MA303: Practice Test 3

Answer all questions, showing your working. Write your name on the question paper and hand it in together with your solutions. No books, notes or calculators are allowed.

Question 1

For each of the following

\begin{align*}
  y'' - xy' - y &= 0 \quad (x_0 = 1) \\
  y'' + x^2 y &= 0 \quad (x_0 = 0)
\end{align*}  \quad (1)  \quad (2)

seek power series solutions about the given point \( x_0 \). Then

(a) (8 points) find the recurrence relation;

(b) (8 points) find the first three terms in each of two linearly independent solutions \( y_1 \) and \( y_2 \), and state the radius of convergence of each;

(c) (4 points) find the first three terms of the solution corresponding to the initial conditions \( y(0) = 1 = y'(0) \).

Question 2

(a) (4 points) What is the inverse Laplace transform of \( \frac{s-2}{(s^2+4)(s+2)} \)?

(b) (12 points) Find the solution to the equation

\[ y''(t) + 2y(t) = f(t) = \begin{cases} 
  \frac{t}{2} & 0 \leq t < 6 \\
  3 & t \geq 6
\end{cases} \]

with initial conditions \( y(0) = 0, \ y'(0) = 1 \).

(c) (4 points) State, with a reason, whether or not your solution to (b) is continuous.

Question 3

(a) (6 points) What are the eigenvalues and eigenvectors of the matrix \( A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \)?
(b) (6 points) Find the general solution to the following system of equations (expressed in terms of a linear combination of real-valued functions):

\[
\begin{align*}
    x_1' &= -x_1 - 4x_2 \\
    x_2' &= x_1 - x_2
\end{align*}
\]

(c) (3 points) Sketch a phase portrait.

(d) (5 points) Find the general solution to the system \( Ax = g(t) = \begin{pmatrix} 0 \\ 2e^{-t} \end{pmatrix} \).

**Question 4**

(a) (10 points) Sketch the even and odd extensions of the function \( f(x) = 1 - x, \quad 0 < x < 1 \), to a periodic function \( f : \mathbb{R} \to \mathbb{R} \) of period 2.

(b) (10 points) In each case, find the corresponding Fourier series.

**Question 5**

(a) (7 points) What are the eigenvalues and eigenfunctions of the boundary value problem

\[
X''(x) + \lambda X(x) = 0, \quad X'(0) = 0 = X'(L).
\]

(You may assume that \( \lambda \) is real.)

(b) (6 points) The heat equation for a thin completely insulated rod (of unit length and unit thermal diffusivity constant) is \( u_{xx} = u_t \). The boundary conditions are

\[
u_x(0, t) = 0 = u_x(1, t), \quad t > 0.
\]

Use the method of separation of variables to find linearly independent solutions \( u_n(x, t), \quad n = 1, 2, 3, \ldots \), to this problem.

(c) (4 points) What is the solution corresponding to the initial temperature distribution \( f(x) = 1 - x, \quad 0 < x < 1 \)? What is the steady state solution?

(d) (3 points) Show that the temperature at the midpoint of the rod is constant.

**Table of Laplace transforms**

| \( t^n \) | \( \frac{n!}{s^{n+1}} \) |
| \( \sin \omega t \) | \( \frac{\omega}{s^2 + \omega^2} \) |
| \( \cos \omega t \) | \( \frac{s}{s^2 + \omega^2} \) |
| \( \sinh \omega t \) | \( \frac{s\omega}{s^2 - \omega^2} \) |
| \( \cosh \omega t \) | \( \frac{s}{s^2 - \omega^2} \) |
| \( f(t)e^{ct} \) | \( F(s - c) \) |
| \( u_c(t)f(t - c) \) | \( e^{-cs}F(s) \) |
| \( y'(t) \) | \( sY(s) - y(0) \) |
| \( y''(t) \) | \( s^2Y(s) - sy(0) - y'(0) \) |