A complex valued function $f(z)$ of $z = x + iy$ in a region of the complex plane is said to be differentiable at an element $w$ of the region if the function

$$[f(z) - f(w)]/(z - w)$$

is continuous at $w$ when suitably defined at $w$. The value at $w$ is taken as the definition of the derivative $f'(w)$ at $w$. A function is continuous at $w$ if it is differentiable at $w$.

A square summable power series $f(z)$ with complex coefficients converges in the unit disk and defines a function in the unit disk. The value

$$f(w) = \langle f(z), (1 - w^- z)^{-1} \rangle$$

at $w$ of the function represented by a square summable power series $f(z)$ is a scalar product in the space of square summable power series with the square summable power series

$$(1 - w^- z)^{-1} = 1 + (w^-)z + (w^-)^2 z^2 + \dots$$

The function represented by a square summable power series is continuous since the identity

$$f(\beta) - f(\alpha) = \langle f(z), (1 - \beta^- z)^{-1} - (1 - \alpha^- z)^{-1} \rangle$$

holds when $\alpha$ and $\beta$ are in the unit disk and since the square summable power series

$$(1 - \beta^- z)^{-1} - (1 - \alpha^- z)^{-1} = (\beta - \alpha)^- z + (\beta^2 - \alpha^2)^- z^2 + \dots$$

satisfies the inequality

$$\|(1 - \beta^- z)^{-1} - (1 - \alpha^- z)^{-1}\|^2 \le |\beta - \alpha|^2 (1 + |\alpha + \beta|^2 + |\alpha^2 + \alpha\beta + \beta^2|^2 + \dots)$$

If $f(z)$ is a square summable power series, a sequence of square summable power series $f_n(z)$ is defined inductively by

$$f_0(z) = f(z)$$

and

$$f_{n+1}(z) = [f_n(z) - f_n(0)]/z$$

for every nonnegative integer $n$. Since the inequality

$$\|f_n(z)\| \le \|f(z)\|$$

holds for every nonnegative integer $n$, the square summable power series

$$[f(z) - f(\alpha)]/(z - \alpha) = f_1(z) + \alpha f_2(z) + \alpha^2 f_3(z) + \dots$$

is a sum in the metric topology of the space of square summable power series when $\alpha$ is in the unit disk. Since the power series represents a continuous function in the disk, the power series $f(z)$ represents a differentiable function in the disk. The function

$$[f(w) - f(\alpha)]/(w - \alpha)$$

of $w$ in the disk is continuous at $\alpha$ when given a definition $f'(\alpha)$ at $\alpha$.

Square summable power series which represent the same function are identical since the coefficients of a square summable power series are all zero if the function represented vanishes identically. A square summable power series is identified with the function it represents. The reproducing kernel function

$$(1 - w^- z)^{-1}$$

for function values at $w$ in the space of square summable power series is the element of the space which in a scalar product determines the value of the represented function at $w$ when $w$ is in the unit disk.

If $W(z)$ is a nontrivial power series such that multiplication by $W(z)$ is a contractive transformation of the space of square summable power series into itself, then

$$W(z)W(w)^-/(1 - w^- z)$$

is the reproducing kernel function for function values at $w$ in the range space $\mathcal{M}(W)$ when $w$ is in the unit disk. For if

$$g(z) = W(z)f(z)$$

is an element of the space $\mathcal{M}(W)$, the identity

$$g(w) = \langle g(z), W(z)W(w)^-/(1 - w^- z)\rangle_{\mathcal{M}(W)}$$

is a consequence of the identity

$$f(w) = \langle f(z), (1 - w^- z)^{-1}\rangle$$

since multiplication by $W(z)$ is an isometric transformation of the space $\mathcal{C}(z)$ onto the space $\mathcal{M}(W)$ and since the identity

$$g(w) = W(w)f(w)$$

is satisfied. The reproducing kernel function

$$W(z)W(w)^-/(1 - w^- z)$$

for function values at $w$ in the space $\mathcal{M}(W)$ is obtained from the reproducing kernel function

$$(1 - w^- z)^{-1}$$

for function values at $w$ in the space of square summable power series under the adjoint of the inclusion of $\mathcal{M}(W)$ in $\mathcal{C}(z)$.

The reproducing kernel function

$$[1 - W(z)W(w)^-]/(1 - w^- z)$$

for function values at $w$ in the space $\mathcal{H}(W)$ is obtained from the reproducing kernel function

$$(1 - w^- z)^{-1}$$

for function values at $w$ in the space of square summable power series under the adjoint of the inclusion of the space $\mathcal{H}(W)$ in $\mathcal{C}(z)$. The identity

$$f(w) = \langle f(z), [1 - W(z)W(w)^-]/(1 - w^- z)\rangle_{\mathcal{H}(W)}$$

holds for every element $f(z)$ of the space $\mathcal{H}(W)$. Since the identity applies when

$$f(z) = [1 - W(z)W(w)^-]/(1 - w^- z),$$

the function represented by the power series $W(z)$ is bounded by one in the unit disk.

Reproducing kernel functions are applied to determine the structure of a Hilbert space $\mathcal{H}$ whose elements are functions in the unit disk. A continuous linear functional on the space is assumed to be defined for every element $w$ of the unit disk by taking function values at $w$. The reproducing kernel function for function values at $w$ is th unique element $K(w, z)$ of the space which represents the value

$$f(w) = \langle f(z), K(w, z)\rangle_{\mathcal{H}}$$

for every element $f(z)$ of the space. The indeterminate $z$ is treated as a dummy variable in the notation for a function. The function

$$K(\alpha, \beta) = \langle K(\alpha, z), K(\beta, z)\rangle_{\mathcal{H}}$$

of $\alpha$ and $\beta$ in the unit disk is treated as an infinite matrix. The symmetry of a scalar product implies the Hermitian symmetry

$$K(\beta, \alpha) = K(\alpha, \beta)^-$$

of the matrix. The infinite matrix is nonnegative in a sense which is determined by its finite square submatrices. If $\gamma_1, \ldots, \gamma_r$ are in the unit disk, then the $r \times r$ matrix with entry

$$K(\gamma_i, \gamma_j)$$

in the $i$–th row and $j$–th column is nonnegative. A nonnegative number results when the matrix is multiplied on the right by a column vector with $r$ entries and on the left by the conjugate transpose row vector. The nonnegative number is a sum of products

$$c_i^- K(\gamma_i, \gamma_j) c_j$$

taken over $i$ and $j$ equal to $1, \ldots, r$ for complex numbers $c_1, \ldots, c_r$.

Reproducing kernel functions are applied in interpolation. If $\gamma_1, \ldots, \gamma_r$ are distinct elements of disk, the set of elements of the Hilbert space which vanish at these elements is a closed vector subspace whose orthogonal complement consists of functions which are determined by their values at these elements. A function on the finite set is extended to the unit disk so as to be orthogonal to functions which vanish on the finite set. The space of functions on the finite set is a Hilbert space in the scalar product inherited from the full space. Every function on the finite set is a linear combination of reproducing kernel functions which represent values taken on the set. A reproducing kernel function for values on a set is its own extrapolation to the full space. The nonnegativity of a reproducing kernel function is the condition for the existence of a scalar product for the functions on the finite set which creates a Hilbert space compatible with the reproducing property. The finite linear combinations of reproducing kernel functions form a dense vector subspace of the Hilbert space of functions defined on the unit disk. The Hilbert space is the metric completion of the dense subspace. The reproducing property permits the elements of the completion to be realized as functions defined on the unit disk.

The Jordan curve theorem states that the complex complement of a simple closed curve in the complex plane is the union of a bounded region and an unbounded region. The Cauchy formula states that the Stieltjes integral

$$\int f(z)dz = 0$$

of a continuous function over the closed curve is equal to zero if the curve has finite length, if the function has a continuous extension to the closure of the bounded region, and if the function is differentiable at all but a finite number of elements of the bounded region. An example of a simple closed curve is the unit circle, which bounds the unit disk. The Cauchy formula for the unit circle is proved by decomposing the unit disk into regions which are bounded by circles centered at the origin and straight lines through the origin.

Points of nondifferentiability are constructed for a function $f(z)$ of $z$ in the unit disk, which has a continuous extension to the closed disk, when the Cauchy integral

$$S(1) = \int_0^{2\pi} f(e^{i\theta}) ie^{i\theta} d\theta$$

for the unit circle is nonzero. A point of nondifferentiability is constructed in the annulus

$$a < |z| < b$$

when the inequality

$$(b - a)|S(1)| \leq |\int_0^{2\pi} f(be^{i\theta}) ibe^{i\theta} d\theta - \int_0^{2\pi} f(ae^{i\theta}) iae^{i\theta} d\theta|$$

is satisfied. If the length of an interval $(\alpha, \beta)$ is less than $2\pi$, a simple closed curve is constructed from $ae^{i\alpha}$ to $be^{i\alpha}$ along a radial line away from the origin, from $be^{i\alpha}$ to $be^{i\beta}$

counterclockwise along a circle of radius $b$ centered at the origin, from $be^{i\beta}$ to $ae^{i\beta}$ along a radial line towards the origin, and from $ae^{i\beta}$ to $ae^{i\alpha}$ clockwise along a circle of radius $a$ about the origin. The Cauchy integral for the curve is

$$S(a, b; \alpha, \beta)$$
$$= \int_a^b f(re^{i\alpha})e^{i\alpha}dr - \int_a^b f(re^{i\beta})e^{i\beta}dr + \int_\alpha^\beta f(be^{i\theta})ibe^{i\theta}d\theta - \int_\alpha^\beta f(ae^{i\theta})iae^{i\theta}d\theta.$$

The Cauchy integral is zero for a linear function since it is zero for a constant and for $z$. The nonzero nature of the integral measures the difficulty in approximating the given function by a linear function.

A point of nondifferentiability is found in the region bounded by the curve when the inequality
$$(\beta - \alpha)(b - a)|S(1)| \le 2\pi|S(a, b; \alpha, \beta)|$$
is satisfied. A point $w$ of nondifferentiability is obtained when the regions containing $w$ and satisfying the inequality form a basis for the neighborhoods of $w$. If the inequality

$$|f(z) - g(z)| \le \epsilon|z - w|$$

holds in the region for some linear function $g(z)$ for a positive number $\epsilon$, then

$$|S(1)| \le \epsilon$$

since the inequality
$$2\pi|S(a, b; \alpha, \beta)| \le (\beta - \alpha)(b - a)\epsilon$$
is satisfied.

The maximum principle states that the real part of a function $f(z)$ of $z$ in the unit disk, which is differentiable at all but a finite number of points in the disk and which has a continuous extension to the closed disk, vanishes in the unit disk if it is nonpositive on the unit circle and nonnegative at the origin. The function $f(z)/z$ is differentiable at all but a finite number of points in the annulus

$$a < |z| < 1$$

when $a$ is in the interval $(0, 1)$. Since the identity

$$\int_0^{2\pi} f(ae^{i\theta})d\theta = \int_0^{2\pi} f(e^{i\theta})d\theta$$

holds by the proof of the Cauchy formula, the value of the function at the origin is an average
$$2\pi f(0) = \int_0^{2\pi} f(e^{i\theta})d\theta$$

of values on the boundary. If the real part of the integrand is nonpositive and real part of the integral is nonnegative, then the real part of the integral and the real part of the integrand are zero. The function is a constant since its real part vanishes in the unit disk.

An example of a function which is differentiable and bounded by one in the unit disk is

$$W(z) = (\alpha - z)/(1 - \alpha^- z)$$

when $\alpha$ is in the unit disk. A Hilbert space $\mathcal{H}$ of functions in the unit disk exists whose reproducing kernel function for function values at $w$ is

$$[1 - W(z)W(w)^-]/(1 - w^- z) = (1 - \alpha^- \alpha)(1 - \alpha^- z)^{-1}(1 - \alpha w^-)^{-1}$$

when $w$ is in the unit disk. The space is contained isometrically in the space of square summable power series since

$$(1 - \alpha^- z)^{-1}$$

is the reproducing kernel function for function values at $\alpha$ in $\mathcal{C}(z)$. The orthogonal complement of $\mathcal{H}$ in $\mathcal{C}(z)$ is a Hilbert space $\mathcal{M}$ which is contained isometrically in $\mathcal{C}(z)$ and which contains the functions which vanish at $\alpha$. Since the reproducing kernel function for function values at $w$ in $\mathcal{M}$ is

$$W(z)W(w)^-/(1 - w^- z),$$

multiplication by $W(z)$ is an isometric transformation of $\mathcal{C}(z)$ onto $\mathcal{M}$. Since $\mathcal{M}$ is contained isometrically in $\mathcal{C}(z)$, multiplication by $W(z)$ is an isometric transformation of $\mathcal{C}(z)$ into itself.

