shall mean the vector space of holomorphic functions on Ω generated by functions h of the form $h(z) = K(z, w)$, $w \in \Omega$.

If $\mathcal{K}$ is a compact subset of Ω, let $\widehat{\mathcal{K}}$ denote the hull of $\mathcal{K}$ with respect to holomorphic functions on Ω.

Theorem 5.1. *Suppose that Ω is a bounded pseudoconvex domain with C^∞ smooth boundary that is of finite type in the sense of D'Angelo, and suppose that w_0 is a boundary point of Ω that satisfies the ϑ-UCP. Suppose that $\mathcal{K}$ is a compact subset of Ω such that $\widehat{\mathcal{K}} = \mathcal{K}$. Given a holomorphic function f defined on a neighborhood of $\mathcal{K}$ and a number $\epsilon > 0$, there is a function κ in the complex linear span $\mathcal{S}$ of*

$$\left\{ \frac{\partial^{|\beta|}}{\partial \bar{w}^\beta} K(z, w_0) \ : \ |\beta| \geq 0 \right\}$$

such that $|f - \kappa| < \epsilon$ on $\mathcal{K}$.

Proof. In what follows, to streamline the writing, we shall frequently exhibit mathematical bad taste by thinking of one function space as a subspace of another, even though the domains on which the functions are defined are different. Thus, for example, we shall speak of $A^\infty(\Omega)$ as if it were a subspace of $H^2(D)$ when D is an open subset of Ω without mentioning that we are actually restricting functions to subsets.

Suppose that f is a holomorphic function on a neighborhood U of $\mathcal{K}$ which cannot be approximated uniformly on $\mathcal{K}$ by functions in $\mathcal{S}$. Then there would exist a complex finite Borel measure $d\mu$ on $\mathcal{K}$ such that

$$\int_{\mathcal{K}} h\, d\mu = 0 \qquad \text{for all } h \in \mathcal{S}, \tag{5.1}$$

but $\int_{\mathcal{K}} f\, d\mu \neq 0$. Since Ω is pseudoconvex and of finite type, the Bergman kernel is in $C^\infty((\overline{\Omega} \times \overline{\Omega}) - \{(z,z) : z \in b\Omega\})$, and it follows that $P\, d\mu$ is in $C^\infty(\overline{\Omega})$. The orthogonality condition (5.1) implies that $P\, d\mu$ vanishes to infinite order at w_0. The remark after Theorem 4.1 yields a function $\psi \in C_0^\infty(\Omega)$ such that $P\psi = P\, d\mu$ and Theorem 4.1 shows that $P\, d\mu \equiv 0$. This means that $d\mu$ is orthogonal to the complex linear span of the set of functions

$$\{K(z,w) \,:\, w \in \Omega\}.$$

But this linear span is dense in $A^\infty(\Omega)$ (see [10]), and so $d\mu$ is orthogonal to $A^\infty(\Omega)$. Catlin proved [13, Theorem 3.2.1] (note the third remark on page 618) that there is an open set V with $\mathcal{K} \subset V \subset U$, and functions $f_j \in A^\infty(\Omega)$ such that $f_j \to f$ in $H^2(V)$. Convergence in $H^2(V)$ implies uniform convergence on $\mathcal{K}$. A contradiction is now obtained by writing

$$0 = \int_{\mathcal{K}} f_j\, d\mu \to \int_{\mathcal{K}} f\, d\mu \neq 0$$

and the conclusion of the theorem is proved.

We shall prove a generalized version of Theorem 5.1 at the end of this section in which the set $\mathcal{K}$ is allowed to intersect the boundary of the domain. Next, however, we shall prove a density theorem in which functions can be approximated in a much stronger sense by functions in $\mathcal{S}$.

Theorem 5.2. *Suppose Ω is a bounded pseudoconvex domain of finite type in the sense of D'Angelo with C^∞ smooth boundary. Suppose z_0 and w_0 are boundary points of Ω, $z_0 \neq w_0$, and the ϑ-UCP holds at w_0. Given an $\delta > 0$ and a function h which is holomorphic on $B_\delta(z_0) \cap \Omega$ and in $C^\infty(B_\delta(z_0) \cap \overline{\Omega})$, there is a sequence of functions in the linear span $\mathcal{S}$ (as defined in Theorem 5.1) which tends to h in $C^\infty(B_\epsilon(z_0) \cap \overline{\Omega})$ for some $\epsilon \leq \delta$.*

Proof. To prove this theorem, we shall need to use the duality theory developed in [6,7,8] (see also Ligocka [32,33]). If D is a bounded domain in $\mathbb{C}^n$ with C^∞ smooth boundary, $A^\infty(D)$ denotes the space of holomorphic functions in $C^\infty(\overline{D})$ equipped with the topology inherited from that space. If D is further assumed to be pseudoconvex and of finite type in the sense of D'Angelo, then D satisfies Condition R and it follows that the dual of $A^\infty(D)$ is given by the space $A^{-\infty}(D)$ which is defined as the space of holomorphic functions g on D that satisfy a growth estimate of the form

$$|g(z)| d(z)^s \leq C$$

where $d(z)$ denotes the distance from z to bD, s is some positive integer, and C is a constant. The duality is expressed via an extension of the usual $L^2(\Omega)$ inner product (see [8]).

Suppose that h is a holomorphic function in $C^\infty(B_\delta(z_0)\cap\overline{\Omega})$ where $|z_0-w_0|>\delta>0$. Since z_0 is a point of finite type, it is possible to construct arbitrarily small domains D such that D is a C^∞ smooth pseudoconvex domain of finite type, $D\subset\Omega$, and $D\cap B_r(z_0)=\Omega\cap B_r(z_0)$ for some small $r>0$. Furthermore, such domains can be constructed which are strictly star-like (and arbitrarily C^1 close to being a ball [1]) so that we may assume that the space of holomorphic polynomials is dense in $A^\infty(D)$. We now choose one such domain D_1 so that $D_1\subset B_\delta(z_0)\cap\Omega$. Let $r>0$ be small enough that $D_1\cap B_r(z_0)=\Omega\cap B_r(z_0)$, and choose another small domain D_2 of finite type so that $D_2\subset D_1\cap B_r(z_0)$, and $(bD_2-b\Omega)\subset D_1$, and the boundary of D_2 agrees with that of D_1 (and Ω) near z_0. We shall prove that h can be approximated by functions in $\mathcal{S}$ as described in the statement of the theorem by proving that $\mathcal{S}$ is dense in $A^\infty(D_2)$.

