COMPLEX ANALYSIS

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Preface

The present formulation of complex analysis in Hilbert spaces originates in lecture notes of courses given in postdoctoral years after a thesis written at Cornell University in 1957 which prepares a half century of research with the Riemann hypothesis. An approach to the Riemann hypothesis due to Hermite and Stieltjes in the last half of the nineteenth century is implemented by techniques introduced for the same purpose by Hardy and Hilbert in the first half of the twentieth century. The continuation in the second half of the twentieth century draws on earlier publications with minimal guidance from teachers or colleagues.

These lecture notes which originate as preparation for the Riemann hypothesis retain their value for that purpose. But they acquire another significance by applying the factorization of analytic functions to the construction of invariant subspaces and by reformulating conformal mapping.

The Riemann mapping theorem is generally acknowledged as one of the most fundamental contributions of complex analysis. Although the theorem was stated by Riemann, his formal proof omits estimates of injective analytic functions required for passage to a limit.

The proof given by Schwarz introduces estimates of bounded analytic functions which retain their importance in later arguments. The subsequent proof given by Bieberbach applies estimates of injective analytic functions which complement the treatment of Schwarz. A proof is now given which applies both boundedness and injectivity. The reader who is prepared to do the work needed to understand this sophisticated treatment is rewarded with an understanding of complex analysis which meets the challenges of the twenty–first century.

The Bieberbach conjecture applies to a power series

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

with vanishing constant coefficient which represents an injective function in the unit disk. The estimate

$$|a_n| \leq n$$

is conjectured to apply for every positive integer n if it applies for the least positive integer. Although Bieberbach could prove the conjecture only for the second coefficient, this information is sufficient for a proof of the Riemann mapping theorem.

The Bieberbach conjecture was stated before the first of two world wars and was proved only in special cases before the second. Complex analysis was unprepared for the general proof of the Bieberbach conjecture proposed in 1984 by the present author. Chance allowed a presentation of the argument to the Leningrad Seminar in Geometric Function Theory which published the proceedings in Russian and in English leaving the burden of confirmation to the American Mathematical Society. A misunderstanding resulted that the argument was simplified during verification. Since the Leningrad Seminar carefully avoids a claim to simplification, the claim is made by minor contributors to the verification procedure. The priority in publication accorded to their argument monopolized the attention of the mathematical public during the brief moment when publicity made it receptive to the proof.

The desire of the public to be informed as quickly as possible rather than as accurately as possible undermines the academic procedures which have in the past been found necessary for sustained research. The implication encouraged by the verification procedure is the view of the Bieberbach conjecture as an eccentric terminal result whose proof eliminates the need for further funding.

No less an authority than Lars Ahlfors has expressed himself in this vein during an international symposium on the occasion of the proof of the Bieberbach conjecture. Since his classical text on complex analysis inspires the present work, it is remarkable that he overlooks the application of the Bieberbach conjecture to the proof of the Riemann mapping theorem. He seems to have degraded this major contribution to complex analysis of the nineteenth century to an issue which has been clarified after the turn of the century.

The proof of the Bieberbach conjecture continues an estimation theory due to Helmut Grunsky, a student of Bieberbach. The injective property of analytic functions creates contractive transformations in Hilbert spaces whose scalar product is defined as an area integral of the complex derivative. For this reason the contractive property is known as the area theorem. Hilbert spaces are constructed whose elements are functions analytic in the unit disk. The spaces appear in pairs which are related to each other by the Grunsky transformation. Estimates of coefficients are obtained which need to be refined in two ways for a proof of the Bieberbach conjecture. The estimates need to be localized so as to apply to an initial segment of coefficients. The estimates need to be exponentiated since they are obtained for the coefficients of a logarithm of the desired function rather than the function itself.

The proof of the Bieberbach conjecture was completed in the comparatively short time of four years. Preparations for the proof of the Riemann hypothesis were adapted to a context which was not anticipated twenty–five years previously when they began. That such a transition is possible testifies to the coherence of complex analysis.

Invariant subspaces are an underlying concept in the classical approaches to the Riemann hypothesis. The existence of invariant subspaces for linear transformations of a complex vector space of finite dimension into itself is a theorem of Gauss, who applied a factorization of polynomials. Hilbert showed that an isometric linear transformation of a Hilbert space into itself admits a nontrivial invariant subspace. The subspaces constructed by Gauss and by Hilbert are invariant subspaces for continuous linear transformations which commute with the given transformation. The Hilbert construction adapts the Stieltjes representation of nonnegative linear functionals on polynomials. Stieltjes applied finite–dimensional Hilbert spaces of polynomials which are generalized for the Riemann hypothesis by the present author. An invariant subspace construction for contractive transformations of a Hilbert space into itself is indicated by Beurling in the middle of the twentieth century as an application of the factorization of functions which are analytic and bounded by one in the unit disk. Hilbert spaces are constructed whose elements are functions analytic in the unit disk.

Hilbert spaces whose elements are functions analytic in the upper half–plane are introduced in Fourier analysis by Hardy for application to the Riemann hypothesis. The Hardy space for the unit disk is the Hilbert space of functions which are represented in the disk by square summable power series. The space is chosen by Beurling for the construction of invariant subspaces. The contractive transformations to which the construction applies are characterized by functions which are analytic and bounded by one in the unit disk. An invariant subspace is constructed by factorization of the function in the class of functions which are analytic and bounded by one in the disk.