Applications of the maximum principle are made when a continuous function $W(z)$ of $z$ in the unit disk is bounded by one and differentiable at all but a finite number of points in the disk. If the inequality

$$|W(\alpha)| < 1$$

holds for some $\alpha$ in the disk, then it holds for all $\alpha$ in the disk. If the inequality holds for a point $\alpha$ of differentiability, then a continuous function $W'(z)$ of $z$ in the unit disk, which is bounded by one and differentiable at all but a finite number of points in the disk, is defined by the identity

$$W'(z)(\alpha - z)/(1 - \alpha^- z) = [W(\alpha) - W(z)]/[1 - W(\alpha)^- W(z)].$$

The identity is applied as a parametrization of the continuous functions $V(z)$, which are bounded by one in the unit disk and differentiable at all but a finite number of points in the disk, such that

$$V(\alpha) = W(\alpha).$$

Such a function is obtained on replacing $W(z)$ by $V(z)$ in the identity and replacing $W'(z)$ by a continuous function $V'(z)$ which is bounded by one in the unit disk and differentiable at all but a finite number of points in the disk.

If a continuous function $W(z)$ of $z$ in the unit disk is bounded by one in the disk and is differentiable at all but a finite number of points in the disk and if a Hilbert space $\mathcal{H}$ exists whose elements are functions of $z$ in the disk and which has the function

$$[1 - W(z)W(w)^-]/(1 - w^- z)$$

of $z$ as reproducing kernel function for function values at $w$ when $w$ is in the unit disk, then multiplication by $W(z)$ is an isometric transformation of $\mathcal{C}(z)$ onto a Hilbert space $\mathcal{M}$ whose elements are functions of $z$ in the unit disk and which has the function

$$W(z)W(w)^-/(1 - w^- z)$$

of $z$ as reproducing kernel function for function values at $w$ when $w$ is in the unit disk. A Hilbert space $\mathcal{H} \vee \mathcal{M}$ exists in which the spaces $\mathcal{H}$ and $\mathcal{M}$ are contained contractively as complementary spaces. The elements of the space $\mathcal{H} \vee \mathcal{M}$ are functions defined in the unit disk. Since the reproducing kernel function for function values at $w$ in the space $\mathcal{H} \vee \mathcal{M}$ is the sum of the reproducing kernel functions for function values at $w$ in the spaces $\mathcal{H}$ and $\mathcal{M}$, the function

$$(1 - w^- z)^{-1}$$

of $z$ is the reproducing kernel function for function values at $w$ in the space $\mathcal{H} \vee \mathcal{M}$ when $w$ is in the unit disk. The space $\mathcal{H} \vee \mathcal{M}$ is isometrically equal to $\mathcal{C}(z)$ since the space of square summable power series has the same reproducing kernel functions. Since the space $\mathcal{M}$ is contained contractively in $\mathcal{C}(z)$, multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. The function $W(z)$ is represented by a square summable power series. The space $\mathcal{H}$ is isometrically equal to the space $\mathcal{H}(W)$. The space $\mathcal{H}(W)$ is interpreted as $\mathcal{C}(z)$ when $W(z)$ is identically zero.

If a continuous function $U(z)$ of $z$ in the unit disk is bounded by one and is differentiable at all but a finite number of points in the disk and if the inequality

$$|U(\alpha)| < 1$$

holds at a point $\alpha$ of the disk, then the continuous function

$$V(z) = [U(\alpha) - U(z)]/[1 - U(z)U(\alpha)^-]$$

of $z$ is bounded by one in the disk and is differentiable at all but a finite number of points in the disk. Multiplication by $U(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself if, and only if, multiplication by $V(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. For a Hilbert space $\mathcal{H}(U)$ exists whose elements are functions of $z$ in the disk and which contains the function

$$[1 - U(z)U(w)^-]/(1 - w^- z)$$

of $z$ as reproducing kernel function for function values at $w$ when $w$ is in the disk if, and only if, a Hilbert space $\mathcal{H}(V)$ exists whose elements are functions of $z$ in the disk and which contains the function

$$[1 - V(z)V(w)^-]/(1 - w^- z)$$

of $z$ as reproducing kernel function for function values at $w$ when $w$ is in the disk. Since the identity

$$[1 - U(z)U(\alpha)^-][1 - V(z)V(w)^-][1 - U(\alpha)U(w)^-] = [1 - U(\alpha)U(\alpha)^-][1 - U(z)U(w)^-]$$

is satisfied, multiplication by

$$[1 - U(\alpha)U(\alpha)^-]^{-\frac{1}{2}}[1 - U(z)U(\alpha)^-]$$

is an isometric transformation of the space $\mathcal{H}(V)$ onto the space $\mathcal{H}(U)$.

If a continuous function $U(z)$ of $z$ in the disk is bounded by one and differentiable at all but a finite number of points in the disk and if

$$U(\alpha) = 0$$

at a point $\alpha$ of differentiability, then the identity

$$U(z) = V(z)(\alpha - z)/(1 - \alpha^- z)$$

holds for a continuous function $V(z)$ of $z$ in the disk which is bounded by one and which is differentiable at all but a finite number of points in the disk. Multiplication by $U(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself if, and only if, multiplication by $V(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. A space $\mathcal{H}(U)$, whose elements are functions of $z$ in the unit disk and which contains the function

$$[1 - U(z)U(w)^-]/(1 - w^- z)$$

of $z$ as reproducing kernel function for function values at $w$ when $w$ is in the disk, exists if, and only if, a Hilbert space $\mathcal{H}(V)$ exists whose elements are functions of $z$ in the disk and which contains the function

$$[1 - V(z)V(w)^-]/(1 - w^- z)$$

of $z$ as reproducing kernel function for function values at $w$ when $w$ is in the disk. The space $\mathcal{H}(V)$ is contained isometrically in the space $\mathcal{H}(U)$ and contains the elements of the space $\mathcal{H}(U)$ which vanish at $\alpha$.

If a continuous function $W(z)$ of $z$ in the unit disk is bounded by one in the disk and is differentiable at all but a finite number of points in the disk and if $\alpha_1, \dots, \alpha_r$ are distinct points of differentiability in the disk, then continuous functions $W_n(z)$ of $z$ in the disk, which are bounded by one in the disk and which are differentiable at all but a finite number of points in the disk, are defined inductively by

$$W_0(z) = W(z)$$

and

$$W_n(z)(\alpha_n - z)/(1 - \alpha_n^- z) = [W_{n-1}(\alpha_n) - W_{n-1}(z)]/[1 - W_{n-1}(z)W_{n-1}(\alpha_n)^-]$$

when $n$ is positive and $W_{n-1}(z)$ is not a constant of absolute value one. A parametrization results of the continuous functions of $z$ in the unit disk, which are bounded by one in the disk and which are differentiable at all but a finite number of points in the disk, having the same values as $W(z)$ at the points $\alpha_1, \dots, \alpha_r$. Such functions are obtained on replacing $W_r(z)$ by an arbitrary continuous function of $z$ which is bounded by one in the unit disk and which is differentiable at all but a finite number of points in the disk. A Hilbert space $\mathcal{H}(W)$, whose elements are functions of $z$ in the disk and which contains the function

$$[1 - W(z)W(w)^-]/(1 - w^- z)$$

of $z$ as reproducing kernel function for function values at $w$ when $w$ is in the disk, exists if, and only if, a Hilbert space $\mathcal{H}(W_r)$ exists whose elements are functions of $z$ in the disk and which contains the function

$$[1 - W_r(z)W_r(w)^-]/(1 - w^- z)$$

of $z$ as reproducing kernel function for function values at $w$ when $w$ is in the disk. If $W_r(z)$ is a constant of absolute value one, the space $\mathcal{H}(W_r)$ contains no nonzero element and the space $\mathcal{H}(W)$ has dimension $r$. The condition that the space $\mathcal{H}(W)$ has dimension at least $r$ is necessary and sufficient for the construction of the function $W_r(z)$.

A theorem of Cauchy states that a continuous function of $z$ in the unit disk is represented by a power series if it is differentiable at all but a finite number of points in the disk. If a continuous function $W(z)$ of $z$ is bounded by one in the disk and is differentiable at all but a finite number of points in the disk, then multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. A proof is given by showing that for every finite set of distinct points $\alpha_1, \dots, \alpha_r$ in the disk the matrix whose entry in the $i$–th row and $j$–th column is

$$[1 - W(\alpha_i)W(\alpha_j)^-]/(1 - \alpha_j^- \alpha_i)$$

is nonnegative. The conclusion is immediate when $\alpha_1, \dots, \alpha_r$ are points of differentiability since multiplication by $V(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself for a power series $V(z)$ representing a function which agrees with $W(z)$ at the given points. The same conclusion holds by continuity when the points are not points of differentiability.

A function $f(z)$ of $z$ is said to be analytic in the unit disk if it is represented by a power series. The Cauchy theorem states that a function $f(z)$ of $z$ is analytic in the unit disk if it is continuous in the disk and is differentiable at all but a finite number of points in the disk.

A function $\phi(z)$ of $z$, which is analytic and has nonnegative real part in the unit disk, admits a Poisson representation. When the function is continuous in the closed disk, the integral representation

$$2\pi \, \frac{\phi(z) + \phi(w)^-}{1 - w^- z} = \int_0^{2\pi} \frac{\phi(e^{i\theta}) + \varphi(e^{i\theta})^-}{(1 - e^{-i\theta} z)(1 - w^- e^{i\theta})} \, d\theta$$

holds when $z$ and $w$ are in the unit disk. The Poisson representation is an application of the Cauchy integrals

$$2\pi\phi(z) = \int_0^{2\pi} \frac{\phi(e^{i\theta})d\theta}{1 - e^{-i\theta}z}$$

and

$$0 = \int_0^{2\pi} \frac{\phi(e^{i\theta})e^{i\theta}d\theta}{1 - w^- e^{i\theta}}.$$

When the function $\phi(z)$ of $z$ is not continuous in the closed disk, a nonnegative measure $\mu$ on the Baire subsets of the real line is constructed whose value

$$\mu(E) = \lim \int_E \tfrac{1}{2}[\varphi(e^{ix-y}) + \varphi(e^{ix-y})^-]dx$$

is a limit as $y$ decreases to zero of integrals of the real part of

$$\varphi(e^{ix-y}).$$

The Poisson representation reads

$$\pi \, \frac{\varphi(z) + \varphi(w)^-}{1 - w^- z} = \int_0^{2\pi} \frac{d\mu(e^{i\theta})}{(1 - e^{-i\theta}z)(1 - w^- e^{i\theta})}$$

when $z$ and $w$ are in the unit disk.

A Hilbert space is constructed whose elements are equivalence classes of Baire measurable functions $f(e^{i\theta})$ of $e^{i\theta}$ on the unit circle for which the integral

$$2\pi\|f\|^2 = \int_0^{2\pi} |f(e^{i\theta})|^2 d\mu(e^{i\theta})$$

is finite. A partially isometric transformation of the space onto the Herglotz space $\mathcal{L}(\phi)$ is defined by taking a function $f(e^{i\theta})$ of $e^{i\theta}$ on the unit circle into the function

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})d\mu(e^{i\theta})}{1 - e^{-i\theta}z}$$

of $z$ in the unit disk. Multiplication by $e^{-i\theta}$ in the Hilbert space of functions on the boundary corresponds to the difference–quotient transformation in the Herglotz space. A related isometric transformation exists of the Hilbert space of functions on the unit circle onto the extension space of the Herglotz space. Multiplication by $e^{i\theta}$ in the Hilbert space of functions on the unit circle corresponds to multiplication by $z$ in the extension space $\mathcal{E}(\phi)$ to the Herglotz space $\mathcal{L}(\phi)$.

A Riemann mapping function is a power series

$$f(z) = \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 + \dots$$

with vanishing constant coefficient which represents an injective mapping of the unit disk into the complex plane.

The area theorem is the source of estimates of coefficients of Riemann mapping functions. Analyticity and injectivity imply a contractive property of composition in a Hilbert space whose elements are functions analytic in the unit disk.

An isomorphic Hilbert space $\mathcal{G}$ is the set of equivalence classes of power series

$$h(z) = c_0 + c_1 z + c_2 z^2 + \ldots$$

such that the sum

$$\|h(z)\|_{\mathcal{G}}^2 = |c_1|^2 + 2|c_2|^2 + 3|c_3|^2 + \ldots$$

converges. Power series are defined a equivalent if they have equal coefficients of $z^n$ for every positive integer $n$. Representatives are chosen in equivalence classes with vanishing constant coefficient for the definition of analytic functions. An element of the space represents an analytic function $h(z)$ of $z$ in the unit disk such that the integral

$$\pi \|h(z)\|_{\mathcal{G}}^2 = \iint |h'(z)|^2 dxdy$$

with respect to area measure in the unit disk computes the scalar self–product.