Suppose that $\mathcal{S}$ is not dense in $A^\infty(D_2)$. Then there would exist a function $g\not\equiv 0$ in $A^{-\infty}(D_2)$ such that the extended inner product

$$\langle g,\kappa\rangle_{D_2}=0 \qquad \text{for all } \kappa\in\mathcal{S}.$$

Define

$$\mathcal{G}(z)=\langle g,K(\cdot,z)\rangle_{D_2} \qquad \text{for } z\in\Omega.$$

Since $K(z,w)$ extends C^∞ smoothly to $b\Omega\times b\Omega$ minus the boundary diagonal, it is easy to check that $\mathcal{G}$ is a holomorphic function on Ω that extends C^∞ smoothly up to the boundary of Ω near w_0. Notice that the orthogonality condition implies that $\mathcal{G}$ vanishes to infinite order at w_0. We shall prove the theorem by showing that we may think of $\mathcal{G}$ as being equal to the Bergman projection on Ω of the function that is equal to g on D_2 and equal to zero on $\Omega-D_2$ and that the ϑ-UCP applies in this generalized setting to yield that $\mathcal{G}\equiv 0$ on Ω. From this it will follow that

$$\langle g,K(\cdot,z)\rangle_{D_2}=0 \qquad \text{for all } z\in\Omega.$$

But the linear span of the set of functions $K(\cdot,z)$ as z ranges over Ω is dense in $A^\infty(\Omega)$ (see [10]). We know that polynomials are dense in $A^\infty(D_2)$ and hence it follows that the linear span of the functions $K(\cdot,z)$ as z ranges over Ω is dense in $A^\infty(D_2)$. Hence, it will follow that g must be orthogonal to $A^\infty(D_2)$, and since the extended pairing is non-degenerate, this will imply that $g\equiv 0$, contrary to hypothesis, and the proof will be complete. To summarize, the proof will be accomplished if we prove that $\mathcal{G}\equiv 0$ on Ω.

Since $A^\infty(D_2)$ is dense in $A^{-\infty}(D_2)$ (see [8]), there exists a sequence of functions g_j in $A^\infty(D_2)$ converging to g in $A^{-\infty}(D_2)$. Let P denote the Bergman projection on Ω, P_1 the Bergman projection on D_1, and P_2 the Bergman projection on D_2. Let $K(z,w)$, $K_1(z,w)$ and $K_2(z,w)$ denote the respective Bergman kernel functions associated to Ω, D_1, and D_2. We may think of the functions g and g_j as also being defined on D_1 or Ω by setting these functions to be zero on $\Omega-D_2$. We now claim that P_1g_j tends to a function G in $A^{-\infty}(D_1)$ and that we may think of G as being equal to P_1g. A sequence convergenes in the space $A^{-\infty}(D_1)$ if it converges in some negative Sobolev norm as described in [8]. If s is a positive integer, the Sobolev $-s$ norm of a holomorphic function f on D_1 is given by

$$\|f\|_{-s}=\sup\left\{\left|\int_{D_1} f\,\varphi\,dV\right| \;:\; \varphi\in C_0^\infty(D_1),\ \|\varphi\|_s=1\right\},$$

where $\|\varphi\|_s$ denotes the usual Sobolev s norm of φ. The space $A^{-s}(D_1)$ consisting of functions in $A^{-\infty}(D_1)$ with finite $-s$ norm is a Banach space under the $-s$ norm. We shall also need to know that the $-s$ norm of a holomorphic function $f \in H^2(D_1)$ can be estimated by means of the L^2 inner product (which, incidentally, agrees with the extended L^2 inner product when the functions involved are in L^2). There is a constant $C = C(s, D_1)$ such that

$$\|f\|_{-s} \leq C \sup \left\{ \left| \int_{D_1} f\, \overline{h}\, dV \right| \; : \; h \in A^\infty(D_1),\; \|h\|_s = 1 \right\}.$$

There is also a constant $c = c(s, D_1)$ such that

$$\left| \int_{D_1} f\, \overline{h}\, dV \right| \leq c \|f\|_{-s} \|h\|_s$$

for $f \in H^2(D_1)$ and $h \in A^\infty(D_1)$.

Because $g_j \in A^\infty(D_2)$, there is a function $\varphi_j \in C^\infty(\overline{D_2})$ which vanishes to infinite order on bD_2 such that $P_2 \varphi_j = g_j$ (see [6,9]). We may think of φ_j as also being in the space $C^\infty(\overline{D_1})$ by extending φ_j to be zero on $D_1 - D_2$, and it is easy to verify that $P_1 g_j = P_1 \varphi_j$. Since pseudoconvex domains of finite type satisfy Condition R, it follows that $P_1 g_j \in A^\infty(D_1)$. We may now estimate

$$\begin{aligned} &\|P_1 g_j - P_1 g_k\|_{-s}^{D_1} \\ &\leq C \sup \left\{ \left| \int_{D_1} P_1(g_j - g_k)\, \overline{h}\, dV \right| : h \in A^\infty(D_1),\; \|h\|_s = 1 \right\} \\ &\leq C \sup \left\{ \left| \int_{D_2} (g_j - g_k)\, \overline{h}\, dV \right| \; : \; h \in A^\infty(D_1),\; \|h\|_s = 1 \right\} \\ &\leq (\text{constant}) \|g_j - g_k\|_{-s}^{D_2}. \end{aligned}$$

Since g_j is a Cauchy sequence in $A^{-s}(D_2)$, this estimate shows that $P_1 g_j$ is a Cauchy sequence in $A^{-s}(D_1)$, and hence that $P_1 g_j$ converges to some function G in $A^{-\infty}(D_1)$. Because the Bergman kernel function of D_1 extends C^∞ smoothly up to $bD_1 \times bD_1$ away from the boundary diagonal, and because

$$G(z) = \lim_{j \to \infty} \int_{w \in D_2} K_1(z, w) g_j(w)\; dV = \langle g, K_1(\cdot, z) \rangle_{D_2},$$

it is easy to see that G extends C^∞ smoothly up to the part of the boundary of D_1 given by $bD_1 - bD_2$. In fact, given a point $\zeta_0 \in bD_1 - bD_2$, there is a radius $r > 0$ such that $P_1 g_j$ tends to G in $C^\infty(B_r(\zeta_0) \cap \overline{D_1})$. Let G_j denote the function which is equal to $P_1 g_j$ on D_1 and equal to zero on $\Omega - D_1$, and let $\mathcal{G}_j = P G_j$, i.e.,

$$\mathcal{G}_j(z) = \langle G_j, K(\cdot, z) \rangle_{D_1} \qquad \text{for } z \in \Omega.$$