Multiplication by a function is contractive as a transformation of the Hardy space into itself if the function is analytic and bounded by one in the disk. Beurling applies his construction under a restrictive hypothesis since the multiplication is assumed to be isometric. The range of the transformation is a Hilbert space which is contained isometrically in the Hardy space. Since the range is invariant under multiplication by z, the orthogonal complement of the range is an invariant subspace for the adjoint transformation, which takes $f(z)$ into $[f(z) - f(0)]/z$. The orthogonal complement of the range is a Hilbert space which is contained isometrically in the Hardy space. An invariant subspace is constructed for the transformation of the orthogonal complement into itself which takes $f(z)$ into $[f(z) - f(0)]/z$. The transformation is contractive and not isometric, nor does it have an isometric adjoint.

When multiplication is contractive, the range of the transformation is a Hilbert space which is contained contractively in the Hardy space when given the scalar product which originates from the domain. A generalization of orthogonal complementation is introduced to produce a Hilbert space which is contained contractively in the Hardy space. A larger class of contractive transformations is made accessible to the construction of invariant subspaces by factorization.

Every contractive transformation of a Hilbert space into itself becomes accessible when the Hardy space of functions analytic in the unit disk is replaced by the isomorphic Hilbert space of square summable power series. It is sufficient to replace square summable power series with coefficients in the complex numbers by square summable power series with coefficients in a given Hilbert space. Multiplication is defined by power series whose coefficients are operators on the coefficient space. A formal construction of invariant subspaces is obtained since there is a classical procedure for factorization.

The Gauss construction of invariant subspaces applies to transformations, which can be assumed contractive, of a Hilbert space of finite dimension into itself. The spectrum of the transformation determines invariant subspaces. When the procedure is applied to Hilbert spaces of infinite dimension, Hilbert spaces are obtained which are contained contractively in the given space such that the restricted transformation takes the subspace contractively into itself.

The generalization of the Beurling construction of invariant subspaces was made in joint work with James Rovnyak. Collaboration began when he was an undergraduate at Lafayette College, Easton, Pennsylvania, and continued when he began doctoral studies at Yale University. His doctoral thesis was prepared in a joint seminar held at the Courant Institute of Mathematical Sciences. The thesis was written under the nominal supervision of Kakutani who gave the project free rein. The application to invariant subspaces was completed in a postdoctoral position at Purdue University.

The remaining obstacle in the construction of invariant subspaces is to obtain an isometric inclusion from a contractive inclusion. The Hilbert spaces which are contained contractively in a given Hilbert space have the structure of a convex set because of an identification of spaces with self–adjoint transformations which are nonnegative and contractive. The convex set is determined by its extreme points since it is compact. The issue of isometric inclusion reduces to the properties of extreme points of the set of contractively contained Hilbert spaces which are invariant for the given transformation and in which the restriction is contractive.

The reception of results would have met no obstacle had they been presented in Ukraine. The relationship between factorization and invariant subspaces was treated there by Moshe Livšic in work which had an international impact after his emigration to Israel. In Ukraine the results would have been received for discussion in seminar before being submitted for publication. The invariant subspace construction would have been publishable without justification by a general solution of the invariant subspace problem.

A proof of the existence of invariant subspaces for contractive transformations of a Hilbert space into itself seemed necessary for meaningful publication. Since Beurling does not require a discussion of extreme points, the properties of extreme points were expected to be irrelevant in a general context were similar hypotheses are satisfied. An erroneous announcement to this effect was published in the Bulletin of the American Mathematical Society.

The properties of extreme points may be unavoidable for the construction of invariant subspaces. A sufficient condition for an extreme point is an identity for difference quotients which permits the construction of an invariant subspace. A proof of the existence of invariant subspaces is obtained by showing that the condition is also necessary. The conjecture that extreme points satisfy the identity for difference quotients was accepted by Ky Fan for publication in the Journal of Mathematical Analysis and Applications. The proof of the conjecture (in essential cases), given here for the first time, was delayed by priority accorded to the Riemann hypothesis.

The Bieberbach conjecture displaced the existence of invariant subspaces as a research goal when a resemblance was observed between the Hilbert spaces of analytic functions introduced by Grunsky and the Hilbert spaces of analytic functions of the invariant subspace construction. The construction of Hilbert spaces by complementation applies to the Grunsky spaces. Contractive transformations defined by multiplication are replaced by contractive transformations defined by composition.

The setting of contractive composition is a Hilbert space of functions analytic in the unit disk and vanishing at the origin whose scalar product is defined as an area integral of the square of the absolute value of the derivative in the unit disk. An isomorphic Hilbert space is the set of power series with vanishing constant coefficients which are square integrable with the contribution of the n–th coefficient given weight n.

Exponentiation is contractive from the initial Grunsky space to the space of square summable power series with complex coefficients. If

$$f(z) = \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 + \ldots$$

and

$$\exp f(z) = \beta_0 + \beta_1 z + \beta_2 z^2 + \ldots ,$$

then the inequality

$$|\beta_0|^2 + |\beta_1|^2 + |\beta_2|^2 + \ldots \leq \exp(|\alpha_1|^2 + 2|\alpha_2|^2 + 3|\alpha_3|^2 + \ldots)$$

is satisfied.

Related inequalities are found when the Hilbert spaces of analytic functions of the invariant subspace construction are known. A function which is analytic and bounded by one in the unit disk is applied in the construction of a Grunsky space and in the construction of a space for the construction of invariant subspaces. Exponentiation is contractive from a Grunsky space to the corresponding space of the invariant subspace construction. The relationship between mated spaces is a stimulus for research which is rewarded by a proof of the Bieberbach conjecture.