Contractive composition is obtained for a Riemann mapping function $f(z)$ which maps the unit disk onto a region which is contained in the unit disk. If

$$h(z) = c_0 + c_1 z + c_2 z^2 + \ldots$$

is an element of the space $\mathcal{G}$,

$$g(z) = c_0 + c_1 f(z) + c_2 f(z)^2 + \ldots$$

is an element of the space whose scalar self–product is computed by the integral

$$\pi \|g(z)\|_{\mathcal{G}}^2 = \iint |g'(z)|^2 dxdy$$

with respect to area measure for the unit disk. Since the chain rule

$$g'(z) = h'(f(z)) f'(z)$$

applies to complex differentiation and since the mapping defined by $f(z)$ is injective, the change of variable theorem produces the integral

$$\pi \|g(z)\|_{\mathcal{G}}^2 = \iint |h'(z)|^2 dxdy$$

with respect to area measure over the region onto which $f(z)$ maps the unit disk. Since the region is contained in the unit disk, the integral

$$\pi \|h(z)\|_{\mathcal{G}}^2 - \|g(z)\|_{\mathcal{G}}^2 = \iint |h'(z)|^2 dxdy$$

with respect to area measure over the complement of the region in the unit disk verifies the contractive property of composition.

The Hilbert space $\mathcal{G}$ is contained isometrically in a Krein space ext $\mathcal{G}$ whose elements are equivalence classes of Laurent series. Laurent series are defined as equivalent if the coefficients of $z^n$ are equal for every nonzero integer $n$. The orthogonal complement of the Hilbert space $\mathcal{G}$ in the Krein space ext $\mathcal{G}$ is the anti–space of a Hilbert space which is the anti–isometric image of $\mathcal{G}$ under the transformation which takes $f(z)$ into $f(z^{-1})$.

If $h(z)$ is an element of ext $\mathcal{G}$ whose coefficient of $z^n$ vanishes for all but a finite number of negative integers $n$, then $h(z)$ represents a function which is analytic in the region obtained from the unit disk on deleting the origin. The composition

$$g(z) = h(f(z))$$

is an element of ext $\mathcal{G}$ whose coefficient of $z^n$ vanishes for all but a finite number of negative integers $n$. The integral

$$\pi \langle h(z), h(z) \rangle_{\text{ext } \mathcal{G}} - \pi \langle g(z), g(z) \rangle_{\text{ext } \mathcal{G}} = \iint |g'(z)|^2 dxdy$$

with respect to area over the complement in the unit disk of the region onto which $f(z)$ maps the unit disk verifies the contractive property of composition on a dense set of elements of ext $\mathcal{G}$. The contractive property follows by continuity for all elements of ext $\mathcal{G}$.

A proof of the contractive property of composition in the Krein space is not essential at the outset since this property is taken as a hypothesis.

The Grunsky transformation is defined under hypotheses of contractivity. If $W(z)$ is a power series with vanishing constant coefficient such that a contractive transformation of the space $\mathcal{G}$ into itself is defined by taking $f(z)$ into $f(W(z))$, then the composition acts a a partially isometric transformation of the Hilbert space $\mathcal{G}$ onto a Hilbert space which is contained contractively in $\mathcal{G}$. The Grunsky space $\mathcal{G}(W)$ is defined as the Hilbert space which is the complementary space in $\mathcal{G}$ to the range of the transformation.

Elements of $\mathcal{G}$ define functions analytic in the unit disk when representatives with vanishing constant coefficient are chosen in equivalence classes. The reproducing kernel function for function values at $w$ is the function

$$\log \frac{1}{1 - zw^-} = \frac{(zw^-)}{1} + \frac{(zw^-)^2}{2} + \frac{(zw^-)^3}{3} + \ldots$$

of $z$ when $w$ is in the unit disk. The reproducing kernel function for function values at $w$ in the range of the transformation is the function

$$\log \frac{1}{1 - W(z)W(w)^-} = \frac{[W(z)W(w)^-]}{1} + \frac{[W(z)W(w)^-]^2}{2} + \frac{[W(z)W(w)^-]^3}{3} + \ldots$$

of $z$ when $w$ is in the unit disk. The reproducing kernel function for function values at $w$ in the space $\mathcal{G}(W)$ is the function

$$K(w, z) = \log \frac{1 - W(z)W(w)^-}{1 - zw^-}$$

of $z$ when $w$ is in the unit disk.

Multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself since $W(z)$ represents a function which is analytic in the unit disk and which is bounded by one by the positivity properties of reproducing kernel functions. A relationship between the Grunsky space $\mathcal{G}(W)$ and the space $\mathcal{H}(W)$ of the invariant subspace construction is implied by the resemblance between reproducing kernel functions.

For every positive integer $r$ a Hilbert space is constructed whose elements are functions of the complex variables $z_1, \ldots, z_r$ in the unit disk for each variable. The reproducing kernel function at $w_1, \ldots, w_r$ is the function

$$K(w_1, z_1) \ldots K(w_r, z_r)$$

of $z_1, \ldots, z_r$ for $w_1, \ldots, w_r$ in the unit disk. A partially isometric transformation of the product space onto a Hilbert space $\mathcal{G}^r(W)$ whose elements are functions analytic in the unit disk is defined by taking a function $f(z_1, \ldots, z_r)$ of $z_1, \ldots, z_r$ into the function

$$f(z, \ldots, z)$$

of $z$. The reproducing kernel function for function values at $w$ in the space $\mathcal{G}^r(W)$ is the function

$$K(w, z)^r$$

of $z$ when $w$ is in the unit disk.

The complex numbers are a Hilbert space $\mathcal{G}^0(W)$ of functions analytic in the unit disk whose reproducing kernel function for function values at $w$ is the function

$$1 = K(w, z)^0$$

of $z$ in the unit disk when the scalar product is determined by the choice of absolute value as norm.

If an element $f_r(z)$ of the space $\mathcal{G}^r(W)$ is chosen for every nonnegative integer $r$, the sum

$$f(z) = f_0(z) + \frac{1}{1!}\, f_1(z) + \frac{1}{2!}\, f_2(z) + \ldots$$

is an element of the space $\mathcal{H}(W)$ which satisfies the inequality

$$\|f(z)\|^2_{\mathcal{H}(W)} \leq \|f_0(z)\|^2_{\mathcal{G}^0(W)} + \frac{1}{1!}\, \|f_1(z)\|^2_{\mathcal{G}'(W)} + \frac{1}{2!} \|f_2(z)\|^2_{\mathcal{G}^2(W)} + \ldots$$

whenever the sum converges. Every element $f(z)$ of the space $\mathcal{H}(W)$ admits a representation for which equality holds. If $f(z)$ is an element of the space $\mathcal{G}(W)$, then

$$\exp f(z)$$

is an element of the space $\mathcal{H}(W)$ which satisfies the inequality

$$\| \exp f(z) \|^2_{\mathcal{H}(W)} \leq \exp \| f(z) \|^2_{\mathcal{G}(W)}.$$

If $W(z)$ is a power series with vanishing constant coefficient such that a contractive transformation of the space $\mathcal{G}$ into itself is defined by taking $f(z)$ into $f(W(z))$, then

$$W^*(z) = W(z^-)^-$$

is a power series with vanishing constant coefficient such that a contractive transformation of the space $\mathcal{G}$ into itself is defined by taking $f(z)$ into $f(W^*(z))$. If a contractive transformation of the space ext $\mathcal{G}$ into itself is defined by taking $f(z)$ into $f(W(z))$, then a contractive transformation of the space ext $\mathcal{G}$ into itself is defined by taking $f(z)$ into $f(W^*(z))$.

The Grunsky transformation of the space $\mathcal{G}(W)$ into the space $\mathcal{G}(W^*)$ is defined when the composition $f(z)$ into $f(W(z))$ is contractive in ext $\mathcal{G}$.

**Theorem 18.** *If for a power series $W(z)$ with vanishing constant coefficient a contractive transformation of ext $\mathcal{G}$ into itself is defined by taking $f(z)$ into $f(W(z))$, then the function*

$$\log \frac{1 - W(w^-)/W(z)}{1 - w^-/z}$$

*of $z$ is represented by an element of the space $\mathcal{G}(W)$ and the function*

$$\log \frac{1 - W^*(z)/W(w)^-}{1 - z/w^-}$$

*of $z$ is represented by an element of the space $\mathcal{G}(W^*)$ when $w$ is in the unit disk. The Grunsky transformation is a contractive transformation of the space $\mathcal{G}(W)$ into the space $\mathcal{G}(W^*)$ which takes $f(z)$ into $g(z)$ when the identity*

$$g(w) = \langle f(z), \log \frac{1 - W(w^-)/W(z)}{1 - w^-/z} \rangle_{\mathcal{G}(W)}$$

*holds for $w$ in the unit disk and whose adjoint is a contractive transformation of the space $\mathcal{G}(W^*)$ into the space $\mathcal{G}(W)$ which takes $f(z)$ into $g(z)$ when the identity*

$$g(w) = \langle f(z), \log \frac{1 - W^*(z)/W(w)^-}{1 - z/w^-} \rangle_{\mathcal{G}(W^*)}$$

*holds for $w$ in the unit disk.*

*Proof of Theorem 18.* Since a contractive transformation of ext $\mathcal{G}$ into itself is defined by taking $f(z)$ into $f(W(z))$, the transformation acts as a partially isometric transformation of ext $\mathcal{G}$ onto a Krein space which is contained contractively in ext $\mathcal{G}$. Since the transformation takes $\mathcal{G}$ contractively into itself, it acts as a partially isometric transformation of $\mathcal{G}$ onto a Hilbert space which is contained contractively in $\mathcal{G}$ and whose complementary space in ext $\mathcal{G}$ is the orthogonal sum of $\mathcal{G}(W)$ and the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$. The transformation acts as a partially isometric transformation of the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$ onto a Krein space $\mathcal{M}$ which is contained contractively in the orthogonal sum of the space $\mathcal{G}(W)$ and the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$.

An element

$$f(z) + g(z)$$

of $\mathcal{M}$ is the sum of an element $f(z)$ of the space $\mathcal{G}(W)$ and an element $g(z)$ of the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$ which satisfies the inequality

$$\|f(z)\|^2_{\mathcal{G}(W)} + \langle g(z), g(z)\rangle_{\text{ext }\mathcal{G}} \leq \langle f(z) + g(z), f(z) + g(z)\rangle_{\mathcal{M}}.$$

An anti–isometric transformation of $\mathcal{G}$ onto the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$ is defined by taking $f(z)$ into $f(z^{-1})$. The transformation takes

$$\log(1 - zw^-)^{-1}$$

into

$$\log(1 - w^-/z)^{-1}$$

when $w$ is in the unit disk. Since the identity

$$f(w) = \langle f(z), \log(1 - zw^-)^{-1}\rangle_{\mathcal{G}}$$

holds for every element $f(z)$ of $\mathcal{G}$, the identity

$$f(1/w) = \langle f(z), \log(1 - w^-/z)\rangle_{\text{ext }\mathcal{G}}$$

holds for every element $f(z)$ of the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$. Since the function represented by $W(z)$ maps the unit disk into itself,

$$\log(1 - W(w^-)/z)$$

is an element of the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$ which satisfies the identity

$$f(1/W^*(w)) = \langle f(z), \log(1 - W(w^-)/z)\rangle_{\text{ext }\mathcal{G}}$$

for every element $f(z)$ of the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$.

Since a partially isometric transformation of the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$ onto $\mathcal{M}$ is defined by taking $f(z)$ into

$$g(z) = f(W(z)),$$

the element

$$\log(1 - W(w^-)/W(z))$$

of $\mathcal{M}$ satisfies the identity

$$f(1/W^*(w)) = \langle g(z), \log(1 - W(w^-)/W(z))\rangle_{\mathcal{M}}$$

for every element $g(z)$ of $\mathcal{M}$. The element $f(z)$ of the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$ is uniquely determined by its image $g(z)$ in $\mathcal{M}$.