It is easy to see that $\mathcal{G}_j = P G_j = P g_j$ since all three of these functions, when paired in the $H^2(\Omega)$ inner product with a function $f \in H^2(\Omega)$, give $\int_{D_2} g_j\, \overline{f}\, dV$. The same reasoning we used above can be applied to see that $\mathcal{G}_j$ tends to $\mathcal{G}$ in $A^{-\infty}(\Omega)$ and that this sequence also convergenes in $C^\infty(B_r(w_0) \cap \overline{\Omega})$ for small r. Let χ be a

C^∞ function that is equal to one on a neighborhood of $\overline{D_2}$ and equal to zero on a neighborhood of the closure of $bD_1 \cap \Omega$. We shall split $\mathcal{G}_j$ into two pieces via

$$\mathcal{G}_j = P[\chi G_j] + P[(1-\chi)G_j].$$

Since $P = I - \vartheta N \bar{\partial}$, and since G_j is zero near w_0, we may further write

$$\mathcal{G}_j = -\vartheta N[\bar{\partial}(\chi G_j)] + -\vartheta N \bar{\partial}[(1-\chi)G_j] \qquad \text{near } w_0.$$

Now, the functions $(1-\chi)G_j$ tend in $L^2(\Omega)$ to $(1-\chi)G$ (which is in $L^2(\Omega)$ because G extends smoothly to $bD_1 - bD_2$), and the $(0,1)$-forms $\bar{\partial}[\chi G_j]$ tend in $C^\infty_{(0,1)}(\overline{\Omega})$ to $\bar{\partial}[\chi G]$. Hence, by letting $j \to \infty$, we obtain

$$\mathcal{G} = -(\vartheta N)[\bar{\partial}(\chi G)] - (\vartheta N \bar{\partial})[(1-\chi)G] \qquad \text{near } w_0.$$

Since $(\vartheta N)[\bar{\partial}(\chi G)]$ and $(\vartheta N \bar{\partial})[(1-\chi)G]$ are both functions in $L^2(\Omega)$ that are orthogonal to $H^2(\Omega)$ and that extend C^∞ smoothly to $b\Omega$ near w_0, we deduce via Theorem 3.2 and the ϑ-UCP that $\mathcal{G} \equiv 0$ and the proof is finished.

We conclude this section by showing how the proof of Theorem 5.1 can be modified to yield an improved result. In the statement of the next theorem, we shall use the following notation. If U is a relatively open subset of $\overline{\Omega}$, we let $A^\infty(U)$ denote the set of holomorphic functions on U°, the interior of U, which are bounded and which have bounded derivatives of all orders on U°.

Theorem 5.3. *Suppose that Ω is a bounded pseudoconvex domain with C^∞ smooth boundary that is of finite type in the sense of D'Angelo, and suppose that w_0 is a boundary point of Ω that satisfies the ϑ-UCP. Suppose that $\mathcal{K}$ is a compact subset of $\overline{\Omega}$ which is convex with respect to $A^\infty(\Omega)$ such that $w_0 \notin \mathcal{K}$. Suppose that f is a function which is defined and holomorphic on a relatively open subset U of $\overline{\Omega}$ containing $\mathcal{K}$ and $f \in A^\infty(U)$. Given a number $\epsilon > 0$, there is a function κ in the complex linear span $\mathcal{S}$ of*

$$\left\{ \frac{\partial^{|\beta|}}{\partial \bar{w}^\beta} K(z, w_0) \, : \, |\beta| \geq 0 \right\}$$

such that $|f - \kappa| < \epsilon$ on $\mathcal{K}$.

The proof of this result follows the same steps as the proof of Theorem 5.1. There are two points in the proof that need additional attention because $\mathcal{K}$ might intersect the boundary. The first point concerns the projection of the Borel measure $d\mu$ on $\mathcal{K}$. We may define $P\,d\mu$ as before. The smoothness properties of the Bergman kernel yield that $P\,d\mu$ is a holomorphic function on Ω that extends C^∞ smoothly up to the boundary of Ω near w_0. We must prove that if $P\,d\mu$ vanishes to infinite order at w_0, then $P\,d\mu \equiv 0$. To see this, we shall show that there is a function ψ in $C^\infty(\overline{\Omega})$ that is orthogonal to $H^2(\Omega)$ such that $\psi = P\,d\mu$ near w_0. The uniqueness property we need will then follow from Rosay's Lemma and the ϑ-UCP at w_0. If we can construct such a ψ for measures supported on very small compact subsets, then we can use a partition of unity to obtain such a function for $d\mu$ on $\mathcal{K}$. We have constructed such a ψ when $\mathcal{K} \subset\subset \Omega$ (see the remark after Theorem 4.1). Hence, we may assume that $\mathcal{K}$ is a very small compact subset of $\overline{\Omega}$ that does not contain

w_0. The key to the construction of ψ is formula (4.1), taking β to be the null multi-index. We may integrate formula (4.1) with respect to $d\mu$ in the w variable over $\mathcal{K}$. The resulting function is in the range of ϑN and is therefore orthogonal to $H^2(\Omega)$. The proof of the uniqueness property is complete.

The second point in the proof that needs attention concerns the application of Catlin's theorem [13, Theorem 3.2.1]. Here, we take a relatively open subset V of $\overline{\Omega}$ containing $\mathcal{K}$ and a sequence of functions f_j in $A^\infty(\Omega)$ such that f_j converges to f *in a Sobolev space* $H^s(V)$, where s is chosen to be larger than n so that the Sobolev lemma yields that f_j converges uniformly to f on $\mathcal{K}$. All the rest of the proof of Theorem 5.1 carries over and the proof of Theorem 5.2 is complete.

6. Applications to mapping problems. If $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between bounded *non-pseudoconvex* domains with C^∞ smooth boundaries in $\mathbb{C}^n$, it is not currently known if f must extend smoothly to the boundary near even a single boundary point. In this section, we shall show that such a map must extend smoothly to certain types of boundary points provided that they satisfy the ϑ-UCP.