The localization of inequalities in exponentiation is a fundamental contribution of the Leningrad Seminar in Geometric Function Theory. Lebedev and Milin obtain the inequality

$$(\sum_{n=0}^{\infty} \rho_n |\beta_n|^2) \exp(\sum_{n=1}^{\infty} \sigma_n / n) \leq \exp(\sum_{n=1}^{\infty} n \sigma_n |\alpha_n|^2)(\sum_{n=0}^{\infty} \rho_n)$$

when the weights ρ_n are a nonincreasing sequence of nonnegative numbers such that the sum

$$\sum_{n=0}^{\infty} \rho_n$$

converges and when the weights

$$\sigma_r = \sum_{n=r}^{\infty} \rho_n / \sum_{n=0}^{\infty} \rho_n$$

defined for every positive number r give a convergent sum

$$\sum_{n=1}^{\infty} \sigma_n/n.$$

Essential cases are obtained when the numbers ρ_n are zeros and ones.

An example of a Riemann mapping is defined by the Koebe function

$$f(z) = z + 2z^2 + 3z^3 + \ldots$$

which maps the unit disk onto the region obtained from the complex plane on deleting the half–line of real numbers which are not greater than minus one–quarter. The Koebe function belongs to a family of analytic functions vanishing at the origin which map the unit disk injectively onto regions which are star–like with respect to the origin: A convex combination of an element of the region with the origin is an element of the region. The Bieberbach conjecture is immediate for members of the family. The Koebe function is an extremal function for the Bieberbach conjecture since equality holds for the n–th coefficient for every positive integer n.

A proof of the Bieberbach conjecture for the third coefficient was obtained by Karl Löwner, a student in Berlin where Bieberbach lectured on complex analysis. Bieberbach praised the manuscript when he gave his approval for publication.

Löwner showed that every Riemann mapping function is a limit of finite compositions of Riemann mapping functions which apply to star–like regions. He failed to obtain a proof of the Bieberbach conjecture for all coefficients because he was unable to propagate estimates under composition.

Information cannot be propagated when it is unsuitably coded. The need for good coding of information appears in Hilbert spaces of entire functions applied to the Riemann hypothesis. Contractive inclusions of Hilbert spaces of entire functions resemble isometric inclusions in essential properties. Complementation was discovered in this elementary context for which the contractive inclusions are almost isometric. A more substantial application of complementation is applied in the construction of invariant subspaces. A close resemblance to isometric inclusions underlies the proof of the existence of invariant subspaces. The adaptation to the proof of the Bieberbach conjecture causes no difficulty.

The proof of the Bieberbach conjecture and its application to the proof of the Riemann mapping theorem can be read with a superficial knowledge of invariant subspaces. The Hilbert spaces of analytic functions applied to the construction of invariant subspaces are applied only in the case of power series with complex coefficients. Vector and operator coefficients are however natural to the construction of invariant subspaces. The application of factorization for analytic functions to invariant subspaces for contractive transformations is as fundamental to complex analysis as the Riemann mapping theorem and the Riemann hypothesis.

The author thanks the Department of Mathematics of Purdue University for its continued support of research on the Riemann hypothesis. The lecture notes on square summable

power series are preparation for the methods of complex analysis applied to this fundamental problem.

CONTENTS

Chapter 1. The Existence of Invariant Subspaces

Complex numbers are constructed from the Gauss numbers

$$x + iy$$

whose coordinates x and y are rational numbers. The addition and multiplication of Gauss numbers is derived in the obvious way from the addition and multiplication of rational numbers and from the defining identity

$$i^2 = -1$$

of the imaginary unit i. Conjugation is the homomorphism of additive and multiplicative structure which takes

$$z = x + iy$$

into

$$z^- = x - iy.$$

The product

$$z^- z = x^2 + y^2$$

of a nonzero Gauss number and its conjugate is a positive rational number.

The Gauss plane admits topologies which are compatible with addition and multiplication. The Dedekind topology is derived from convex structure.

A convex combination

$$(1 - t)z + tw$$

of elements z and w of the Gauss plane is an element of the Gauss plane when t is a rational number in the interval $[0, 1]$. A subset of the Gauss plane is said to be preconvex if it contains all elements of the Gauss plane which are convex combinations of elements of the set. The preconvex span of a subset of the Gauss plane is defined as the smallest preconvex subset of the Gauss plane which contains the given set.

The closure in the Gauss plane of a preconvex subset B is the set B^- of elements w of the Gauss plane such that the set whose elements are w and the elements of B is preconvex. The closure of a preconvex set is a preconvex set which is its own closure.

A nonempty preconvex set is defined as open if it is disjoint from the closure of every disjoint nonempty preconvex set. The intersection of two nonempty open preconvex sets is an open preconvex set when it is nonempty.

A subset of the Gauss plane is said to be open if it is a union of nonempty open preconvex sets. The empty set is open since it is an empty union of such sets. Unions of open sets are open. Finite intersections of open sets are open.

An example of an open set is the complement in the Gauss plane of the closure of a nonempty preconvex set. A subset of the Gauss plane is said to be closed if it is the complement in the Gauss plane of an open set. Intersections of closed sets are closed. Finite unions of closed sets are closed. The Gauss plane is a Hausdorff space in the topology whose open and closed sets are defined by convexity. These open and closed sets define the Dedekind topology of the Gauss plane.

If a nonempty open preconvex set A is disjoint from a nonempty preconvex set B, then a maximal preconvex set exists which contains B and is disjoint from A. The maximal preconvex set is closed and has a preconvex complement. The existence of the maximal preconvex set is an application of the Kuratowski–Zorn lemma.

Addition and multiplication are continuous as transformations of the Cartesian product of the Gauss plane with itself into the Gauss plane when the Gauss plane is given the Dedekind topology. The complex plane is the completion of the Gauss plane in the uniform Dedekind topology. Uniformity of topology refers to the determination of neighborhoods of a Gauss number by neighborhoods of the origin. If A is a neighborhood of the origin and if w is a Gauss number, then the set of sums of w with elements of A is a neighborhood of w. Every neighborhood of w is derived from a neighborhood of the origin.