Since the element

$$\log(1 - W(w^-)/W(z)) = \log \frac{1 - W(w^-)/W(z)}{1 - w^-/z} + \log\ (1 - w^-/z)$$

of $\mathcal{M}$ is the sum of an element of $\mathcal{G}$ and an element of the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$ and since the identities

$$\langle \log(1 - w^-/z), \log(1 - w^-/z)\rangle_{\text{ext }\mathcal{G}} = \log(1 - ww^-)$$

and

$$\langle \log(1 - W(w^-)/W(z)), \log(1 - W(w^-)/W(z))\rangle_{\mathcal{M}} = \log(1 - W(w^-)W^*(w))$$

are satisfied, the element

$$\log \frac{1 - W(w^-)/W(z)}{1 - w^-/z}$$

of $\mathcal{G}$ is an element of the space $\mathcal{G}(W)$ which satisfies the inequality

$$\| \log \frac{1 - W(w^-)/W(z)}{1 - w^-/z} \|^2_{\mathcal{G}(W)} \leq \log \frac{1 - W(w^-)W^*(w)}{1 - ww^-}$$

A contractive transformation of the space $\mathcal{G}^*(W)$ into the space $\mathcal{G}(W)$ exists which takes a finite linear combination

$$\sum c_k \log \frac{1 - W^*(z)W(w_k^-)}{1 - zw_k^-}$$

of reproducing kernel functions for the space $\mathcal{G}(W^*)$ into the finite linear combination

$$\sum c_k \ \log \frac{1 - W(w_k^-)/W(z)}{1 - w_k^-/z}$$

of elements of the space $\mathcal{G}(W)$ since the identity

$$\| \sum c_k \log \frac{1 - W^*(z)W(w_k^-)}{1 - zw_k^-} \|^2_{\mathcal{G}(W^*)} = \sum c_k c_i^- \log \frac{1 - W^*(w_i)W(w_k^-)}{1 - w_i w_k^-}$$

and the inequality

$$\| \sum c_k \log \frac{1 - W(w_k^-)/W(z)}{1 - w_k^-/z} \|^2_{\mathcal{G}(W)} \leq \sum c_k c_i^- \log \frac{1 - W^*(w_i)W(w_k^-)}{1 - w_i w_k^-}$$

are satisfied.

The adjoint transformation of the space $\mathcal{G}(W)$ into the space $\mathcal{G}(W^*)$ takes $f(z)$ into $g(z)$ when the identity

$$g(w) = \langle f(z), \log \frac{1 - W(w^-)/W(z)}{1 - w^-/z} \rangle_{\mathcal{G}(W)}$$

holds for $w$ in the unit disk. This completes the construction of the Grunsky transformation of the space $\mathcal{G}(W)$ into the space $\mathcal{G}(W^*)$.

Since the transformation takes

$$\log \frac{1 - W(z)W(w)^-}{1 - zw^-}$$

into

$$\log \frac{1 - W^*(z)/W(w)^-}{1 - z/w^-}$$

when $w$ is in the unit disk, the adjoint transformation of the space $\mathcal{G}(W^*)$ into the space $\mathcal{G}(W)$ takes $f(z)$ into $g(z)$ when the identity

$$g(w) = \langle f(z), \log \frac{1 - W^*(z)/W(w)^-}{1 - z/w^-} \rangle_{\mathcal{G}(W^*)}$$

holds for $w$ in the unit disk.

The Grunsky transformation originates as a characterization of power series $W(z)$ with vanishing constant coefficient which represent injective mappings of the unit disk. Since the function

$$\frac{1 - W(w^-)/W(z)}{1 - w^-/z}$$

of $z$ admits an analytic logarithm in the unit disk when $w$ is in the unit disk, the numerator is nonzero whenever the denominator is nonzero. In the present formulation the contractive property of the composition $f(z)$ into $f(w(z))$ in ext $\mathcal{G}$ implies that the function represented by $W(z)$ is not only injective but bounded by one in the unit disk. The converse implication has not yet been verified. The original Grunsky transformation is a limiting case of the present transformation which gives a weaker conclusion under a weaker hypothesis.

The Koebe function as a power series

$$f(z) = z + 2z^2 + 3z^3 + \ldots$$

represents a function

$$f(z) = z/(1-z)^2$$

which maps the unit disk injectively onto a region obtained from the complex plane on deleting the real numbers not greater than minus one–quarter. The analytic function

$$zf'(z)/f(z) = (1+z)/(1-z)$$

of $z$ in the unit disk has positive real part and has value one at the origin.

A related power series

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

with vanishing constant coefficient is defined by

$$zf'(z)/f(z) = 1/\phi(z)$$

for every analytic function of $z$ in the unit disk which has positive real part and which has value one at the origin. The series represents an injective mapping of the unit disk onto a region which contains the origin and which contains every convex combination of one of its elements with the origin. When $t$ is positive and not greater than one, the function

$$tf(z)$$

of $z$ maps the unit disk injectively onto a region which is contained in the given region. A power series $W(t, z)$ with vanishing constant coefficient which represents an injective mapping of the unit disk into itself is defined by the composition

$$tf(z) = f(W(t, z)).$$

The composing functions form a semi–group under composition: The identity

$$W(ab, z) = W(a, W(b, z))$$

holds when $a$ and $b$ are positive and not greater than one. The evolution equation

$$t \frac{\partial}{\partial t} W(t, z) = \phi(z) \ z \frac{\partial}{\partial z} W(t, z)$$

generates the functions belonging to the semi–group. The function $W(t, z)$ has derivative at the origin equal to $t$.

The Grunsky spaces of analytic functions are Hilbert spaces of analytic functions derived from the spaces applied in the construction of invariant subspaces on a hypothesis of injectivity for the transfer function. Exponentiation is contractive from the initial Grunsky space $\mathcal{G}$ into the initial space $\mathcal{C}(z)$ of the invariant subspace construction. If

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

is an element of $\mathcal{G}$, then

$$\exp f(z) = b_0 + b_1 z + b_2 z^2 + \ldots$$

is an element of $\mathcal{C}(z)$ which satisfies the inequality

$$\sum |b_n|^2 \leq \exp(\sum n(a_n|^2).$$

A generalization is due to Lebedev and Milin.

**Theorem 19.** *Assume that a nonincreasing sequence of nonnegative numbers $\rho_n$ has a convergent positive sum, that*

$$\sigma_r = \sum_{n=r}^{\infty} \rho_n / \sum_{n=0}^{\infty} \rho_n$$

*is defined for every positive integer $r$, and that the sum*

$$\sum \sigma_n / n$$

*over the positive integers $n$ converges. If*

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

*and*

$$\exp f(z) = b_0 + b_1 z + b_2 z^2 + \ldots,$$

*then the inequality*

$$(\sum \rho_n |b_n|^2) \exp(\sum \sigma_n / n) \leq (\sum \rho_n) \exp(\sum n \sigma_n |a_n|^2)$$

*is satisfied.*

*Proof of Theorem 19.* The inequality is verified by maximizing

$$\exp(-\sum n \sigma_n |a_n|^2) \sum \rho_n |b_n|^2$$

under the constraint of convergent sums. If a differentiable function $\alpha_n(t)$ of positive $t$ is given for every positive integer $n$, a differentiable function $\beta_n(t)$ of positive $t$ is defined for every nonnegative integer $n$ by the equation

$$\sum \beta_n(t) z^n = \exp(\sum \alpha_n(t) z^n).$$

The differential equation

$$\beta_n'(t) = \sum \beta_{n-k}(t)\alpha_k'(t)$$

is satisfied for every nonnegative integer $n$ with summation over the positive integers $k$ which are not greater than $n$.

The derivative with respect to $t$ of the sum

$$\log(\sum \rho_n \beta_n(t)^- \beta_n(t)) - \sum n\sigma_n \alpha_n(t)^- \alpha_n(t)$$

is the sum

$$\sum [s_k(t) - k\sigma_k \alpha_k(t)]^- \alpha_k'(t) + \sum [s_k(t) - k\sigma_k \alpha_k(t)] \alpha_k'(t)^-$$

over the positive integers $k$ with

$$s_k(t) = \frac{\sum \rho_{n+k} \beta_{n+k}(t) \beta_n(t)^-}{\sum \rho_n \beta_n(t) \beta_n(t)^-}$$

defined by sums over the nonnegative integers $n$.

The derivative is nonnegative when $\alpha_k(t)$ is defined as the solution of the differential equation

$$\alpha_k'(t) = s_k(t) - k\sigma_k \alpha_k(t)$$

with initial condition

$$\alpha_k(0) = a_k$$

for every positive integer $k$. The inequality

$$|\alpha_k(t) - a_k \exp(-k\sigma_k t)| \leq \frac{1 - \exp(-k\sigma_k t)}{k\sigma_k}$$

applies when $t$ is positive since

$$|s_k(t)| \leq 1.$$

Since the inequality

$$\exp(-\sum n\sigma_n |a_n|^2) \sum \rho_n |b_n|^2 \leq \exp(-\sum n\sigma_n |\alpha_n(t)|^2) \sum \rho_n |\beta_n(t)|^2$$

is satisfied, it is sufficient to obtain an estimate of

$$\exp(-\sum n\sigma_n |a_n|^2) \sum \rho_n |b_n|^2$$

when the inequality

$$k\sigma_k |a_k| \leq 1$$

is satisfied. A maximum of the continuous function is obtained by compactness. The maximum is attained on the set of coefficients defined by the equations

$$k\sigma_k a_k = \frac{\sum \rho_{n+k} b_{n+k} b_n^-}{\sum \rho_n b_n^- b_n}.$$

The set of noncritical points is mapped continuously into the set of critical points by the solutions of the differential equations. Since the set of noncritical points is connected, the set of noncritical points is mapped onto a compact connected set of critical points. The function

$$\exp(-\sum k\sigma_k|a_k|^2)\sum \rho_n|b_n|^2$$

of coefficients is a constant on the set of critical points. This computes the maximum value since an element of the set of critical points is defined by

$$ka_k = \omega^k$$

for every positive integer $k$ with

$$b_n = \omega^n$$

for every nonnegative integer $n$.

This completes the proof of the theorem.

Lebedev and Milin state the inequality only when the coefficients $\rho_n$ are a sequence of zeros and ones. The inequality is the motivation for contractive properties of composition which are found in the Koebe function and related mappings defining composition semigroups.

If a power series $f(z)$ has vanishing constant coefficient, the power series

$$f(tz/(1+z)^2) = \sum \alpha_n(t)z^n$$

has vanishing constant coefficient for every positive number $t$. Since the differential equation

$$t\ \frac{\partial}{\partial t}\ f(tz/(1+z)^2) = \frac{1+z}{1-z}\ z\ \frac{\partial}{\partial z}\ f(tz/(1+z)^2)$$

is satisfied, the coefficients $\alpha_n(t)$ satisfy the differential equations

$$t\alpha_n'(t) = s_n(t) + s_{n-1}(t)$$

in terms of the coefficients $s_n(t)$ of the power series

$$(1+z)z\ \frac{\partial}{\partial z}\ f(tz/(1+z)^2) = \sum\ s_n(t)z^n$$

which satisfy the equations

$$n\ \alpha_n(t) = s_n(t) - s_{n-1}(t).$$

Nonnegative differentiable functions $\sigma_n(t)$ of $t \geq 1$, defined for positive integers $n$, are said to be admissible as a family if the differential equations

$$\sigma_n(t) + \frac{t\sigma_n'(t)}{n} = \sigma_{n+1}(t) - \frac{t\sigma_{n+1}'(t)}{n+1}$$

are satisfied and if the solutions are nonincreasing functions of $t$. These conditions imply that the sum

$$\sum \sigma_n(t) \frac{[s_n(t) - s_{n-1}(t)]^-[s_n(t) - s_{n-1}(t)]}{n}$$

is a nondecreasing function of $t$ since the inequality

$$[s_n(t) - s_{n-1}(t)]^-[s_n(t) - s_{n-1}(t)] \le 2s_n(t)^- s_n(t) + 2s_{n-1}(t)^- s_{n-1}(t)$$

is satisfied. The sum

$$\sum n\sigma_n(t)\alpha_n(t)^-\alpha_n(t)$$

is a nondecreasing function of $t$.

The formal sum

$$\sum \frac{\sigma_n(t)}{n} (z^n + z^{-n})$$

over the positive integers $n$ satisfies the differential equation

$$t \frac{\partial}{\partial t} \sum \frac{\sigma_n(t)}{n} (z^n + z^{-n}) = \frac{1-z}{1+z} z \frac{\partial}{\partial z} \sum \frac{\sigma_n(t)}{n} (z^n + z^{-n})$$
$$= \frac{1-z}{1+z} \sum \sigma_n(t)(z^n - z^{-n}).$$

The equation admits a unique solution defining an admissible family for initial conditions $\sigma_n(1)$ an arbitrary nonincreasing sequence of nonnegative numbers such that the increments

$$\sigma_n(1) - \sigma_{n+1}(1)$$

are nonincreasing and have finite sum. It is sufficient to make the verification when a positive integer $r$ exists such that

$$\sigma_n(1) = r + 1 - n$$

when $n$ is not greater than $r$ and such that $\sigma_n(1)$ vanishes otherwise.