A boundary point z_0 of a bounded domain Ω is called extreme (see Peiming Ma [34]) if

1) the boundary of Ω is C^∞ smooth near z_0, and
2) there is a *pseudoconvex* domain Ω_0 such that $\Omega \subset \Omega_0$ and an $\epsilon > 0$ such that $B_\epsilon(z_0) \cap \Omega = B_\epsilon(z_0) \cap \Omega_0$.

That every bounded domain with C^∞ smooth boundary has an open set in its boundary consisting of strictly pseudoconvex extreme points can be seen by allowing a large ball containing the domain to shrink until the boundary of the ball comes into contact with the boundary of the domain. Boundary points of the domain near contact points with the boundary of the ball are easily seen to be extreme.

Theorem 6.1. *Suppose that Ω_1 and Ω_2 are bounded non-pseudoconvex domains in $\mathbb{C}^n$, and that Ω_1 has a C^∞ smooth boundary, and Ω_2 has a real analytic boundary. Suppose that $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping, and that z_0 is an extreme boundary point of Ω_1 that is of finite type in the sense of D'Angelo. If the ϑ-UCP holds at z_0, then f must extend C^∞ smoothly up to the boundary of Ω_1 near z_0.*

We remark that, since the ϑ-UCP is known to hold at strictly pseudoconvex boundary points that are real analytic, this theorem yields that proper maps between non-pseudoconvex domains with real analytic boundaries must extend C^∞ smoothly up to the boundary near all the strictly pseudoconvex extreme boundary points. Hence, Chern-Moser invariants [17] can be used to prove non-equivalence of domains with real analytic boundaries whose invariants do not match at any strictly pseudoconvex points. This implies that "most" (in the sense of Green and Krantz) pairs of bounded non-pseudoconvex domains with real analytic boundaries are biholomorphically inequivalent.

Proof of the theorem. The first part of the proof is easy and follows the procedure described in [4]. Let P_1 and P_2 denote the Bergman kernels associated to Ω_1 and Ω_2, respectively, and let $u = \det f'$ denote the holomorphic jacobian determinant of f. Suppose that z_0 is an extreme boundary point of Ω_1 that is of finite type.

Since $b\Omega_2$ is real analytic, given a holomorphic polynomial $h(z)$, there is a function $\varphi \in C_0^\infty(\Omega_2)$ such that $P_2\varphi = h$. The transformation formula for the Bergman projections under a proper holomorphic map yields

$$u(h \circ f) = u[(P_2\varphi) \circ f] = P_1(u[\varphi \circ f]).$$

Since $\varphi \in C_0^\infty(\Omega_2)$, and since f is proper holomorphic, it follows that $u[\varphi \circ f]$ is in $C_0^\infty(\Omega_1)$. Peiming Ma [34] proved that the Bergman projection satisfies local regularity estimates near pseudoconvex extreme points that are of finite type in the sense of D'Angelo. Hence, the identity above reveals that if $h(z)$ is a holomorphic polynomial on $\mathbb{C}^n$, then $u(h \circ f)$ extends C^∞ smoothly up to the boundary near z_0. In particular, taking $h \equiv 1$ yields that u extends C^∞ smoothly to the boundary near z_0. If we prove that u can vanish to at most finite order at z_0, then we can apply the division theorem of [9] or [21] to deduce that f extends C^∞ smoothly up to $b\Omega_1$ near z_0.

Let Ω_0 denote a pseudoconvex domain containing Ω_1 with $z_0 \in b\Omega_0$ as described in condition (2) of the definition of extreme boundary point. Let P_0 denote the Bergman projection associated to Ω_0. Let $h \equiv 1$ and let $\varphi \in C_0^\infty(\Omega_2)$ be such that $h = P_2\varphi$ as above. Extend the functions $u(h \circ f)$ and $u(\varphi \circ f)$ to Ω_0 by setting them equal to zero on $\Omega_0 - \Omega_1$. Since $u(h \circ f) - u(\varphi \circ f)$ is orthogonal to $H^2(\Omega_1)$, it is also orthogonal to $H^2(\Omega_0)$. Now Theorem 3.2, the ϑ-UCP at z_0, and the fact that u cannot vanish identically, together imply that u cannot vanish to infinite order at z_0. The proof is complete.

We next describe how the ϑ-UCP property could be used to prove the following strengthened version of another result of Peiming Ma's. The proof of this theorem will also demonstrate the relevance of density theorems of the kind described in §5.

Theorem 6.2. *Suppose that $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping between bounded non-pseudoconvex domains in $\mathbb{C}^n$ with C^∞ smooth boundaries and that the target domain Ω_2 satisfies Condition R. Then f must extend C^∞ smoothly up to the boundary of Ω_1 near any extreme boundary point of finite type at which the ϑ-UCP holds.*

Proof. Let $K_1(z,w)$ and $K_2(z,w)$ denote the Bergman kernels associated to Ω_1 and Ω_2, respectively. Let $F_1, F_2, \ldots, F_m$ denote the local inverses to f which are defined locally on Ω_2 minus the image of the branch locus of f and let $U_k = \det F_k'$. Suppose that z_0 is a pseudoconvex extreme boundary point of Ω_1 that is of finite type. Ma [34] proved that if $h \in A^\infty(\Omega_2)$, then $u(h \circ f)$ extends C^∞ smoothly up to $b\Omega_1$ near z_0. In order to conclude that f extends smoothly up to $b\Omega_1$ near z_0, it remains only to show that u cannot vanish to infinite order at z_0. We shall now show that this follows from the transformation formula for the Bergman kernels under proper holomorphic mappings and the ϑ-UCP. The transformation formula is

$$u(z)K_2(f(z),w) = \sum_{j=1}^{m} K_1(z, F_j(w))\overline{U_j(w)}. \tag{6.1}$$

The function on the left hand side of this identity extends C^∞ up to $b\Omega_1$ near z_0 by Ma's result and the fact that $K_2(z,\zeta)$ is in $A^\infty(\Omega_2)$ as a function of z for each fixed $\zeta \in \Omega_2$. Pick a point $w_0 \in \Omega_2$ so that the functions $F_j(w)$ are all holomorphic on a neighborhood of w_0, $U_j(w_0) \neq 0$ for each j, and $F_i(w_0) \neq F_j(w_0)$ if $i \neq j$. Let $p(w)$ be a holomorphic polynomial on $\mathbb{C}^n$ such that

$$\sum_{j=1}^{m} p(F_j(w_0))U_j(w_0) \neq 0.$$

We must next modify the proof of Theorem 5.1 to prove that there is an element κ in the linear span of the set of functions

$$\left\{ \frac{\partial^{|\beta|}}{\partial \bar{z}^{\beta}} K_1(w, z_0) \; : \; |\beta| \geq 0 \right\}$$

such that κ is so close to $p(w)$ on the set $\{F_j(w_0)\}_{j=1}^{m}$ that

$$\sum_{j=1}^{m} \kappa(F_j(w_0))U_j(w_0) \neq 0. \tag{6.2}$$

Let us assume, for the moment, that there is such a κ given by

$$\kappa(w) = \sum_{|\beta|<M} c_\beta \frac{\partial^{|\beta|}}{\partial \bar{z}^{\beta}} K_1(w, z_0).$$

