Answer all questions, showing your working. Write your name on the question paper and hand it in together with your solutions. No books, notes or calculators are allowed.

Question 1
(a) (3 points) For which values of $x$ does the power series $\sum_{n=0}^{\infty} \frac{n!}{2^n} (x + 2)^n$ converge? Justify your answer.
(b) (3 points) What is the Laplace transform of $f(t) = (t + 1)^2$?
(c) (4 points) What is the inverse Laplace transform of $\frac{s}{s^2 - 4s + 8}$?

Solution 1
(a) With $a_n = \frac{n!}{2^n}$, we have $\lim_{n \to \infty} \frac{|a_{n+1}|}{a_n} = \lim_{n \to \infty} \frac{n+1}{2} = \infty$, so by the ratio test the radius of convergence of the given power series is $\frac{1}{\infty} = 0$. This means that the series converges only for $x = x_0 = -2$.
(b) $\mathcal{L}((t + 1)^2) = \mathcal{L}(t^2 + 2t + 1) = \mathcal{L}(t^2) + 2\mathcal{L}(t) + \mathcal{L}(1) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$.
(c) $\mathcal{L}^{-1}\left(\frac{s}{s^2 - 4s + 8}\right) = \mathcal{L}^{-1}\left(\frac{s-2}{(s-2)^2 + 2}\right) + \mathcal{L}^{-1}\left(\frac{2}{(s-2)^2 + 2}\right) = e^{2t}(\cos 2t + \sin 2t)$.

Question 2
(a) (2 points) Show that $x_0 = 0$ is a regular singular point of the differential equation

$$2x^2y''(x) - xy'(x) + (1 + x)y(x) = 0.$$ 

(b) (8 points) By solving an appropriate indicial equation and corresponding recurrence relation, find ONE solution to (a), valid for positive values of $x$. (Your solution should look like $x^r(1 + \sum_{n=1}^{\infty} a_n x^n)$, where $a_n$ is of the form exponential divided by factorial. HINT: multiply above and below by $2 \cdot 4 \cdots 2n = 2^n n!$).

Solution 2
(a) The singular points are the roots of $P(x) = 2x^2 = 0$, namely $x = 0$. With $p(x) = \frac{-x}{2x^2} = -\frac{1}{2}$ and $q(x) = \frac{1+x}{2x^2}$, both $xp(x)$ and $x^2q(x)$ are analytic at 0 (the limits as $x \to 0$ are $-\frac{1}{2}$ and $\frac{1}{2}$ respectively). This means that 0 is a regular singular point.
(b) By (a), we can expect a solution (valid for \( x > 0 \)) of the form

\[
y(x) = x^r \sum_{n=0}^{\infty} a_n x^{n+r}.
\]

Substituting this into the given equation, we require

\[
0 = \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r}
\]

\[
= (2r(r-1) - r + 1)a_0 x^r + \sum_{n=1}^{\infty} \left( (2(n+r)(n+r-1) - (n+r) + 1) a_n + a_{n-1} \right) x^{n+r}.
\]

This yields the indicial equation

\[
0 = 2r(r-1) - r + 1 = (2r-1)(r-1).
\]

Taking the root \( r = 1 \), the corresponding recurrence relation is

\[
a_n = -\frac{1}{2(n+1) - n} a_{n-1} = -\frac{1}{n(2n+1)} a_{n-1}.
\]

From this we see that

\[
a_n = \frac{(-1)^n}{n!(3.5(2n+1))} a_0 = \frac{(-1)^n 2^n}{(2n+1)!} a_0.
\]

Taking \( a_0 = 1 \) we obtain the solution

\[
y(x) = x \left( \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^n \right) = x \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^n \right).
\]

A similar solution is obtained when we take \( r = \frac{1}{2} \).

**Question 3**

(a) (9 points) Use the Laplace transform to find the solution to the equation

\[
y''(t) + y(t) = f(t) = \begin{cases} 1 & t < \pi \\ 0 & t \geq \pi \end{cases}
\]

with initial conditions \( y(0) = y'(0) = 0 \).

(b) (1 point) Sketch your solution (for \( t \geq 0 \)).

**Solution 3**

(a) Applying the Laplace transform \( \mathcal{L} \), we have

\[
s^2 Y(s) + Y(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt = \int_0^\pi e^{-st} dt = \frac{1}{s} \left( 1 - e^{-\pi s} \right)
\]

giving

\[
Y(s) = \frac{1}{s(s^2 + 1)} \left( 1 - e^{-\pi s} \right) = \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) \left( 1 - e^{-\pi s} \right) = \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) - e^{-\pi s} \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right).
\]

From the table we therefore have

\[
y(t) = (1 - \cos t) - u_\pi(t)(1 - \cos(t - \pi))
\]

(b) Note that, using the definition of \( u_\pi(t) \) and the fact that \( -\cos(t - \pi) = \cos t \), we have

\[
y(t) = \begin{cases} 1 - \cos t & 0 \leq t < \pi \\ -2 \cos t & t \geq \pi \end{cases}
\]

which is now straightforward to sketch.
Table of Laplace transforms

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<th>Function</th>
<th>Laplace Transform</th>
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<td>$\frac{\omega}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>$\cos \omega t$</td>
<td>$\frac{s}{s^2 + \omega^2}$</td>
</tr>
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<td>$\sinh \omega t$</td>
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</tr>
<tr>
<td>$\cosh \omega t$</td>
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<td>$f(t)e^{ct}$</td>
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<td>$u_c(t)f(t - c)$</td>
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<td>$y''(t)$</td>
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