Since the identity

$$\sum (r+1-n)z^n = \sum \frac{z^{n+1} - z}{z-1} = \frac{z^{r+2} - z^2}{(z-1)^2} - \frac{rz}{z-1}$$

holds with summation over the positive integers $n$ which are not greater than $r_1$ the identity

$$\sum (r+1-n)(z^n - z^{-n}) = \frac{z^{r+1} - z^{-r-1} - (r+1)(z - z^{-1})}{(z^{\frac{1}{2}} - z^{-\frac{1}{2}})^2}$$

holds with summation over the positive integers $n$ which are not greater than $r$.

Since the identity

$$(2r+2)\sum \frac{(2r+1-k)!}{k!(2r+1-2k)!}\,(z^{\frac{1}{2}}-z^{-\frac{1}{2}})^{2r+2-2k} = z^{r+1}+z^{-r-1}$$

holds with summation over the nonnegative integers $k$ which are not greater than $r+1$, the identity

$$-\frac{1-z}{1+z}\,(z^{r+1}-z^{-r-1}) = \sum \frac{(2r+1-k)!}{k!(2r+1-2k)!}\,(z^{\frac{1}{2}}-z^{-\frac{1}{2}})^{2r+2-2k}$$

holds with summation over the nonnegative integers $k$ which are not greater than $r$.

The solution of the differential equation is

$$-t\,\frac{\partial}{\partial t}\sum \frac{\sigma_n(t)}{n}\,(z^n+z^{-n}) = \sum \frac{(2r+1-k)!}{k!(2r+1-2k)!}\,t^{k-r}(z^{\frac{1}{2}}-z^{-\frac{1}{2}})^{2r-2k}$$

with summation over the nonnegative integers $k$ which are not greater than $r$. Since the binomial expansion

$$(z^{\frac{1}{2}}-z^{-\frac{1}{2}})^{2r-2k} = \sum (-1)^m\,\frac{(2r-2k)!}{m!(2r-2k-m)!}\,z^{r-k-m}$$

applies with summation of the integers $m$ such that

$$k-r \le m \le r-k,$$

the identity

$$-t\,\frac{\partial}{\partial t}\,\frac{\sigma_n(t)}{n} = \sum (-1)^k\,\frac{(r+n+1+k)!}{k!(r-n-k)!(2n+k)!}\,\frac{t^{-n-k}}{2n+1+2k}$$

holds for every positive integer $n$ which is not greater than $r$ with summation over the nonnegative integers $k$ which are not greater than $r-n$. The equation reads

$$-t\,\frac{\partial}{\partial t}\,\frac{\sigma_n(t)}{n} = \frac{(r+n+1)!\,t^{-n}}{(r-n)!(2n+1)!}\,F(n-r, n+2+r, n+\tfrac{1}{2}; n+\tfrac{3}{2}, 2n+1; t^{-1})$$

in the hypergeometric notation

$$F(a,b,c;d,e;z) = 1+\frac{abc}{1!de}\,z+\frac{a(a+1)b(b+1)c(c+1)}{2!d(d+1)e(e+1)}\,z^2+\dots.$$

Another derivation of the equation appears in *A proof of the Bieberbach conjecture*, Acta Mathematica 154 (1985), 137–152.

Since

$$\frac{(r+n+1+k)!}{(r-n-k)!} - \frac{(r+n+k)!}{(r-1-n-k)!} = (2n+1+2k) \ \frac{(r+n+k)!}{(r-n-k)!}$$

when $n+k$ is less than $r$, the identity reads

$$-(2n)! \ t^{n+1} \ \frac{\partial}{\partial t} \ \frac{\sigma_n(t)}{n} = \sum \ \frac{(m+n)!}{(m-n)!} \ F(n-m, n+1+m; 2n+1; t^{-1})$$

with summation over the positive integers $m$ which are not greater than $r$.

The hypergeometric series

$$F(a,b;c;z) = 1 + \frac{ab}{1!c} \ z + \frac{a(a+1)b(b+1)}{2!c(c+1)} \ z^2 + \ldots$$

satisfies the differential equations

$$F'(a,b;c;z) = a[F(a+1,b;c;z) - F(a,b;c;z)]/z$$

and

$$F'(a,b;c;z) = b[F(a,b+1,c;z) - F(a,b;c;z)]/z$$

and

$$F'(a,b;c;z) = (c-1)[F(a,b;c-1;z) - F(a,b;c;z)]/z$$

as well as the differential equations

$$(1-z)F'(a,b;c;z) - bF(a,b;c;z) = (a-c)[F(a,b;c;z) - F(a-1,b;c;z)]/z$$

and

$$(1-z)F'(a,b;c;z) - aF(a,b;c;z) = (b-c)[F(a,b;c;z) - F(a,b-1;c)z)]/z$$

and

$$(1-z)F'(a,b;c;z) - (a+b-c)F(a,b;c;z) = \frac{(c-a)(c-b)}{c} \ F(a,b;c+1;z)$$

which imply the differential equation

$$z(1-z)F''(a,b;c;z) + [c-(a+b+1)z]F'(a,b;c;z) - abF(a,b;c;z) = 0$$

and the recurrence relation

$$F(a,b;c;z) = \frac{b}{a-b-1} \ \frac{a-c}{a-b} \ [F(a-1,b+1;c;z) - F(a,b;c;z)]/z$$
$$+ \frac{a}{b-a-1} \ \frac{b-c}{b-a} \ [F(a+1,b-1;c;z) - F(a,b;c;z)]/z.$$

Another consequence is the identity

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

when $c-a-b$ has positive real part.

For every integer $r$ which is not less than a given positive integer $n$ the polynomial

$$F(n-r, n+1+r; 2n+1; z)$$

of degree $r-n$ is an eigenfunction of the differential operator taking $F(z)$ into

$$z(1-z)F''(z) + [2n+1-(2n+2)z]F'(z)$$

for the eigenvalue

$$(n-r)(n+1+r).$$

The operator on polynomials admits a unique self–adjoint extension in the Hilbert space of functions defined in the interval $(0,1)$ which are square integrable with respect to the measure whose value on a Baire subset of the interval is the integral

$$\frac{2n+1}{(2n)!(2n)!}\int t^{2n}dt$$

taken over the set. An orthonormal basis for the Hilbert space is the set of polynomials

$$\frac{(r+n)!}{(r-n)!}\, F(n-r, n+1+r; 2n+1; z)$$

for integers $r$ which are not less than $n$. A computation of scalar products is made from the identity

$$\left[\frac{(n+r+1)^2}{(2r+1)(2r+2)} + \frac{(r-n)^2}{(2r)(2r+1)} - z\right] F(n-r, n+1+r; 2n+1; z)$$
$$= \frac{(n+r+1)^2}{(2r+1)(2r+2)}\, F(n-r-1, n+2+r; 2n+1; z)$$
$$+ \frac{(r-n)^2}{(2r)(2r+1)}\, F(n-r+1, n+r; 2n+1; z)$$

from which the recurrence relation

$$(n+r+1)^2 \int_0^1 t^{2n}|F(n-r-1, n+2+r; 2n+1; t)|^2 dt$$
$$= (r+1-n)^2 \int_0^1 t^{2n}|F(n-r, n+1+r; 2n+1; t)|^2 dt$$

follows.

A theorem of Richard Askey and George Gasper, *Positive Jacobi sums II*, American Journal of Mathematics 98 (1976), 709–737, states that, for every positive integer $n$ and every integer $r$ which is not less than $n$, the sum

$$\sum \frac{(m+n)!}{(m-n)!}\, F(n-m, n+1+m; 2n+1; z)$$

over the integers $m$ such that $n \le m \le r$ is a polynomial whose values in the interval $(0,1)$ are positive.

## Chapter 3. Conformal Mapping

The Lagrange skew–plane is a generalization of the Gauss plane. A Lagrange number

$$\xi = d + ia + jb + kc$$

has rational numbers $a, b, c$, and $d$ as coordinates. The addition and multiplication of Lagrange numbers are defined from the addition and multiplication of rational numbers by the multiplication table

$$\begin{aligned} ij &= k, & jk &= i, & ki &= j, \\ ji &= -k, & kj &= -i, & ik &= -j, \\ ii &= -1, & jj &= -1, & kk &= -1. \end{aligned}$$

The properties of the Lagrange skew–plane resemble those of the Gauss plane except for the noncommutativity of multiplication.

The associative law

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

holds for all Lagrange numbers $\alpha, \beta$, and $\gamma$. The commutative law

$$\alpha + \beta = \beta + \alpha$$

holds for all Lagrange numbers $\alpha$ and $\beta$. The origin 0 of the Lagrange skew–plane, which has vanishing coordinates, satisfies the identity

$$0 + \gamma = \gamma = \gamma + 0$$

for every element $\gamma$ of the Lagrange skew–plane. For every element $\alpha$ of the Lagrange skew–plane a unique element

$$\beta = -\alpha$$

of the Lagrange skew–plane exists such that

$$\alpha + \beta = 0 = \beta + \alpha.$$

The identity

$$(\alpha + \beta)^- = \alpha^- + \beta^-$$

holds for all Lagrange numbers $\alpha$ and $\beta$.

Multiplication by a Lagrange number $\gamma$ is a homomorphism of additive structure. The identity

$$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$$

holds for all Lagrange numbers $\alpha$ and $\beta$. The parametrization of homomorphisms is consistent with additive structure: The identity

$$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$$

holds for all Lagrange numbers $\alpha, \beta$, and $\gamma$. Multiplication by $\gamma$ is the homomorphism which annihilates every element of the Lagrange skew–plane when $\gamma$ is the origin. Multiplication by $\gamma$ is the identity homomorphism when $\gamma$ is the unit.

The composition of homomorphisms is consistent with multiplicative structure: The associative law

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

holds for all Lagrange numbers $\alpha, \beta$, and $\gamma$. Conjugation is an anti–homomorphism of multiplicative structure: The identity

$$(\alpha\beta)^- = \beta^- \alpha^-$$

holds for all Lagrange numbers $\alpha$ and $\beta$.

A rational number is a Lagrange number

$$\gamma = \gamma^-$$

which is self–conjugate. If

$$\gamma = d + ia + jb + kc$$

is a nonzero Lagrange number, then

$$\gamma^- \gamma = a^2 + b^2 + c^2 + d^2$$

is a positive rational number. A nonzero Lagrange number $\alpha$ has an inverse

$$\beta = \alpha^- / (\alpha^- \alpha)$$

such that

$$\beta\alpha = 1 = \alpha\beta.$$

A Lagrange number is said to be integral if its coordinates are either all integers or all halves of odd integers. Sums and products of integral Lagrange numbers are integral. The conjugate of an integral Lagrange number is integral. If $\xi$ is a nonzero integral Lagrange number, $\xi^- \xi$ is a positive integer. The Euclidean algorithm is adapted to the search for integral Lagrange numbers $\xi$ which represent a given positive integer

$$r = \xi^- \xi.$$

If $\alpha$ is an integral Lagrange number and if $\beta$ is a nonzero integral Lagrange number, then an integral Lagrange number $\gamma$ exists which satisfies the inequality

$$(\alpha - \beta\gamma)^- (\alpha - \beta\gamma) < \beta^- \beta.$$

The choice of the coordinates of $\gamma$ is made so that the coordinates of

$$\beta^- \alpha - \beta^- \beta\gamma = d + ia + jb + kc$$

satisfy the inequalities

$$-\beta^-\beta \leq 2a \leq \beta^-\beta,$$

and

$$-\beta^-\beta \leq 2b \leq \beta^-\beta,$$

and

$$-\beta^-\beta \leq 2c \leq \beta^-\beta,$$

and

$$-\beta^-\beta \leq 2d \leq \beta^-\beta$$

and so that a strict inequality

$$(\beta^-\alpha - \beta^-\beta\gamma)(\beta^-\alpha - \beta^-\beta\gamma) < (\beta^-\beta)^2$$

is obtained.

A nonempty set of integral Lagrange numbers is said to be a left ideal if it contains the sum

$$\alpha + \beta$$

of any elements $\alpha$ and $\beta$ and if it contains the product

$$\alpha\beta$$

of any element $\beta$ with an integral Lagrange number $\alpha$.

A nonempty set of integral Lagrange numbers is said to be a right ideal if it contains the sum

$$\alpha + \beta$$

of any elements $\alpha$ and $\beta$ and if it contains the product

$$\alpha\beta$$

of any element $\alpha$ with an integral Lagrange number $\beta$.