A Cauchy class of closed subsets of the Gauss plane is a nonempty class of closed subsets such that the intersection of the members of any finite subclass is nonempty and such that for every neighborhood A of the origin some member B of the class exists such that all differences of elements of B belong to A.

A Cauchy class of closed subsets is contained in a maximal Cauchy class of closed subsets. A Cauchy sequence is a sequence of elements $w_1, w_2, w_3, \dots$ of the Gauss plane such that a Cauchy class of closed subsets is defined whose members are indexed by the positive integers r. The r–th member is the closed preconvex span of the elements w_n with n not less than r. A Cauchy sequence determines a maximal Cauchy class. Every maximal Cauchy class is determined by a Cauchy sequence.

An element of the Gauss plane determines the maximal Cauchy class of closed sets which contain the element. An element of the complex plane is determined by a maximal Cauchy class. The Gauss plane is a subset of the complex plane.

If B is a closed subset of the Gauss plane, the closure B^- of B in the complex plane is defined as the set of all elements of the complex plane whose maximal Cauchy class has B as a member. A subset of the complex plane is defined as open if it is disjoint from the closure in the complex plane of every disjoint closed subset of the Gauss plane. Unions of open subsets of the complex plane are open. Finite intersections of open subsets of the complex plane are open. A subset of the Gauss plane is open in the Dedekind topology of the Gauss plane if, and only if, it is the intersection with the Gauss plane of an open subset of the complex plane.

A subset of the complex plane is defined as closed if its complement in the complex plane is open. Intersections of closed subsets of the complex plane are closed. Finite unions of closed subsets of the complex plane are closed. The closure of a subset of the complex plane is defined as the smallest closed set containing the given set. The closure in the Gauss plane of a subset of the Gauss plane is the intersection with the Gauss plane of the closure of the subset in the complex plane.

The complex plane is a Hausdorff space in the topology whose open sets and closed sets are determined by convexity. These open sets and closed sets define the Dedekind topology of the complex plane.

The Gauss plane is dense in the complex plane. Addition and multiplication admit unique continuous extensions as transformations of the Cartesian product of the complex plane with itself into the complex plane. Conjugation admits a unique continuous extension as a transformation of the complex plane into itself.

Properties of addition in the Gauss plane are preserved in the complex plane. The associative law

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

holds for all complex numbers α, β, and γ. The commutative law

$$\alpha + \beta = \beta + \alpha$$

holds for all complex numbers α and β. The origin 0 of the Gauss plane satisfies the identities

$$0 + \gamma = \gamma = \gamma + 0$$

for every element γ of the complex plane. For every element α of the complex plane a unique element

$$\beta = -\alpha$$

of the complex plane exists such that

$$\alpha + \beta = 0 = \beta + \alpha.$$

Conjugation is a homomorphism γ into γ^- of additive structure: The identity

$$(\alpha + \beta)^- = \alpha^- + \beta^-$$

holds for all complex numbers α and β.

Multiplication by a complex number γ is a homomorphism of additive structure: The identity

$$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$$

holds for all complex numbers α and β. The parametrization of homomorphisms is consistent with additive structure: The identity

$$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$$

holds for all complex numbers α, β, and γ. Multiplication by γ is the homomorphism which annihilates every element of the complex plane when γ is the origin. Multiplication by γ is the identity homomorphism when γ is the unit 1 of the Gauss plane.

The composition of homomorphisms is consistent with multiplicative structure: The associative law

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

holds for all complex numbers α, β, and γ. Multiplication is commutative: The identity

$$\alpha\beta = \beta\alpha$$

holds for all complex numbers α and β. Conjugation is a homomorphism of multiplicative structure: The identity

$$(\alpha\beta)^- = \alpha^-\beta^-$$

holds for all complex numbers α and β.

A real number is a complex number γ which is self–conjugate:

$$\gamma^- = \gamma.$$

Sums and products of real numbers are real. A real number is said to be nonnegative if it can be written

$$\gamma^-\gamma$$

for a complex number γ. The sum of two nonnegative numbers is nonnegative. A nonnegative number is said to be positive if it is nonzero. A positive number α has a positive inverse

$$\beta = \alpha^{-1}$$

such that

$$\beta\alpha = 1 = \alpha\beta.$$

A nonzero complex number α has a nonzero complex inverse

$$\beta = \alpha^{-1}$$

such that

$$\beta\alpha = 1 = \alpha\beta.$$

The complex plane is complete in the uniform Dedekind topology. Cauchy completion produces the same space. Closed and bounded subsets of the complex plane are compact. Boundedness of a set means that the elements α of the set satisfy the inequality

$$\alpha^-\alpha \leq c$$

for some nonnegative number c. A nonempty class of closed sets has a nonempty intersection if the members of every finite subclass have a nonempty intersection and if some member of the class is bounded.

The space $\mathcal{C}(z)$ of square summable power series is the set of power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$$

with complex coefficients a_n such that the sum

$$\|f(z)\|^2 = |a_0|^2 + |a_1|^2 + |a_2|^2 + \dots$$

converges. If w is a complex number, the power series

$$wf(z) = wa_0 + wa_1 z + wa_2 z^2 + \dots$$

is square summable since the identity

$$\|wf(z)\|^2 = |w|^2 \|f(z)\|^2$$

is satisfied. If power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots$$

and

$$g(z) = b_0 + b_1 z + b_2 z^2 + \dots$$

are square summable, then the convex combination

$$(1-t)f(z) + tg(z)$$
$$= [(1-t)a_0 + tb_0] + [(1-t)a_1 + tb_1]z + [(1-t)a_2 + tb_2]z^2 + \dots$$

and the difference

$$g(z) - f(z) = (b_0 - a_0) + (b_1 - a_1)z + (b_2 - a_2)z^2 + \dots$$

are square summable power series since the convexity identity

$$\|(1-t)f(z) + tg(z)\|^2 + t(1-t)\|g(z) - f(z)\|^2 = (1-t)\|f(z)\|^2 + t\|g(z)\|^2$$

holds when t is in the interval $[0, 1]$.