Formula (6.1) shows that the complex conjugate of the left hand side of (6.2) is equal to

$$\sum_{|\beta|<M} c_\beta \frac{\partial^{|\beta|}}{\partial z^{\beta}} \left[u(z) K_2(f(z), w_0) \right]$$

evaluated at $z = z_0$. If u vanishes to infinite order at z_0, this last quantity would necessarily be zero. This contradiction forces us to conclude that u can vanish to at most finite order at z_0 and the proof would be finished. To finish the proof, we shall invoke the following lemma to show that such a κ exists.

Lemma 6.3. *Suppose that Ω is a bounded domain in $\mathbb{C}^n$ and that z_0 is a boundary point of Ω such that the boundary of Ω is C^∞ smooth near z_0, z_0 is a point of finite type in the sense of D'Angelo, and that z_0 is an extreme boundary point. Suppose that the ϑ-UCP holds at z_0. Given a polynomially convex compact subset $\mathcal{K}$ of Ω, a polynomial $p(w)$, and an $\epsilon > 0$, there is an element κ in the complex linear span $\mathcal{S}$ of*

$$\left\{ \frac{\partial^{|\beta|}}{\partial \bar{z}^{\beta}} K(w, z_0) \; : \; |\beta| \geq 0 \right\}$$

such that $|\kappa(w) - p(w)| < \epsilon$ for $w \in \mathcal{K}$.

Remark. We remark that if Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ and z_0 is a boundary point of Ω such that the boundary of Ω is C^∞ smooth near z_0, z_0 is a point of finite type in the sense of D'Angelo, and the ϑ-UCP holds at z_0, then Ω and z_0 satisfy the hypotheses of the Lemma.

Proof. We shall continue to use the notation that we set up in the proof of Theorem 6.2. Thus, Ω_0 denotes the pseudoconvex domain containing Ω satisfying condition (2) in the definition of extreme boundary point. However, we no longer need subscript ones and twos, and so we let $K(z,w)$ denote the Bergman kernel associated to Ω.

Let D be a relatively compact subdomain of Ω containing $\mathcal{K}$. It will be enough to prove that $\mathcal{S}$ is dense in the $H^2(D)$ closure of the space of holomorphic polynomials because convergence in H^2 implies uniform convergence on compact subsets. Let $\mathcal{P}$ denote the closure in $H^2(D)$ of the space of holomorphic polynomials. If $\mathcal{S}$ is not dense in $\mathcal{P}$, there would be a function $G \in \mathcal{P}$, $G \not\equiv 0$, such that G is orthogonal to the generating set of $\mathcal{S}$. Extend G to be defined on Ω and Ω_0 by setting G to be equal to zero outside D and consider the Bergman projection PG on Ω. Extend PG to Ω_0 by setting it to be equal to zero outside Ω. P. Ma proved [34] that the Bergman kernel of Ω is in $C^\infty((\overline{\Omega}\cap B_r(z_0))\times\Omega)$ for some small r. This shows that PG is C^∞ smooth up to the boundary near z_0 and that

$$\frac{\partial^{|\beta|}}{\partial z^\beta}PG(z_0) = \int_{w\in D} G(w)\,\frac{\partial^{|\beta|}}{\partial z^\beta}K(z_0,w)\,dV.$$

The orthogonality condition therefore yields that PG vanishes to infinite order at z_0. Now $G-PG$ is orthogonal to $H^2(\Omega)$, and hence, when viewed as a function on Ω_0, $G-PG$ is also orthogonal to $H^2(\Omega_0)$. Hence, Theorem 3.2 shows that $G-PG=\vartheta\alpha$ near z_0 where α vanishes on $b\Omega$ near z_0 and is C^∞ smooth up to the boundary there. The ϑ-UCP now yields that PG must vanish near z_0, and hence that $PG\equiv 0$. This implies that G is orthogonal to the linear span of $\{K(z,w) : w\in\Omega\}$, which is dense in $H^2(\Omega)$. Hence G is certainly also orthogonal to all holomorphic polynomials. Since G is contained in the space $\mathcal{P}$ we are forced to conclude that $G\equiv 0$, contrary to hypotheses, and the proof is complete.

7. Unique continuation properties of the Szegő projection. We have been studying unique continuation properties of the $\bar\partial$-problem and the Bergman projection and kernel. Most of these properties have interesting analogues when phrased for the $\bar\partial_b$-problem and the Szegő projection and kernel. In this last section, I will demonstrate the nature of these questions by answering some of them in the plane.

We now assume that Ω is a bounded domain in the plane with C^∞ smooth boundary, i.e., that Ω is bounded by finitely many non-intersecting simple closed C^∞ curves. Let $L^2(b\Omega)$ denote the space of complex valued functions on $b\Omega$ which are square integrable with respect to arc length measure and let $H^2(b\Omega)$ denote the classical Hardy space of holomorphic functions on Ω whose boundary values are in $L^2(b\Omega)$. We now let the symbol P denote the Szegő projection, which is the orthogonal projection of $L^2(b\Omega)$ onto the closed subspace $H^2(b\Omega)$, and we let $S(z,w)$ denote the Szegő kernel (see [2,11,35] for definitions and basic properties of these objects). It is known that P maps $C^\infty(b\Omega)$ into itself and that $S(z,w)$ extends to be a function in $C^\infty((\overline{\Omega}\times\overline{\Omega})-\{(z,z) : z\in b\Omega\})$. Let $D_r(z_0)$ denote the disc of radius r about z_0. Let $\langle\cdot,\cdot\rangle_b$ denote the inner product in $H^2(b\Omega)$ and let $\langle\cdot,\cdot\rangle_\Omega$ denote both the inner product in $H^2(\Omega)$ and the extended inner product expressing the duality between $A^\infty(\Omega)$ and $A^{-\infty}(\Omega)$.