Conjugation transforms a left ideal into a right ideal and a right ideal into a left ideal. A determination of structure is made for right ideals.

A nonzero integral Lagrange number $\beta$ belongs to a right ideal whose elements are the products $\beta\gamma$ with integral Lagrange numbers $\gamma$. A right ideal which contains a nonzero element contains a nonzero element $\beta$ which minimizes the positive integer $\beta^-\beta$. If $\alpha$ is an element of the ideal, an integral Lagrange number $\gamma$ exists which satisfies the inequality

$$(\alpha - \beta\gamma)^-(\alpha - \beta\gamma) < \beta^-\beta.$$

The identity

$$\alpha = \beta\gamma$$

follows since $\alpha - \beta\gamma$ is an element of the ideal which is not nonzero.

The Euclidean algorithm solves the equation

$$r = \xi^- \xi$$

for an integral Lagrange number $\xi$ when $r$ is a given positive integer. The solution is obtained from an approximate solution in a quotient ring of the ring of integral Lagrange numbers.

A ring of Lagrange numbers is a nonempty set of Lagrange numbers which contains the difference

$$\alpha - \beta$$

and the product

$$\alpha\beta$$

of any elements $\alpha$ and $\beta$ of the set. The set of integral Lagrange numbers is a conjugated ring: The ring contains $\xi^-$ whenever it contains $\xi$.

A quotient ring of the ring of integral Lagrange numbers is defined for every positive integer $r$. Integral Lagrange numbers $\alpha$ and $\beta$ are said to be congruent modulo $r$ if

$$\beta - \alpha = r\gamma$$

is divisible by $r$: The equation admits an integral Lagrange number $\gamma$ as solution. Congruence modulo $r$ is an equivalence relation on integral Lagrange numbers. The ring is a union of disjoint equivalence classes.

Equivalence classes inherit addition and multiplication since $\alpha_1 + \beta_1$ and $\alpha_2 + \beta_2$ are congruent modulo $r$ and since $\alpha_1\beta_1$ and $\alpha_2\beta_2$ are congruent modulo $r$ whenever $\alpha_1$ and $\alpha_2$ are congruent modulo $r$ and $\beta_1$ and $\beta_2$ are congruent modulo $r$. Equivalence classes inherit conjugation since $\gamma_1^-$ and $\gamma_2^-$ are congruent modulo $r$ whenever $\gamma_1$ and $\gamma_2$ are congruent modulo $r$. Addition and multiplication of equivalence classes have the properties required of a ring:

The associative law

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

holds for all integral Lagrange numbers $\alpha, \beta$, and $\gamma$ modulo $r$. The commutative law

$$\alpha + \beta = \beta + \alpha$$

holds for all integral Lagrange numbers $\alpha$ and $\beta$ modulo $r$. The image of the origin of the Lagrange numbers is an origin 0 for the Lagrange numbers modulo $r$: The identity

$$0 + \gamma = \gamma = \gamma + 0$$

holds for every integral Lagrange number $\gamma$ modulo $r$. For every integral Lagrange number $\alpha$ modulo $r$ an integral Lagrange number

$$\beta = -\alpha$$

modulo $r$ exists such that

$$\alpha + \beta = 0 = \beta + \alpha.$$

Multiplication by an integral Lagrange number $\gamma$ modulo $r$ is a homomorphism of additive structure: The identity

$$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$$

holds for all integral Lagrange numbers $\alpha$ and $\beta$ modulo $r$. The parametrization of homomorphisms is consistent with additive structure: The identity

$$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$$

holds for all integral Lagrange numbers $\alpha, \beta$, and $\gamma$ modulo $r$. Multiplication by $\gamma$ is the homomorphism which annihilates every integral Lagrange number modulo $r$ when $\gamma$ is the origin. Multiplication by $\gamma$ is the identity homomorphism when $\gamma$ is the image 1 of the unit of the Lagrange numbers.

The composition of homomorphisms is consistent with multiplicative structure: The associative law

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

holds for all integral Lagrange numbers $\alpha, \beta$, and $\gamma$ modulo $r$.

The ring of integral Lagrange numbers modulo $r$ is conjugated: The identity

$$(\alpha\beta)^- = \beta^-\alpha^-$$

holds for all integral Lagrange numbers $\alpha$ and $\beta$ modulo $r$.

There are twenty–four integral Lagrange numbers $\xi$ which represent

$$1 = \xi^-\xi.$$

These Lagrange units form a group under multiplication. The eight elements of the group which are fourth roots of unity form a normal subgroup whose quotient is a cyclic group of three elements.

If $r$ is an odd positive integer, every integral Lagrange number is congruent modulo $r$ to a unique Lagrange number whose coordinates are nonnegative integers less than $r$. The number of integral Lagrange numbers modulo $r$ is equal to $r^4$.

If $r$ and $s$ are relatively prime positive integers, the equation

$$1 = ra + sb$$

admits a solution in integers $a$ and $b$. A canonical homomorphism of the ring of integral Lagrange numbers modulo $rs$ onto the ring of integral Lagrange numbers modulo $r$ exists whose kernel is the conjugated ideal of elements divisible by $s$. A canonical homomorphism of the ring of integral Lagrange numbers modulo $rs$ onto the ring of integral Lagrange numbers modulo $s$ exists whose kernel is the conjugated ideal of elements divisible by $r$. The conjugated ring of integral Lagrange numbers modulo $rs$ is canonically isomorphic to the Cartesian product of the conjugated ring of integral Lagrange numbers modulo $r$ and the conjugated ring of integral Lagrange numbers modulo $s$.

The ring of integral Lagrange numbers modulo two contains sixteen elements. The invertible elements of the ring are represented by Lagrange units. There are twelve integral Lagrange numbers modulo two since a Lagrange unit $\omega$ and its negative $-\omega$ are congruent modulo two. A canonical homomorphism exists of the ring of integral Lagrange numbers modulo $2r$ onto the ring of integral Lagrange numbers modulo $r$ whose kernel is the conjugated ideal of elements divisible by $r$. Since the ideal contains sixteen elements, every integral Lagrange number modulo $r$ is represented by sixteen integral Lagrange numbers modulo $2r$. The number of integral Lagrange numbers modulo $r$ is equal to $r^4$ for every positive integer $r$.

The multiplicative group of nonzero integers modulo $p$ is cyclic for every odd prime $p$. The number of nonzero integers modulo $p$ which are square of integers modulo $p$ is $\frac{1}{2}(p-1)$ as is the number of integers modulo $p$ which are nonsquares. The product of two squares and the product of two nonsquares are squares. The product of a square and a nonsquare is a nonsquare. Since a nonsquare exists, some sum of two squares exists which is a nonsquare.

A skew–conjugate integral Lagrange number

$$\iota = ia + jb$$

modulo $p$ is defined by the choice of integers $a$ and $b$ modulo $p$ such that the equation

$$a^2 + b^2 = c^2$$

admits no solution $c$ in the integers modulo $p$. If $u$ and $v$ are integers modulo $p$ such that

$$(u + iv)^-(u + iv) = u^2 - \iota^2 v^2$$

vanishes, then $u$ and $v$ both vanish. A conjugated field of $p^2$ elements is obtained whose elements are integral Lagrange numbers

$$u + \iota v$$

modulo $p$ with integers $u$ and $v$ modulo $p$ as coordinates.

An integer $a$ modulo $p$ exists such that

$$-1 - a^2$$

is a square since $\frac{1}{2}(p+1)$ integers modulo $p$ are represented whereas there are only $\frac{1}{2}(p-1)$ nonsquares. A skew–conjugate integral Lagrange number

$$\kappa = ia + jb + k$$

modulo $p$ exists for some integer $b$ modulo $p$ such that

$$\kappa^- \kappa = 0.$$

Every integral Lagrange number is represented as

$$\alpha + \kappa\beta$$

for unique elements $\alpha$ and $\beta$ of the field. The identity

$$(\alpha + \kappa\beta)^-(\alpha + \kappa\beta) = \alpha^- \alpha$$

is satisfied.

If $p$ is a prime, a canonical homomorphism of the ring of integral Lagrange numbers modulo $rp$ onto the ring of integral Lagrange numbers modulo $r$ exists whose kernel is the conjugated ideal of elements divisible by $r$.

If $I$ is a right ideal of the ring of integral Lagrange numbers modulo $r$, then the set of integral Lagrange numbers which represent elements of the ideal is a right ideal which contains $r$. An integral Lagrange number $\xi$ exists such that the elements of the ideal are the products $\xi\eta$ with $\eta$ an integral Lagrange number. The representation

$$r = \xi^- \xi$$

holds if $I$ contains no nonzero element which is self–conjugate.

The number of right ideals of the ring of integral Lagrange numbers modulo $r$ which contain no nonzero self–conjugate element is equal to the sum of the odd divisors of $r$. The number of integral Lagrange numbers $\xi$ which represent

$$r = \xi^- \xi$$

is equal to twenty–four times the sum of the odd divisors of $r$.

The Lagrange skew–plane admits topologies which are compatible with addition and multiplication. The Dedekind topology is derived from convex structure.

A convex combination

$$(1 - t)\xi + t\eta$$

of elements $\xi$ and $\eta$ of the Lagrange skew–plane is an element of the Lagrange skew–plane when $t$ is a rational number in the interval $[0, 1]$. A subset of the Lagrange skew–plane is said to be preconvex if it contains all elements of the Lagrange skew–plane which are convex combinations of elements of the set. The preconvex span of a subset of the Lagrange skew–plane is defined as the smallest preconvex subset of the Lagrange skew–plane which contains the given set.

The closure in the Lagrange skew–plane of a preconvex subset $B$ is the set $B^-$ of elements $\xi$ of the Lagrange skew–plane such that the set whose elements are $\xi$ and the elements of $B$ is preconvex. The closure of a preconvex set is a preconvex set which is its own closure.

A nonempty preconvex set is defined as open if it is disjoint from the closure of every disjoint nonempty preconvex set. The intersection of two nonempty open preconvex sets is an open preconvex set if it is nonempty.

A subset of the Lagrange skew–plane is said to be open if it is a union of nonempty open preconvex sets. The empty set is open since it is an empty union of such sets. Unions of open subsets are open. Finite intersections of open sets are open.

An example of an open set is the complement in the Lagrange skew–plane of the closure of a nonempty preconvex set. A subset of the Lagrange skew–plane is said to be closed if it is the complement in the Lagrange skew–plane of an open set. Intersections of closed sets are closed. Finite unions of closed sets are closed. The Lagrange skew–plane is a Hausdorff space in the topology whose open and closed sets are defined by convexity. These open and closed sets define the Dedekind topology of the Lagrange skew–plane.

If a nonempty open preconvex set $A$ is disjoint from a nonempty preconvex set $B$, then a maximal preconvex set exists which contains $B$ and is disjoint from $A$. The maximal preconvex set is closed and has preconvex complement. The existence of the maximal preconvex set is an application of the Kuratowski–Zorn lemma.

Addition and multiplication are continuous as transformations of the Cartesian product of the Lagrange skew–plane with itself into the Lagrange skew–plane. Conjugation is continuous as a transformation of the Lagrange skew–plane into the Lagrange skew–plane. The Dedekind skew–plane is the completion of the Lagrange skew–plane in the uniform Dedekind topology. Neighborhoods of a Lagrange number are determined by neighborhoods of the origin. If an open set $A$ contains the origin and if $\xi$ is a Lagrange number, then the set of sums of $\xi$ and elements of $A$ is an open set which contains $\xi$. Every open set which contains $\xi$ is obtained from an open set which contains the origin.

A Cauchy class of closed subsets of the Lagrange skew–plane is a nonempty class of closed subsets such that the intersection of the members of any finite subclass is nonempty and such that for every open set $A$ containing the origin some member $B$ of the class exists such that all differences of elements of $B$ belong to $A$.

A Cauchy class of closed subsets is contained in a maximal Cauchy class of closed subsets. A Cauchy sequence is a sequence of elements $\xi_1, \xi_2, \xi_3, \dots$ of the Lagrange skew–plane such that a Cauchy class of closed subsets is defined whose members are the closed preconvex spans of $\xi_r, \xi_{r+1}, \xi_{r+2}, \dots$ for every positive integer $r$. A Cauchy sequence determines a maximal Cauchy class. Every maximal Cauchy class is determined by a Cauchy sequence.

An element of the Dedekind skew–plane is defined by a maximal Cauchy class of elements of the Lagrange skew–plane. An element of the Lagrange skew–plane determines the maximal Cauchy class of closed sets which contain the element. The Lagrange skew–plane is contained in the Dedekind skew–plane.