The space of square summable power series is a vector space over the complex numbers which admits a scalar product. The scalar product of square summable power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots$$

and

$$g(z) = b_0 + b_1 z + b_2 z^2 + \dots$$

is the complex number

$$\langle f(z), g(z)\rangle = b_0^- a_0 + b_1^- a_1 + b_2^- a_2 + \dots$$

Convergence of the sum is a consequence of the identity

$$4\langle f(z), g(z)\rangle = \|f(z) + g(z)\|^2 - \|f(z) - g(z)\|^2 + i\|f(z) + ig(z)\|^2 - i\|f(z) - ig(z)\|^2.$$

Linearity of a scalar product states that the identity

$$\langle af(z) + bg(z), h(z)\rangle = a\langle f(z), h(z)\rangle + b\langle g(z), h(z)\rangle$$

holds for all complex numbers a and b when $f(z), g(z)$, and $h(z)$ are square summable power series. Symmetry of a scalar product states that the identity

$$\langle g(z), f(z)\rangle = \langle f(z), g(z)\rangle^-$$

holds for all square summable power series $f(z)$ and $g(z)$. Positivity of a scalar product states that the scalar self–product

$$\langle f(z), f(z)\rangle = \|f(z)\|^2$$

is positive for every nonzero square summable power series $f(z)$.

Linearity, symmetry, and positivity of a scalar product imply the Cauchy inequality

$$|\langle f(z), g(z)\rangle| \leq \|f(z)\|\|g(z)\|$$

for all square summable power series $f(z)$ and $g(z)$. Equality holds in the Cauchy inequality if, and only if, the power series $f(z)$ and $g(z)$ are linearly dependent. The triangle inequality

$$\|h(z) - f(z)\| \leq \|g(z) - f(z)\| + \|h(z) - g(z)\|$$

follows for all square summable power series $f(z), g(z)$, and $h(z)$. The space of square summable power series is a metric space with

$$\|g(z) - f(z)\| = \|f(z) - g(z)\|$$

as the distance between square summable power series $f(z)$ and $g(z)$. The space of square summable power series is a Hilbert space since the metric space is complete. If a Cauchy sequence $f_0(z), f_1(z), f_2(z), \ldots$ of square summable power series

$$f_n(z) = a_{n0} + a_{n1}z + a_{n2}z^2 + \ldots$$

is given, the inequality

$$|a_{n0} - a_{m0}|^2 + \ldots + |a_{nr} - a_{mr}|^2 \leq \|f_n(z) - f_m(z)\|^2$$

implies that a Cauchy sequence

$$a_{0k}, a_{1k}, a_{2k}, \ldots$$

of complex numbers is obtained as coefficients of z^k for every nonnegative integer k. Since the complex plane is complete, a power series

$$f(z) = a_0 + a_1z + a_2z^2 + \ldots$$

is defined whose coefficients

$$a_k = \lim a_{nk}$$

are limits. The triangle inequality implies that the sequence of distances

$$\|f_n(z) - f_0(z)\|, \ \|f_n(z) - f_1(z)\|, \ \|f_n(z) - f_2(z)\|, \dots$$

is Cauchy for every nonnegative integer n. A limit exists since the real line is complete. Since the inequality

$$|a_{n0} - a_0|^2 + \dots + |a_{nr} - a_r|^2 \leq \lim \|f_n(z) - f_m(z)\|^2$$

holds for every positive integer r, the power series

$$f_n(z) - f(z)$$

is square summable. Since the inequality

$$\|f_n(z) - f(z)\|^2 \leq \lim \|f_n(z) - f_m(z)\|^2$$

is satisfied, the square summable power series $f(z)$ is the limit of the Cauchy sequence of square summable power series $f_n(z)$.

The distance from an element of a Hilbert space to a nonempty closed convex subset of a Hilbert space is attained by an element of the convex set. If $f(z)$ is a square summable power series and if C is a nonempty closed convex set of square summable power series, then an element $g(z)$ of C exists which minimizes the distance δ from $f(z)$ to elements of C. By definition

$$\delta = \inf \|f(z) - g(z)\|$$

is a greatest lower bound taken over the elements $g(z)$ of C. It needs to be shown that

$$\delta = \|f(z) - g(z)\|$$

for some element $g(z)$ of C. For every positive integer n an element $g_n(z)$ of C exists which satisfies the inequality

$$\|f(z) - g_n(z)\|^2 \leq \delta^2 + n^{-2}.$$

Since

$$(1-t)g_m(z) + tg_n(z)$$

belongs to C when t belongs to the interval $[0, 1]$, the inequality

$$\delta \leq \|f(z) - (1-t)g_m(z) - tg_n(z)\|$$

is then satisfied. The convexity identity

$$\begin{gathered} \|f(z) - (1-t)g_m(z) - tg_n(z)\|^2 \\ + t(1-t)\|g_m(z) - g_n(z)\| = (1-t)\|f(z) - g_m(z)\|^2 + t\|f(z) - g_n(z)^2\| \end{gathered}$$

implies the inequality

$$t(1-t)\|g_m(z)-g_n(z)\|^2 \leq (1-t)m^{-2}+tn^{-2}.$$

Since the square summable power series $g_n(z)$ form a Cauchy sequence, they converge to a square summable power series $g(z)$. Since C is closed, $g(z)$ is an element of C which satisfies the identity