The Szegő projection satisfies a unique continuation property analogous to the one satisfied by the Bergman projection in the plane, and the Szegő kernel function satisfies a density property that seems even stronger than the one satisfied by the Bergman kernel.

Theorem 7.1. *Suppose that Ω is a bounded domain in the plane with C^∞ smooth boundary and suppose that $w_0 \in b\Omega$. If $\varphi \in L^2(b\Omega)$ is such that $\varphi = 0$ near w_0, then infinite order vanishing of $P\varphi$ at w_0 implies that $P\varphi \equiv 0$ in $H^2(b\Omega)$. Furthermore, given an $\epsilon > 0$, the complex linear span of*

$$\left\{ \frac{\partial^m}{\partial \bar{w}^m} S(z, w_0) : m \geq 0 \right\}$$

is dense in $C^\infty(b\Omega - D_\epsilon(w_0))$.

Proof. There is a biholomorphic map $f : \Omega \to \Omega_0$ of Ω onto a bounded domain Ω_0 in $\mathbb{C}$ with real analytic boundary. The derivative f' of this map is known to be the square of a (single valued) function in $A^\infty(\Omega)$. We will use the symbol $\sqrt{f'(z)}$ to denote this function. The Szegő projections transform under f via

$$P\left(\sqrt{f'}(\varphi \circ f)\right) = \sqrt{f'}\,((P_0\varphi) \circ f)$$

and the Szegő kernels transform via

$$S(z, w) = \sqrt{f'(z)} S_0(f(z), f(w)) \overline{\sqrt{f'(w)}}$$

(where we have used the convention that subscript zeroes imply that the object is associated to Ω_0 and no subscripts imply that the object is associated to Ω). These transformation formulas together with the fact that f must extend C^∞ smoothly to the boundary with non-vanishing derivative on $\overline{\Omega}$ allow us to reduce our problem to the case where Ω is assumed to have real analytic boundary. We make this assumption from now on.

The Szegő projection on our bounded domain Ω with real analytic boundary has the virtue of mapping $C^\omega(b\Omega)$ into the space $A(\overline{\Omega})$ of functions on Ω that extend to be holomorphic on a neighborhood of $\overline{\Omega}$. Moreover, there is an open subset of $\mathbb{C} \times \mathbb{C}$ containing $(\overline{\Omega} \times \overline{\Omega}) - \{(z, z) : z \in b\Omega\}$ on which the Szegő kernel $S(z, w)$ associated to Ω extends to be holomorphic in z and antiholomorphic in w. Hence, if $\varphi \in L^2(b\Omega)$ is such that $\varphi = 0$ near w_0, then $P\varphi$ extends holomorphically past the boundary near w_0, and therefore infinite order vanishing of $P\varphi$ at w_0 implies that $P\varphi \equiv 0$ near w_0, which implies that $P\varphi \equiv 0$ in $H^2(b\Omega)$. The statement about the Szegő projection is proved.

Before we can prove the statement about the span of the Szegő kernel, we must set down some groundwork. Given a continuous function u defined on the boundary of Ω, the Cauchy transform of u will be written $\mathcal{C}u$ and is defined to be the holomorphic function on Ω given by

$$(\mathcal{C}u)(z) = \frac{1}{2\pi i} \int_{\zeta \in b\Omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta.$$

The Cauchy transform, like the Szegő projection, maps $C^\omega(b\Omega)$ into the space $A(\overline{\Omega})$.

Suppose that $z(t)$ parameterizes one of the boundary curves of Ω in the standard sense. If $z_0 = z(t_0)$ is a point on this curve, we define $T(z_0)$ to be equal to $z'(t_0)/|z'(t_0)|$. Thus, for $z \in b\Omega$, $T(z)$ denotes the complex number of unit modulus representing the unit tangent vector to the boundary at z pointing in the direction of the standard orientation. Notice also that T is in $C^\omega(b\Omega)$, that $dz = T\,ds$, and that $ds = \overline{T}\,dz$.

We shall need the following lemma due to Schiffer [37].

Lemma 7.2. *The space of functions in $L^2(b\Omega)$ orthogonal to $H^2(b\Omega)$ is equal to the space of functions of the form $\overline{HT}$ where $H \in H^2(b\Omega)$. Consequently, a function $u \in L^2(b\Omega)$ can be expressed uniquely as an orthogonal sum*

$$u = h + \overline{HT}$$

where $h = Pu$ and $H = P(\overline{uT})$. Furthermore, if u is in $C^\omega(b\Omega)$, then h and H are in $A(\overline{\Omega})$.

Before proving the statement about the linear span in Theorem 7.1, we must prove a related result. For fixed $a \in \Omega$, let $S_a(z)$ denote the function of z given by $S_a(z) = S(z, a)$. Let Σ denote the (complex) linear span of the set of functions $\{S_a(z) : a \in \Omega\}$. It is easy to see that Σ is a dense subspace of $H^2(b\Omega)$. Indeed, if $h \in H^2(b\Omega)$ is orthogonal to Σ, then $h(a) = \langle h, S_a \rangle_b = 0$ for each $a \in \Omega$; thus $h \equiv 0$. We shall also need to know that Σ satisfies a much stronger density property.

Lemma 7.3. *The complex linear span of $\{S_a(z) : a \in \Omega\}$ is dense in $A^\infty(\Omega)$.*

To say that Σ is dense in $A^\infty(\Omega)$ means that, given a function $h \in A^\infty(\Omega)$, there is a sequence $H_j \in \Sigma$ such that $H_j(z)$ tends uniformly on $\overline{\Omega}$ to $h(z)$, and each derivative of $H_j(z)$ tends uniformly on $\overline{\Omega}$ to the corresponding derivative of $h(z)$.

We shall need to know that the Szegő kernel is equal to the Szegő projection of the kernel for the Cauchy transform. To be precise, given a point a in Ω, let $C_a(z)$ denote the *complex conjugate of*

$$\frac{T(z)}{(2\pi i)(z-a)}.$$

Given $h \in H^2(b\Omega)$, the value of h at $a \in \Omega$ is given by the Cauchy integral formula, $h(a) = \langle h, C_a \rangle_b$. The Szegő kernel S_a also satisfies the property, $h(a) = \langle h, S_a \rangle_b$, and hence it follows that $S_a = PC_a$.