If $B$ is a closed subset of the Lagrange skew–plane, the closure $B^-$ of $B$ in the Dedekind skew–plane is defined as the set of elements of the Dedekind skew–plane whose maximal Cauchy class has $B$ as a member. A subset of the Dedekind skew–plane is defined as open if it is disjoint from the closure in the Dedekind skew–plane of every disjoint closed subset of the Lagrange skew–plane. Unions of open subsets of the Dedekind skew–plane are open. Finite intersections of open subsets of the Dedekind skew–plane are open. A subset of the Lagrange skew–plane is open if, and only if, it is the intersection with the Lagrange skew–plane of an open subset of the Dedekind skew–plane.

A subset of the Dedekind skew–plane is defined as closed if its complement in the Dedekind skew–plane is open. Intersections of closed subsets of the Dedekind skew–plane are closed. Finite unions of closed subset of the Dedekind skew–plane are closed. The closure of a subset of the Dedekind skew–plane is defined as the smallest closed set containing the given set. The closure in the Lagrange skew–plane of a subset of the Lagrange skew–plane is the intersection with the Lagrange skew–plane of the closure of the set in the Dedekind skew–plane.

The Dedekind skew–plane is a Hausdorff space in the topology whose open sets and closed sets are determined by convexity. These open sets and closed sets define the Dedekind topology of the Dedekind skew–plane.

The Lagrange skew–plane is dense in the Dedekind skew–plane. Addition and multiplication admit unique continuous extensions as transformations of the Cartesian product of the Dedekind skew–plane with itself into the Dedekind skew–plane. Conjugation admits a unique continuous extension as a transformation of the Dedekind skew–plant into itself.

Properties of addition in the Lagrange skew–plane are preserved in the Dedekind skew–plane. The associative law

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

holds for all elements $\alpha, \beta$, and $\gamma$ of the Dedekind skew–plane. The commutative law

$$\alpha + \beta = \beta + \alpha$$

holds for all elements $\alpha$ and $\beta$ of the Dedekind skew–plane. The origin 0 of the Lagrange skew–plane satisfies the identities

$$0 + \gamma = \gamma = \gamma + 0$$

for every element $\gamma$ of the Dedekind skew–plane. For every element $\alpha$ of the Dedekind skew–plane a unique element

$$\beta = -\alpha$$

of the Dedekind skew–plane exists such that

$$\alpha + \beta = 0 = \beta + \alpha.$$

Conjugation is a homomorphism of additive structure: The identity

$$(\alpha + \beta)^- = \alpha^- + \beta^-$$

holds for all elements $\alpha$ and $\beta$ of the Dedekind skew–plane.

Multiplication by an element $\gamma$ of the Dedekind skew–plane is a homomorphism of additive structure: The identity

$$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$$

holds for all elements $\alpha$ and $\beta$ of the Dedekind skew–plane. The parametrization of homomorphisms is consistent with additive structure: The identity

$$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$$

holds for all elements $\alpha, \beta$, and $\gamma$ of the Dedekind skew–plane. Multiplication by $\gamma$ is the homomorphism which annihilates every element of the Dedekind skew–plane when $\gamma$ is the origin. Multiplication by $\gamma$ is the identity homomorphism when $\gamma$ is the unit 1 of the Lagrange skew–plane.

The composition of homomorphisms is consistent with multiplicative structure: The associative law

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

holds for all elements $\alpha, \beta$, and $\gamma$ of the Dedekind skew–plane. Conjugation is an anti–homomorphism of multiplicative structure: The identity

$$(\alpha\beta)^- = \beta^-\alpha^-$$

holds for all elements $\alpha$ and $\beta$ of the Dedekind skew–plane.

The inclusion of the complex plane in the Dedekind skew–plane is a homomorphism of additive and multiplicative structure which commutes with conjugation. The complex plane is a closed subset of the Dedekind skew–plane. The Dedekind topology of the Dedekind plane is the subspace topology of the Dedekind topology of the Dedekind skew–plane.

If $\gamma$ is a nonzero element of the Dedekind skew–plane, the real number

$$\gamma^-\gamma$$

is positive. If $\alpha$ is a nonzero element of the Dedekind skew–plane, the nonzero element

$$\beta = \alpha^-/(\alpha^-\alpha)$$

satisfies the identities

$$\beta\alpha = 1 = \alpha\beta.$$

The Dedekind skew–plane is complete in the uniform Dedekind topology: Every Cauchy class of closed subsets of the Dedekind skew–plane has a nonempty intersection. Closed and bounded subsets of the Dedekind skew–plane compact: A subset of the Dedekind skew–plane is said to be bounded if a positive number $c$ exists such that the inequality

$$\gamma^- \gamma \leq c$$

holds for every element $\gamma$ of the set. A nonempty class of closed subsets has a nonempty intersection if every finite subclass has a nonempty intersection and if some member of the class is bounded.

The axiomatization of topology has consequences which are unfamiliar to those whose experience is limited to Dedekind topologies. A topology is defined for a set by a class of subsets which are said to be open or equivalently by a class of subsets which are said to be closed. The two formulations of topology are equivalent since a set is assumed to be open if, and only if, its complement is closed. The union of every class of open sets is assumed to be open. Equivalently the intersection of every class of closed sets is assumed to be closed. The intersection of every finite class of open sets is assumed to be open. Equivalently the union of every finite class of closed sets is assumed to be closed. This definition of topology is supplemented by a condition which defines a Hausdorff space: Distinct elements $a$ and $b$ of the space are contained in disjoint open sets $A$ and $B$, $a$ contained in $A$ and $b$ contained in $B$.

A trivial example of such a topology is defined for a finite set. A finite set is a Hausdorff space in a unique topology: All subsets are both open and closed. This discrete topology of a finite set is applied in the construction of nontrivial topologies.

If a nonempty class $\mathcal{C}$ of nonempty sets is given, the Cartesian product of the sets is defined as the set of all functions defined on the members of the class such that the value of the function on a member set is always an element of the set. The usual function notation is however replaced by the notation applied to sequences: if $N$ is a member of the class, the value of the function at $N$ is written $C_N$. When the members of the class are parametrized by positive integers, the notation $C_n$ means $C_N$ with $n$ the positive integer which parametrizes the member set $N$. The concept of a Cartesian product is applied to classes $\mathcal{C}$ which are unlimited in cardinality. The class $\mathcal{C}$ need not be finite. If it is infinite, it need not be countable. The concept of a Cartesian product can be applied more generally when the class $\mathcal{C}$ is empty or when some member of the class is empty. The Cartesian product is then defined to be empty. (The graph of the function contains no element.)

When the member sets are Hausdorff spaces, the Cartesian product is a Hausdorff space in the Cartesian product topology. The product topology is defined by two conditions: The projection of the product onto each factor space is continuous. A transformation of a topological space into the product space is continuous whenever every composition with a projection into a factor space is continuous.

When the factor spaces are compact Hausdorff spaces, the Cartesian product is a compact Hausdorff space. The proof of compactness is an application of the axiom of choice. The axiom of choice is equivalent to the assertion that a Cartesian product of nonempty sets is nonempty. The Kuratowski–Zorn lemma is a consequence of the axiom of choice: A partially ordered set contains a maximal element if every well–ordered subset admits an upper bound in the set.

Compactness of a Hausdorff space is formulated as the assertion that a nonempty class of closed subsets has a nonempty intersection whenever every finite subclass has the property. Every such class is contained in a maximal such class by the Kuratowski–Zorn lemma. When the class is maximal, the intersection of the members of the class contains a unique element.

If $\mathcal{C}$ is a maximal such class of closed subsets of the Cartesian product, then a maximal such class is seen in every factor space. Seen in a factor space are those closed sets whose inverse image in the Cartesian product are members of the class $\mathcal{C}$. The element determined in every factor space defines the desired element of the Cartesian product.

The adic topology of the Lagrange skew–plane resembles the Dedekind topology in its good relationship to addition and multiplication. The open sets are defined as unions of sets which are both open and closed. The closed sets are defined as intersections of sets which are both open and closed. A basic example of a set which is both open and closed and which contains a given Lagrange number $\xi$ is defined by a positive rational number $\lambda$ and consists of the Lagrange numbers $\eta$ such that

$$\lambda(\xi - \eta)^{-}(\xi - \eta)$$

is integral. Every open set is a union of finite intersections of basic open and closed sets. Every closed set is an intersection of basic open and closed sets.

The Lagrange skew–plane is a Hausdorff space in the adic topology. Addition and multiplication are continuous as transformations of the Cartesian product of the Lagrange skew–plane with itself into the Lagrange skew–plane. Conjugation is continuous as a transformation of the Lagrange skew–plane into itself.

The adic skew–plane is defined as the Cauchy completion of the Lagrange skew–plane in the uniform adic topology. Addition and multiplication admit unique continuous extensions as transformations of the Cartesian product of the adic skew–plane with itself into the adic skew–plane. Conjugation admits a unique continuous extension as a transformation of the adic skew–plane into itself. An element of the adic skew–plane is said to be integral if it belongs to the closure of the integral elements of the Lagrange skew–plane. The adic skew–plane is a conjugated ring which contains the set of integral elements as a conjugated subring. Compactness of the subring is proved by a construction as a closed subset of a Cartesian product of compact Hausdorff spaces.

The Cartesian product of the conjugated ring of integral Lagrange numbers modulo $r$ is taken over the positive integers $r$. The Cartesian product is a conjugated ring whose addition, multiplication, and conjugation are defined by addition, multiplication, and conjugation of projections in factor rings. Since the factor rings are compact Hausdorff spaces in the discrete topology, the Cartesian product is a compact Hausdorff space in the Cartesian product topology. When $r_1$ is a divisor of $r_2$, a canonical homomorphism exists of the factor ring modulo $r_2$ onto the factor ring modulo $r_1$ whose kernel is the conjugated ideal of elements divisible by $r_1$.

A closed subring of the Cartesian product is defined as the set of elements of the Cartesian product such that the projection of the factor ring modulo $r_2$ is mapped into the projection in the factor ring modulo $r_1$ whenever $r_1$ is a divisor of $r_2$. The subring is conjugated and is a compact Hausdorff space in the subspace topology. A continuous conjugated homomorphism of the subring onto the ring of integral elements of the adic skew–plane is defined by taking an element of the subring into the limit of a Cauchy sequence whose $r$–term is an integral element of the Lagrange skew–plane which represents the projection in the factor ring modulo $r$.

The adic skew–plane is a ring of quotients of the subring of its integral elements. A conjugated isomorphism of additive structure of the adic skew–plane onto itself is defined on multiplication by $r$ for every positive integer $r$. The transformation is continuous and has a continuous inverse. Every element of the adic skew–plane is mapped into an integral element on multiplication by some positive integer.

An integral element of the adic skew–plane is said to be $p$–adic for some prime $p$ if its quotient by $r$ is integral for every positive integer $r$ which is not divisible by $p$. The set of $p$–adic elements of the ring of integral elements of the adic skew–plane is a conjugated ideal which is closed in the adic topology. The conjugated ring of integral elements of the adic skew–plane is isomorphic to the Cartesian product of its $p$–adic ideals taken over all primes $p$. The topology of the ring of integral elements is the Cartesian product topology of its $p$–adic ideals.

A decomposition of the adic skew–plane results from the decomposition of its ring of integral elements. An element of the adic skew–plane is said to be $p$–adic if for some prime $p$ its product with a positive integer is a $p$–adic integral element of the adic skew–plane. The set of $p$–adic elements of the adic skew–plane is a conjugated ideal of the adic skew–plane which is closed in the adic topology. The $p$–adic component of an element of the adic skew–plane is integral for all but a finite number of primes $p$. If a $p$–adic element of the adic skew–plane is chosen for every prime $p$ and if all but a finite number of elements are integral, an element of the adic skew–plane exists whose $p$–adic component is the given $p$–adic element for every prime $p$.

The $p$–adic skew–plane is defined for a prime $p$ as the conjugated ring of $p$–adic elements of the adic skew–plane. The $p$–adic topology of the ring is defined as the subspace topology of the adic topology of the adic skew–plane. The set of self–conjugate elements of the ring is the field of $p$–adic numbers. An element

$$\xi = d + ia + jb + kc$$

of the $p$–adic skew–plane has coordinates $a, b, c$, and $d$ in the $p$–adic field which do not all vanish when $\xi$ does not vanish. The product

$$\xi^- \xi = a^2 + b^2 + c^2 + d^2$$

is a $p$–adic number which does not vanish when the coordinates of $\xi$ do not all vanish. An inverse

$$\xi^{-1} = \xi^- / (\xi^- \xi)$$

exists in the $p$–adic skew–plane which satisfies the identities

$$\xi^{-1}\xi = 1 = \xi\xi^{-1}$$

with 1 the unit of the $p$–adic field and also of the $p$–adic skew–plane.