$$\delta = \|f(z)-g(z)\|.$$

A vector subspace of the space of square summable power series is an example of a nonempty convex subset. The orthogonal complement of a vector subspace $\mathcal{M}$ of the space of square summable power series is the set of square summable power series $f(z)$ which are orthogonal

$$\langle f(z), g(z)\rangle = 0$$

to every element $g(z)$ of $\mathcal{M}$. The orthogonal complement of a vector subspace of the space of square summable power series is a closed vector subspace of the space of square summable power series. When $\mathcal{M}$ is a closed vector subspace of the space of square summable power series, and when $f(z)$ is a square summable power series, an element $g(z)$ of $\mathcal{M}$ which is nearest $f(z)$ is unique and is characterized by the orthogonality of $f(z)-g(z)$ to elements of $\mathcal{M}$. Then $f(z)-g(z)$ belongs to the orthogonal complement of $\mathcal{M}$. Every square summable power series $f(z)$ is the sum of an element $g(z)$ of $\mathcal{M}$, called the orthogonal projection of $f(z)$ in $\mathcal{M}$, and an element $f(z)-g(z)$ of the orthogonal complement of $\mathcal{M}$. Orthogonal projection is linear. If $f_0(z)$ and $f_1(z)$ are elements of $\mathcal{M}$ and if c_0 and c_1 are complex numbers, the orthogonal projection of

$$c_0f_0(z)+c_1f_1(z)$$

in $\mathcal{M}$ is

$$c_0g_0(z)+c_1g_1(z)$$

with $g_0(z)$ the orthogonal projection of $f_0(z)$ in $\mathcal{M}$ and $g_1(z)$ the orthogonal projection of $f_1(z)$ in $\mathcal{M}$. The closure of a vector subspace $\mathcal{M}$ of the space of square summable power series is a vector subspace which is the orthogonal complement of the closed vector subspace which is the orthogonal complement of $\mathcal{M}$.

A linear functional on the space of square summable power series is a linear transformation of the space into the complex numbers. A linear functional is continuous if, and only if, its kernel is closed. If a continuous linear functional does not annihilate every square summable power series, then its kernel is a proper closed subspace of the space of square summable power series whose orthogonal complement has dimension one. An element $g(z)$ of the orthogonal complement of the kernel exists such that the linear functional takes $f(z)$ into the scalar product

$$\langle f(z), g(z)\rangle$$

for every square summable power series $f(z)$. If $g(z)$ is a square summable power series, the linear functional which takes $f(z)$ into

$$\langle f(z), g(z)\rangle$$

is continuous by the Cauchy inequality.

Multiplication by z in the space of square summable power series is the transformation which takes

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$$

into

$$zf(z) = 0 + a_0 z + a_1 z^2 + \ldots$$

whenever $f(z)$, and hence $zf(z)$, is square summable. Multiplication by z is an isometric transformation

$$\|zf(z)\| = \|f(z)\|$$

of the space of square summable power series into itself. The range of multiplication by z is the set of square summable power series with constant coefficient zero. The range is a closed vector subspace which is a Hilbert space in the inherited scalar product. The orthogonal complement of the range is the space of constants.

The space of square summable power series with complex coefficients is a fundamental example of a Hilbert space. The constructions made in this space apply in the Hilbert space of square summable power series with coefficients in a Hilbert space. Properties of topology in Hilbert spaces are applied in the generalization.

A Hilbert space is a vector space $\mathcal{H}$ over the complex numbers which is given a scalar product for which self–products of nonzero elements are positive and which is complete in a resulting metric topology.

Addition is a transformation of the Cartesian product of the space with itself into the space whose properties are familiar from the complex plane. The associative law

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

holds for all elements α, β, and γ of the space. The commutative law

$$\alpha + \beta = \beta + \alpha$$

holds for all elements α and β of the space. The space contains an origin 0 which satisfies the identity

$$0 + \gamma = \gamma = \gamma + 0$$

for every element γ of the space. For every element α of the space, a unique element

$$\beta = -\alpha$$

of the space exists such that

$$\alpha + \beta = 0 = \beta + \alpha.$$

Multiplication is a transformation of the Cartesian product of the complex plane with the space into the space whose properties are familiar from the complex plane. Multiplication by a complex number γ is a homomorphism of additive structure: The identity

$$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$$

holds for all elements α and β of the space. The parametrization of homomorphisms is consistent with additive structure: The identity

$$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$$

holds for all complex numbers α and β when γ is an element of the space. Multiplication by γ is the homomorphism which annihilates every element of the space when γ is the origin of the complex plane. Multiplication by γ is the identity homomorphism when γ is the unit of the complex plane. The composition of homomorphisms is consistent with multiplicative structure: The associative law

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

holds for all complex numbers α and β when γ is an element of the space.

The scalar product is a transformation of the Cartesian product of the space with itself into the complex numbers whose properties are familiar from the complex plane. The scalar product defines homomorphisms of additive structure: The identities

$$\langle \alpha + \beta, \gamma \rangle = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$$

and

$$\langle \gamma, \alpha + \beta \rangle = \langle \gamma, \alpha \rangle + \langle \gamma, \beta \rangle$$

hold for all elements α, β, and γ of the space. The scalar product defines a homomorphism and an anti–homomorphism of multiplicative structure for all elements α and β of the space: The identities

$$\langle \gamma\alpha, \beta \rangle = \gamma \langle \alpha, \beta \rangle$$

and

$$\langle \alpha, \gamma\beta \rangle = \gamma^- \langle \alpha, \beta \rangle$$

hold for every complex number γ. The scalar product is symmetric: The identity

$$\langle \beta, \alpha \rangle = \langle \alpha, \beta \rangle^-$$

holds for all elements α and β of the space. The scalar self–product

$$\langle c, c \rangle$$

is positive for every nonzero element c of the space.