Let $u \in C^\omega(b\Omega)$ be given. It is an easy exercise to see that $C^\omega(b\Omega)$ is equal to the space of continuous functions on $b\Omega$ which extend to be *holomorphic* on a neighborhood of $b\Omega$. Hence, there is a function U which is holomorphic on a neighborhood of $b\Omega$ and which is equal to u on $b\Omega$. By multiplying U by a C^∞ function which is compactly supported inside the set where U is holomorphic and which is equal to one on a small neighborhood of $b\Omega$, we may think of U as being a function in $C^\infty(\overline{\Omega})$ which is holomorphic near $b\Omega$. Let Ψ denote the $C_0^\infty(\Omega)$ function given as $\Psi = \partial U/\partial \bar{z}$.

If $v \in C^\infty(\overline{\Omega})$ and $z \in \Omega$, the inhomogeneous Cauchy integral formula (see Hörmander [24, Theorem 1.2.1]) states that

$$v(z) = \frac{1}{2\pi i} \int_{\zeta \in b\Omega} \frac{v(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \iint_{\zeta \in \Omega} \frac{\frac{\partial v}{\partial \bar{\zeta}}}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}.$$

Apply this formula using $v = U$ to obtain the identity

$$U(z) = (\mathcal{C}u)(z) + \frac{1}{2\pi i} \iint_{\zeta \in \Omega} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}.$$

Since Ψ has compact support, we deduce from this formula that $\mathcal{C}u$ extends smoothly to the boundary. Furthermore, the boundary values of $\mathcal{C}u$ are given by $\mathcal{C}u = u - \mathcal{I}$ where, for $z \in b\Omega$,

$$\mathcal{I}(z) = \frac{1}{2\pi i} \iint_{\zeta \in \Omega} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}.$$

Now, because Ψ has compact support, for $z \in b\Omega$, we may approximate the integral defining $\mathcal{I}(z)$ by a (finite) Riemann sum

$$\mathcal{S}(z) = \frac{1}{2\pi i} \sum c_i \frac{1}{a_i - z}$$

in such a way that $\mathcal{S}$ is as close to $\mathcal{I}$ in the topology of $C^\infty(b\Omega)$ as we please.

We have now shown that $u - \mathcal{C}u - \mathcal{S}$ can be constructed to be arbitrarily close to the zero function in $C^\infty(b\Omega)$. If we now multiply $u - \mathcal{C}u - \mathcal{S}$ by T and take the complex conjugate, we see that

$$\overline{Tu} - \overline{T\mathcal{C}u} - \sum \bar{c}_i C_{a_i}$$

can also be made arbitrarily small. Next, we take the Szegő projection of this function and use the fact that Szegő projection is a continuous operator from $C^\infty(b\Omega)$ into itself. Note that $P(\overline{T\mathcal{C}u}) = 0$ because functions of the form $\overline{TH}$, $H \in A^\infty(\Omega)$, are orthogonal to $H^2(b\Omega)$, and keep in mind that $S(z, a) = (PC_a)(z)$. Therefore,

$$P(\overline{Tu}) - \sum \bar{c}_i S(\cdot, a_i)$$

can be made arbitrarily close to zero in $C^\infty(b\Omega)$. To finish the proof, we need only note that a function h in $A^\infty(\Omega)$ can be written as $\overline{Tu}$ where $u = \overline{Th}$. Hence $h = Ph = P(\overline{Tu})$ can be approximated in the $C^\infty(b\Omega)$ topology by functions in Σ and the proof that Σ is dense in $A^\infty(\Omega)$ is finished.

Suppose that $\mathcal{O}$ is an open subset of Ω, and let $\Sigma_{\mathcal{O}}$ denote the complex linear span of $\{S_a(z) : a \in \mathcal{O}\}$. The duality of $A^\infty(\Omega)$ and $A^{-\infty}(\Omega)$ allows us to deduce from the density of Σ in $A^\infty(\Omega)$ that $\Sigma_{\mathcal{O}}$ is also dense in $A^\infty(\Omega)$. Indeed, if $\Sigma_{\mathcal{O}}$ is not dense in $A^\infty(\Omega)$, then there would exist a function $g \in A^{-\infty}(\Omega)$ which is not the zero function such that $\langle g, S_a \rangle_\Omega = 0$ for every $a \in \mathcal{O}$. Let

$$H(a) = \langle g, S_a \rangle_\Omega,$$

and notice that $H(a)$ is a holomorphic function of a on Ω. The orthogonality property of g translates to say that H vanishes on the open set $\mathcal{O}$, and therefore H vanishes identically on Ω, i.e., $\langle g, S_a \rangle_\Omega = 0$ for every $a \in \Omega$. Since Σ is dense in $A^\infty(\Omega)$, and since the pairing between $A^\infty(\Omega)$ and $A^{-\infty}(\Omega)$ is non-degenerate, it follows that $g \equiv 0$, contrary to hypothesis. Hence $\Sigma_{\mathcal{O}}$ is dense in $A^\infty(\Omega)$.

We remark that the same reasoning that we used in the preceding paragraph can be used to show that, given a fixed point $a \in \Omega$, the complex linear span of

$$\left\{ \frac{\partial^m}{\partial \bar{a}^m} S(z, a) \,:\, m \geq 0 \right\}$$

is also dense in $A^\infty(\Omega)$. However, we shall not need this fact to prove the density property in the statement of the theorem.

We have described some useful dense subspaces of $A^\infty(\Omega)$. Let $\mathcal{H}^\perp$ denote the set of functions in $C^\infty(b\Omega)$ that are orthogonal to $H^2(b\Omega)$ in the $L^2(b\Omega)$ inner product. Next, we must describe a useful dense subspace of $\mathcal{H}^\perp$. Lemma 7.2 shows that $\mathcal{H}^\perp$ is equal to the space of functions of the form $\overline{HT}$ where $H \in A^\infty(\Omega)$. The dense subspace of $\mathcal{H}^\perp$ that interests us is expressed in terms of the *Garabedian kernel* $L(z,a)$, which is most easily described in terms of the orthogonal decomposition of the Cauchy kernel $C_a(z)$. Since $S_a = PC_a$, we may write the orthogonal decomposition of C_a in the form $C_a = S_a + \overline{H_a T}$ where $H_a = P(\overline{C_a T})$. Solving this equation for S_a, writing out C_a, and taking complex conjugates gives

$$\overline{S_a(z)} = -i\left(\frac{1}{2\pi}\frac{1}{z-a} - iH_a(z)\right)T(z).$$

The Garabedian kernel L_a (see [23,2]) is defined to be equal to the function in parentheses, i.e.,

$$L_a(z) = \frac{1}{2\pi}\frac{1}{z-a} - iH_a(z).$$

We shall also write $L(z,a)$ for $L_a(z)$. Both S_a and L_a extend holomorphically past the boundary of Ω. In fact, $S_a \in A(\overline{\Omega})$ and L_a is meromorphic on a neighborhood of $\overline{\Omega}$ with a single singularity at a that is a simple pole with residue $1/(2\pi)$. Furthermore, if we define $\ell(z,a)$ via

$$L(z,a) = \frac{1}{2\pi(z-a)} + \ell(z,a),$$

it is known that $\ell(z,a)$ extends to an open subset of $\mathbb{C}\times\mathbb{C}$ containing $\overline{\Omega}\times\overline{\Omega}$ as a holomorphic function of z and a.