The value of the adic skew–plane lies in its relationship to the Dedekind skew–plane which is found in their Cartesian product. The product skew–plane is the set of pairs $\xi = (\xi_+, \xi_-)$ of elements $\xi_+$ of the Dedekind skew–plane and elements $\xi_-$ of the adic skew–plane. The sum

$$\gamma = \alpha + \beta$$

of elements $\alpha$ and $\beta$ is defined by

$$\gamma_+ = \alpha_+ + \beta_+$$

and

$$\gamma_- = \alpha_- + \beta_-.$$

The product

$$\gamma = \alpha\beta$$

of elements $\alpha$ and $\beta$ is defined by

$$\gamma_+ = \alpha_+\beta_+$$

and

$$\gamma_- = \alpha_-\beta_-.$$

The conjugate

$$\beta = \alpha^-$$

of an element $\alpha$ is defined by

$$\beta_+ = \alpha_+^-$$

and

$$\beta_- = \alpha_-^-.$$

The product skew–plane is a Hausdorff space in the Cartesian product topology of the Dedekind skew–plane and the adic skew–plane.

The Dedekind skew–plane and the adic skew–plane are spliced by the construction of a quotient space. A closed subset of the product skew–plane consists of the elements whose components in the Dedekind skew–plane and the adic skew–plane are elements of the Lagrange skew–plane with vanishing sum. If $\alpha$ and $\beta$ are elements of the subset, then so is $\alpha + \beta$. If $\alpha$ is an element of the subset and if $\lambda$ is an element of the Lagrange skew–plane, then an element

$$\beta = \lambda\alpha$$

of the subset is defined by

$$\beta_+ = \lambda\alpha_+$$

and

$$\beta_- = \lambda \alpha_- .$$

If $\alpha$ is an element of the subset, then an element

$$\beta = \alpha^-$$

of the subset is defined by

$$\beta_+ = \alpha_+^-$$

and

$$\beta_- = \alpha_-^- .$$

An equivalence relation is defined of the product skew–plane by defining elements $\alpha$ and $\beta$ to be equivalent when $\beta - \alpha$ belongs to the subset. A fundamental domain for the equivalence relation is the set of elements $\xi$ of the product skew–plane whose adic component is integral and whose Dedekind component satisfies the inequality

$$\xi_+^- \xi_+ < (\xi_+ - \omega)^- (\xi_+ - \omega)$$

for every integral element $\omega$ of the Lagrange skew–plane with integral inverse. Every element of the product skew–plane is equivalent to an element of the closure of the fundamental domain. Equivalent elements of the fundamental domain are equal.

## Appendix. Cardinality

The cardinality of set $A$ is said to be less than or equal to the cardinality of set $B$ if an injective transformation of set $A$ into set $B$ exists. If the cardinality of set $A$ is less than or equal to the cardinality of set $B$ and if the cardinality of set $B$ is less than or equal to the cardinality of set $A$, then an injective transformation exists of set $A$ onto set $B$. Sets $A$ and $B$ are said to have the same cardinality. The cardinality of set $A$ is said to be less than the cardinality of set $B$ if $A$ and $B$ are sets of unequal cardinality such that the cardinality of set $A$ is less than or equal to the cardinality of set $B$.

Experience with finite sets creates the expectation that any two sets are comparable in cardinality. If $A$ and $B$ are sets of unequal cardinality, then either the cardinality of set $A$ is less than the cardinality of set $B$ or the cardinality of set $B$ is less than the cardinality of set $A$. The desired conclusion, or its equivalent, is accepted as a hypothesis in the axiomatic definition of sets.

The axiom of choice is the most plausible of the hypotheses which are equivalent to the desired comparability of cardinalities of sets. If a transformation $T$ takes set $A$ onto set $B$, then a transformation $S$ of set $B$ into set $A$ exists such that the composed transformation $TS$ is the inclusion transformation of set $B$ in itself.

The axiom of choice displaces the previous hypothesis which is equivalent to the comparability of cardinalities of sets. A partial ordering of a set $S$ is determined by distinguished pairs $(a, b)$ of elements $a$ and $b$ of $S$. The inequality $a \leq b$ is written when $(a, b)$ is a distinguished pair. It is assumed that the inequality $a \leq c$ holds whenever $a$ and $c$ are elements of the set for which the inequalities $a \leq b$ and $b \leq c$ hold for some element $b$ of the set. The inequality $c \leq c$ is assumed for every element $c$ of the set. Elements $a$ and $b$ of the set are assumed to be equal if the inequalities $a \leq b$ and $b \leq a$ are satisfied. A set is said to be well–ordered if every nonempty subset contains a least element. An equivalent of the axiom of choice is the hypothesis that every set admits a well–ordering.

The Kuratowski–Zorn lemma is a flexible formulation of the principle of induction implicit in well–ordering. A partially ordered set admits a maximal element if every well–ordered subset has an upper bound in the set.

The proof of the Kuratowski–Zorn lemma from the axiom of choice is an application of induction. Assume that $S$ is a partially ordered set in which every well–ordered subset has an upper bound. An augmentation of a well–ordered subset $A$ is a well–ordered subset $B$ whose elements are the elements of $A$ and some upper bound of $A$ which does not belong to $A$. The axiom of choice is applied to a set whose elements are the pairs $(A, B)$ consisting of an augmentable well–ordered subset $A$ and an augmentation $B$ of $A$. The set is mapped onto the set of augmentable well–ordered subsets by taking $(A, B)$ into $A$. The axiom of choice asserts the existence of a transformation which takes every augmentable well–ordered subset $A$ into an augmentation $(A, A')$ of $A$.

The proof of the Kuratowski–Zorn lemma is facilitated by the introduction of notation. A ladder is well–ordered subset $A$ which is constructed by the chosen augmentation procedure. For every element $b$ of $A$ the augmentation of the set of elements of $A$ which are less than $b$ is the set of elements of $A$ which are less than or equal to $b$. The intersection of ladders $A$ and $B$ is a ladder which is either equal to $A$ or equal to $B$. If $A$ and $B$ are ladders, then either $A$ is contained in $B$ or $B$ is contained in $A$. The union of all ladders is a ladder which contains every ladder. Since the greatest ladder is assumed to have an upper bound, it has a greatest element. The greatest element of the greatest ladder is a maximal element of the given partially ordered set $S$.

Cardinal numbers are constructed by a theorem of Cantor which states that no transformation maps a set onto the class of all its subsets. If a transformation $T$ maps a set $\mathcal{S}$ into the subsets of $\mathcal{S}$, then a subset $\mathcal{S}_\infty$ of $\mathcal{S}$ is constructed which does not belong to the range of $T$. The set $\mathcal{S}_\infty$ is the set of elements $s$ of $\mathcal{S}$ for which no elements $s_n$ of $\mathcal{S}$ can be chosen for every nonnegative integer $n$ so that $s_0$ is equal to $s$ and so that $s_n$ belongs to $Ts_{n-1}$ when $n$ is positive. An element $s$ of $\mathcal{S}$ belongs to $\mathcal{S}_\infty$ if, and only if, $Ts$ is contained in $\mathcal{S}_\infty$. This property implies that $S_\infty$ is not equal to $Ts$ for an element $s$ of $\mathcal{S}$.

If $\gamma$ is a cardinal number, a continuum of order $\gamma$ is defined as a set of least cardinality which has the same cardinality as the class of its subsets which are continua of order less than $\gamma$. The empty set is a continuum of order equal to its cardinality. A set with one element is a continuum of order equal to its cardinality. No other finite set is a continuum of order $\gamma$ for a cardinal number $\gamma$. A countably infinite set is a continuum of order equal to its cardinality.

A parametrization of a continuum $\mathcal{S}$ of order $\gamma$ is an injective transformation $J$ of $\mathcal{S}$ onto the class of its subsets which are continua of order less than $\gamma$ such that no elements $s_n$ of $\mathcal{S}$ can be chosen for every nonnegative integer $n$ so that $s_n$ belongs to $Js_{n-1}$ when $n$ is positive. A continuum of order $\gamma$ admits a parametrization since an injective transformation $T$ exists of $\mathcal{S}$ onto the class of its subsets which are continua of order less than $\gamma$. Since $S_\infty$ is then a continuum of order $\gamma$, it has the same cardinality as $\mathcal{S}$. The restriction of $T$ to $\mathcal{S}_\infty$ is a parametrization of $\mathcal{S}_\infty$. If $W$ is an injective transformation of $\mathcal{S}$ onto $\mathcal{S}_\infty$, then a parametrization $J$ of $\mathcal{S}$ is defined so that $Ja$ is the set of elements $b$ of $\mathcal{S}$ such that $Wb$ belongs to $TWa$.

A parametrization $J$ of a continuum $\mathcal{S}$ of order $\gamma$ is essentially unique. If an injective transformation $T$ maps $\mathcal{S}$ onto the class of its subsets which are continua of order less than $\gamma$, then an injective transformation $W$ of $\mathcal{S}$ onto $\mathcal{S}_\infty$ exists such that $Ja$ is always the set of elements $b$ such that $Wb$ belongs to $TWa$. The construction of $T$ is an application of the Kuratowski–Zorn lemma. Consider the class $\mathcal{C}$ of injective transformations $W$ with domain contained in $\mathcal{S}$ and with range contained in $\mathcal{S}_\infty$ such that every element of $Ja$ belongs to the domain of $W$ whenever $a$ belongs to the domain of $W$ and such that $Ja$ is always the set of elements $b$ of $\mathcal{S}$ such that $Wb$ belongs to $JWa$. The class $\mathcal{C}$ is partially ordered by the inclusion ordering of the graph. A well–ordered subclass of $\mathcal{C}$ has an upper bound in $\mathcal{C}$ whose graph is a union of graphs. A maximal member of the class $\mathcal{C}$ has $\mathcal{S}$ as its domain.

A nonempty set of cardinal numbers contains a least element since a ladder of well–ordered sets can be constructed with these cardinalities.

A continuum of order $\gamma$ exists when $\gamma$ is the cardinality of an uncountable set. It is sufficient to construct a set which has the same cardinality as the class of its subsets which are continua of cardinality less than $\gamma$. If a cardinal number $\alpha$ is greater than the cardinality of every continuum of order less than $\gamma$, it is sufficient to construct a set which has the same cardinality as the class of its subsets of cardinality less than $\alpha$. Such a set is constructed when $\alpha$ is the least cardinality greater than the cardinality of an infinite set $\mathcal{S}$. The class $\mathcal{C}$ of all subsets of $\mathcal{S}$ is a set which has the same cardinality as the class of its subsets of cardinality less than $\alpha$. The cardinality of the class of all subsets of $\mathcal{C}$ of cardinality less than $\alpha$ is less than or equal to the cardinality of all transformations of $\mathcal{S}$ into the set of functions defined on $\mathcal{S}$ with values zero or one. The cardinality of the class of all subsets of $\mathcal{C}$ with values zero or one is less than or equal to the cardinality of the set of all functions defined on the Cartesian product $\mathcal{S} \times \mathcal{S}$ with values zero or one. Since $\mathcal{S}$ is an infinite set, the cardinality of $\mathcal{S} \times \mathcal{S}$ is equal to the cardinality of $\mathcal{S}$. The cardinality of the class of all subsets of $\mathcal{C}$ of cardinality less than $\alpha$ is less than or equal to the cardinality of $\mathcal{C}$.

A hypothesis is required for the determination of cardinalities of continua. The choice of hypothesis depends on the desired applications. When the largest logical structure is wanted in which the accepted methods of analysis apply, then the cardinalities of continua are dependent on hypotheses whose consistency is necessarily untested (as are the accepted hypotheses of analysis). When the smallest logical structure is wanted in which the accepted methods of analysis apply (which is the conventional view in mathematics), then the cardinalities of continua are determined. This is the best choice for a student since it establishes a logical structure with minimal hypotheses which can serve as a guide to generalizations should he want this direction of research. A minimal structure is therefore chosen here.

When a minimal structure is chosen, there are essentially only two ways in which a new cardinality can be constructed from given cardinalities. The cardinality of the class of all subsets of a set is greater than the cardinality of the set. A set of cardinality $\gamma$ can be obtained as a union of a class of cardinality less than $\gamma$ of sets whose cardinalities are less than $\gamma$. Both constructions produce continua from continua. It follows that every infinite set is a continuum whose order is equal to its cardinality. An uncountable continuum is either the class of all subsets of an infinite set in cardinality or it is a union of a class of smaller cardinality of sets of smaller cardinality.

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