A convex combination

$$(1 - t)\alpha + t\beta$$

of elements α and β of the space is an element of the space defined by a real number t in the interval $[0, 1]$. A subset of the space is said to be convex if it contains all convex combinations of pairs of its elements. The convex span of a subset of the space is the smallest convex set which contains the given set.

The norm of an element γ of the space is the nonnegative solution $\|\gamma\|$ of the equation

$$\|\gamma\|^2 = \langle \gamma, \gamma \rangle.$$

The Cauchy inequality

$$|\langle \alpha, \beta \rangle| \leq \|\alpha\| \|\beta\|$$

holds for all elements α and β of the space. Equality holds when, and only when, α and β are linearly dependent. The triangle inequality

$$\|\gamma - \alpha\| \leq \|\gamma - \beta\| + \|\beta - \alpha\|$$

holds for all elements α, β, and γ of the space. The space is a metric space with

$$\|\beta - \alpha\| = \|\alpha - \beta\|$$

as the distance between elements α and β.

The closure B^- of a nonempty convex subset B of the space is defined using the metric topology. The closure of a convex set is a convex set which is its own closure. The closed convex span of a set is the smallest closed convex set which contains the given set.

The metric is not used for the definition of an open set. A nonempty convex set is said to be open if it is disjoint from the closure of every nonempty disjoint convex set. If A is a nonempty open convex set and if B is a nonempty convex set, then the intersection of A with the closure of B is contained in the closure of the intersection of A with B. The intersection of two nonempty open convex sets is an open convex set if it is nonempty.

A subset of the space is said to be open if it is a union of nonempty open convex sets. The empty set is open since it is an empty union. Unions of open sets are open. Finite intersections of open sets are open.

An example of an open set is the complement of the closure of a convex set. A set is said to be closed if its complement is open. Intersections of closed sets are closed. Finite unions of closed sets are closed. The space is a Hausdorff space in the topology whose open and closed sets are defined by convexity. The open and closed sets define the Dedekind topology of the space.

The Dedekind topology is applied in the geometric formulation of the Hahn–Banach theorem: If a nonempty open convex set A is disjoint from a nonempty convex set B, then a maximal convex set exists which contains B and is disjoint from A. The maximal convex set is closed and has a convex complement.

If β is an element of the space, a continuous linear functional β^- is defined on the space by taking α into

$$\beta^- \alpha = \langle \alpha, \beta \rangle$$

for every element α of the space. If c is a real number, the set of elements α which satisfy the inequality

$$\mathcal{R}\, \beta^- \alpha \geq c$$

is a closed convex set whose complement is convex.

Completeness is required of a Hilbert space so that every nonempty closed convex set whose complement is nonempty and convex is determined by an element of the space. A set which is open for the metric topology is open for the Dedekind topology since it is a union of convex sets which are open for the Dedekind topology. The convex set is defined by an element γ of the space and a positive number c as the set of elements α which satisfy the inequality

$$(\alpha - \gamma)^{-}(\alpha - \gamma) < c.$$

When the space is complete in the metric topology, every set which is open for the Dedekind topology is open for the metric topology. The proof applies consequences of completeness in metric spaces observed by Baire.

It needs to be shown that every nonempty convex set which is open for the Dedekind topology is open for the metric topology. The convex set A can be assumed to contain the origin. Argue by contradiction assuming that the set is not open for the metric topology. Since the set is convex, it has an empty interior for the metric topology.

A convex set A_n which is open for the Dedekind topology is defined for every positive integer n as the set of products $n\alpha$ with α in A. The set is convex and has empty interior for the metric topology. A positive number c_n is chosen for every positive integer n so that the sum of the numbers converges. An element α_n of A_n is chosen inductively so that the inequality

$$(\alpha_{n+1} - \alpha_n)^{-}(\alpha_{n+1} - \alpha_n) < c_n^2$$

holds for every positive integer n. The element α_1 of A_1 is chosen arbitrarily. When α_n is chosen in A_n, the element α_{n+1} is chosen in A_{n+1} so that it does not belong to the closure of A_n. The element α_n of the space converge to an element α of the space since a Hilbert space is assumed metrically complete. Properties of open sets for the Dedekind topology are contradicted since the union of the set A_n does not contain every element of the space.

Since the Dedekind topology of a Hilbert space is identical with the metric topology, a Hilbert space is complete in the Dedekind topology.

If B is a nonempty closed convex subset of a Hilbert space and if α is an element of the space which does not belong to B, then an element γ of the convex set exists which is closest to α. The inequality

$$(\alpha - \gamma)^{-}(\alpha - \gamma) \leq (\alpha - \beta)^{-}(\alpha - \beta)$$

holds for every element β of B. The inequality

$$\mathcal{R}\ (\alpha - \gamma)^{-}(\beta - \gamma) \geq 0$$

holds for every element β of B. The set of elements β of the space which satisfy the inequality

$$\mathcal{R}\ (\alpha - \gamma)^{-}\beta \geq \mathcal{R}\ (\alpha - \gamma)^{-}\gamma$$

is a closed convex set which contains B and whose complement is convex and contains α.

A subset B of a Hilbert space is said to be bounded if the linear functional α^- maps B onto a bounded subset of the complex plane for every element α of the space. If B is a nonempty closed and bounded convex subset of a Hilbert space, an element γ of B exists which satisfies the inequality

$$\beta^- \beta \leq \gamma^- \gamma$$

for every element β of B.

The closed convex span of a bounded subset of a Hilbert space is bounded. If B is a nonempty bounded subset of a Hilbert space, an element γ of its closed convex span exists which satisfies the inequality

$$\beta^- \beta \leq \gamma^- \gamma$$

for every element β of B.