It is possible to interpret the Garabedian kernel as being the kernel for the projection $P^\perp$ of $L^2(b\Omega)$ onto the space of functions in $L^2(b\Omega)$ which are orthogonal to $H^2(b\Omega)$, but we shall not do this here (see [2]).

We have just seen that

$$\overline{S_a(z)} = -iL_a(z)T(z)$$

for $a \in \Omega$ and $z \in b\Omega$. Since $\overline{T} = 1/T$ on $b\Omega$, this identity may be rewritten in the form

$$\overline{S_a(z)T(z)} = -iL_a(z). \tag{7.1}$$

This formula allows us to read off that the complex linear span Λ of the set $\{L_a(z) : a \in \Omega\}$ is dense in $\mathcal{H}^\perp$ in the $C^\infty(b\Omega)$ topology. Indeed, a function $u \in C^\infty(b\Omega)$ that is orthogonal to $H^2(b\Omega)$ must be given as $u = \overline{HT}$ for some $H \in A^\infty(\Omega)$. Formula (7.1) reveals that Λ is equal to $\{\overline{\sigma T} : \sigma \in \Sigma\}$, and therefore, since Σ is dense in $A^\infty(\Omega)$, it follows that u can be approximated in $C^\infty(b\Omega)$ by elements in Λ. Similar reasoning shows that, given an open subset $\mathcal{O}$ of Ω, the density of $\Sigma_{\mathcal{O}}$ in $A^\infty(\Omega)$ implies that the complex linear span $\Lambda_{\mathcal{O}}$ of the set $\{L_a(z) : a \in \mathcal{O}\}$ is also dense in $\mathcal{H}^\perp$.

We are finally in a position to prove the rest of the theorem. Let $\mathcal{S}$ denote the linear span mentioned in the statement of Theorem 7.1. Since $S(z,w)$ extends to be holomorphic in z and antiholomorphic in w on a open subset of $\mathbb{C}\times\mathbb{C}$ containing $(\overline{\Omega}\times\overline{\Omega})-\{(z,z):z\in b\Omega\}$, there is a δ with $0<\delta<\epsilon$ so that the expansion

$$S(z,a)=\sum_{j=m}^{\infty}\frac{1}{m!}\left[\frac{\partial^m}{\partial\bar{w}^m}S(z,w_0)\right](\bar{a}-\bar{w}_0)^m$$

is valid for $z\in\overline{\Omega}-D_\epsilon(w_0)$ and $a\in\Omega\cap D_\delta(w_0)$. This shows that the closure of $\mathcal{S}$ in $C^\infty(b\Omega-D_\epsilon(w_0))$ contains the closure of $\Sigma_{\mathcal{O}}$ in $C^\infty(b\Omega-D_\epsilon(w_0))$ where $\mathcal{O}=\Omega\cap D_\delta(w_0)$. We will be finished with the proof if we show that $A^\infty(\Omega)$ is dense in $C^\infty(b\Omega-D_\epsilon(w_0))$. Suppose $u\in C^\infty(b\Omega-D_\epsilon(w_0))$. Let U be a function in $C^\infty(b\Omega))$ that agrees with u on $b\Omega-D_\epsilon(w_0)$. The function U has an orthogonal decomposition given by $U=h+\overline{HT}$ where h and H are in $A^\infty(\Omega)$. The function $\overline{HT}$ can be approximated in $C^\infty(b\Omega)$ by functions in $\Lambda_{\mathcal{O}}$. We will be finished with the proof if we can show that if $a\in\Omega\cap D_\delta(w_0)$, then $L(z,a)$ can be approximated in $C^\infty(b\Omega-D_\epsilon(w_0))$ by functions in $A^\infty(\Omega)$. That this is true follows from Runge's theorem because

$$L(z,a)=\frac{1}{2\pi(z-a)}+\ell_a(z)$$

where $\ell_a\in A^\infty(\Omega)$, and $1/(z-a)$ can be approximated in $C^\infty(b\Omega-D_\epsilon(w_0))$ by rational functions whose poles are outside $\overline{\Omega}$. (Actually, Runge's theorem is not needed here. Just write out the Laurent expansion for $1/(z-a)$ on the complement of $D_\delta(w_0)$ in powers of $z-w_0$. Take a sufficient number of terms in the expansion, then slide the base point from w_0 to a point slightly outside of $\overline{\Omega}$.) The proof is complete.

The density statement in Theorem 7.1 implies a strong form of a unique continuation theorem for the Szegő projection. If Ω is a bounded domain in $\mathbb{C}$ with C^∞ smooth boundary, then the Szegő kernel $S(z,w)$ extends to be in $C^\infty((\overline{\Omega}\times\overline{\Omega})-\{(z,z):z\in b\Omega\})$. It therefore follows that the Szegő projection extends to be defined on the space of distributions on $b\Omega$. Let us also use the symbol P to denote the extended Szegő projection which is understood to map the space of distributions on $b\Omega$ into the space of holomorphic functions on Ω. If λ is a distribution on $b\Omega$ that is supported away from a boundary point w_0, then $P\lambda$ extends C^∞ smoothly up to the boundary near w_0. The density of the linear span in Theorem 7.1 implies the following theorem.

Theorem 7.4. *Suppose that Ω is a bounded domain in $\mathbb{C}$ with C^∞ smooth boundary. If λ is a distribution on $b\Omega$ that is supported away from a boundary point w_0, and if $P\lambda$ vanishes to infinite order at w_0, then $P\lambda\equiv 0$.*

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