MA 303: Extra HW1

**Question 1**

For the differential equation 
\[ xy''(x) + y(x) = 0 \]

(a) (3 points) show that \( x_0 = 0 \) is a regular singular point

(b) (4 points) find the indicial equation, the recurrence relation and the roots of the indicial equation

(c) (3 points) find the solution, valid for positive values of \( x \), corresponding to the larger root.

(Your solution should look like \( x^r \sum_{n=0}^{\infty} a_n x^n \), where \( a_n \) is a fraction involving factorials.)

**Solution 1**

(a) With \( P(x) = x \) we have \( P(0) = 0 \) so 0 is a singular point. The functions \( xp(x) = 0 \) and \( x^2 q(x) = x \) are both analytic at 0 so it is moreover a regular singular point.

(b) Substituting \( y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \) into the equation, we have

\[
0 = \sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r}
\]

\[
= r(r-1)a_0 x^{r-1} + \sum_{n=1}^{\infty} (n + r)(n + r - 1)a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}.
\]

This yields the indicial equation \( r(r-1) = 0 \) (which has roots \( r = 0 \) and \( r = 1 \)) and the recurrence relation \( a_n = -\frac{a_{n-1}}{(n+r)(n+r-1)} \).

(c) Taking \( r = 1 \), we have \( a_n = -\frac{a_{n-1}}{n(n+1)} = \frac{(-1)^n}{n!(n+1)!} \). This gives the solution

\[
y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^{n+1}.
\]

**Question 2**

(a) (6 points) Using the Laplace transform (or otherwise), find the solution to the equation

\[
y''(t) + 2y'(t) + 5y(t) = f(t) = \begin{cases} 0 & 0 \leq t < \pi \\ 1 & t \geq \pi \end{cases}
\]

with initial conditions \( y(0) = 1, y'(0) = 0 \).
(b) (2 points) Rewrite your answer to be in the form
\[ \alpha(t) \cos 2t + \beta(t) \sin 2t + \gamma(t) \]
for some functions \( \alpha(t) \), \( \beta(t) \) and step function \( \gamma(t) \).

c) (2 points) State, with a reason, whether or not your solution is continuous.

**Solution 2**

(a) Applying the Laplace transform \( \mathcal{L} \), we have
\[ \left( s^2 Y(s) - s \right) + 2(sY(s) - 1) + 5Y(s) = \mathcal{L}(f(t)) = \frac{e^{-\pi s}}{s} \]
giving
\[ Y(s) = \frac{s + 2}{s^2 + 2s + 5} + \frac{e^{-\pi s}}{s(s^2 + 2s + 5)} \]
\[ = \frac{s + 1}{(s + 1)^2 + 2^2} + \frac{1}{(s + 1)^2 + 2^2} + \frac{e^{-\pi s}}{5} \left( \frac{1}{s} - \frac{s + 2}{s^2 + 2s + 5} \right) \]
\[ = \frac{s + 1}{(s + 1)^2 + 2^2} + \frac{1}{(s + 1)^2 + 2^2} + \frac{e^{-\pi s}}{5} \left( \frac{1}{s} - \frac{s + 1}{(s + 1)^2 + 2^2} - \frac{1}{(s + 1)^2 + 2^2} \right). \]

Applying the inverse transform and appealing to the table, we have
\[ y(t) = e^{-t} \left( \cos 2t + \frac{1}{2} \sin 2t \right) + \frac{1}{5} u_\pi(t) \left( 1 - e^{-t} \left( \cos 2(t - \pi) + \frac{1}{2} \sin 2(t - \pi) \right) \right). \]

(b) Since \( \cos(2t - 2\pi) = \cos 2t \) and \( \sin(2t - 2\pi) = \sin 2t \), we have
\[ y(t) = \left( e^{-t} - \frac{1}{5} u_\pi(t) e^{\pi t} \right) \cos 2t + \frac{1}{2} \left( e^{-t} - \frac{1}{5} u_\pi(t) e^{\pi t} \right) \sin 2t + \frac{1}{5} u_\pi(t). \]

(c) The solution is continuous at every \( t \neq \pi \) since exponential, trigonometric and constant functions are. The left hand and right hand limits at \( t = \pi \) agree (both are equal to \( e^{-\pi} \)), so the solution is also continuous at \( t = \pi \).

**Table of Laplace transforms**

| \( t^n \) | \( \frac{n!}{s^{n+1}} \) |
| \( \sin \omega t \) | \( \frac{\omega}{s^2 + \omega^2} \) |
| \( \cos \omega t \) | \( \frac{s}{s^2 + \omega^2} \) |
| \( \sinh \omega t \) | \( \frac{s}{s^2 + \omega^2} \) |
| \( \cosh \omega t \) | \( \frac{s}{s^2 - \omega^2} \) |
| \( f(t)e^{ct} \) | \( F(s - c) \) |
| \( u_c(t)f(t - c) \) | \( e^{-cs} F(s) \) |
| \( y(t) \) | \( sY(s) - y(0) \) |
| \( y''(t) \) | \( s^2Y(s) - sy(0) - y'(0) \) |