A Hilbert space admits a topology which is compatible with addition and multiplication and for which every closed and bounded set is compact. The topology is defined as the weakest topology with respect to which the linear functional α^- is continuous for every element α of the space. An open set for the weak topology is a union of open convex sets. A convex set is closed for the weak topology if, and only if, it is closed for the Dedekind topology.

An orthonormal set is a set of elements such that the scalar self–product of every element is one but the scalar product of distinct elements is zero. A Hilbert space admits a maximal orthonormal set as an application of the Kuratowski–Zorn lemma. Two maximal orthonormal sets have the same cardinality. The dimension of the Hilbert space is defined as the cardinality of a maximal orthonormal set.

If $\mathcal{S}$ is a maximal orthonormal set in a Hilbert space, the space determines a subspace of the Cartesian product of the complex numbers taken over the elements of $\mathcal{S}$. An element of the Cartesian product is written as a formal sum

$$\sum f(\iota)\iota$$

over the elements ι of the orthonormal set with coefficients which are a complex valued function $f(\iota)$ of ι in the orthonormal set. An element of the Cartesian product determines an element of the Hilbert space if, and only if, the sum

$$\sum |f(\iota)|^2$$

over the elements of the orthonormal set converges. The weak topology of the Hilbert space agrees with the subspace topology of the Cartesian product topology on bounded sets. The compactness of closed and bounded subsets of the Cartesian product space follows from the compactness of closed and bounded subsets of the complex plane since a Cartesian product of compact Hausdorff spaces is a compact Hausdorff space. The weak compactness of closed and bounded convex subsets of a Hilbert space follows because a closed subset of a compact set is compact.

The Hilbert space $\mathcal{C}(z)$ of square summable power series with coefficients in a Hilbert space $\mathcal{C}$ is advantageous for the relationship between factorization and invariant subspaces. The construction of the space imitates the construction made when $\mathcal{C}$ is the complex numbers.

A vector is an element of the coefficient space $\mathcal{C}$. The scalar product

$$b^- a$$

of vectors a and b is taken in the complex plane. The norm $|c|$ of a vector is the nonnegative solution of the equation

$$|c|^2 = c^- c.$$

The space $\mathcal{C}(z)$ of square summable power series with coefficients in $\mathcal{C}$ is the set of power series

$$f(z) = a_0 + a, z + a_2 z^2 + \ldots$$

with vector coefficients such that the sum

$$\|f(z)\|^2 = |a_0|^2 + |a_1|^2 + |a_2|^2 + \ldots$$

converges. If w is a complex number, the power series

$$wf(z) = wa_0 + wa_1 z + wa_2 z^2 + \ldots$$

with vector coefficients is square summable since the identity

$$\|wf(z)\|^2 = |w|^2 \|f(z)\|^2$$

is satisfied. If power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$$

and

$$g(z) = b_0 + b_1 z + b_2 z^2 + \ldots$$

with vector coefficients are square summable, then the convex combination

$$\begin{gathered}(1-t)f(z) + tg(z) \\ = [(1-t)a_0 + tb_0] + [(1-t)a_1 + tb_1]z + [(1-t)a_2 + tb_2]z^2 + \ldots\end{gathered}$$

and the difference

$$g(z) - f(z) = (b_0 - a_0) + (b_1 - a_1)z + (b_2 - a_2)z^2 + \ldots$$

are power series with vector coefficients which are square summable since the convexity identity

$$\|(1-t)f(z) + tg(z)\|^2 + t(1-t)\|g(z) - f(z)\|^2 = (1-t)\|f(z)\|^2 + t\|g(z)\|^2$$

holds when t is in the interval $[0,1]$.

The space of square summable power series with vector coefficients is a vector space over the complex numbers which admits a scalar product. The scalar product

$$\langle f(z), g(z)\rangle = b_0^- a_0 + b_1^- a_1 + b_2^- a_2 + \ldots$$

of square summable power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$$

and

$$g(z) = b_0 + b_1 z + b_1 z^2 + \ldots$$

with vector coefficients is a complex number. Convergence of the sum is a consequence of the identity

$$4\langle f(z), g(z)\rangle = \|f(z)+g(z)\|^2 - \|f(z)-g(z)\|^2 + i\|f(z)+ig(z)\|^2 - i\|f(z)-ig(z)\|^2.$$

Linearity

$$\langle af(z)+bg(z), h(z)\rangle = a\langle f(z), h(z)\rangle + b\langle g(z), h(z)\rangle$$

holds for all complex numbers a and b when $f(z), g(z)$, and $h(z)$ are square summable power series with vector coefficients. Symmetry

$$\langle g(z), f(z)\rangle = \langle f(z), g(z)\rangle^-$$

holds for all square summable power series $f(z)$ and $g(z)$ with vector coefficients. The scalar self–product

$$\langle f(z), f(z)\rangle$$

of a nonzero square summable power series $f(z)$ with vector coefficients is positive.

The norm of a square summable power series $f(z)$ with vector coefficients is the nonnegative solution $\|f(z)\|$ of the equation

$$\|f(z)\|^2 = \langle f(z), f(z)\rangle.$$

The Cauchy inequality

$$|\langle f(z), g(z)\rangle| \leq \|f(z)\|\|g(z)\|$$

holds for all square summable power series $f(z)$ and $g(z)$ with vector coefficients. Equality holds in the Cauchy inequality if, and only if, $f(z)$ and $g(z)$ are linearly dependent. The triangle inequality

$$\|h(z)-f(z)\| \leq \|g(z)-f(z)\| + \|h(z)-g(z)\|$$

holds for all square summable power series $f(z), g(z)$, and $h(z)$ with vector coefficients. The space of square summable power series with vector coefficients is a metric space with

$$\|g(z)-f(z)\| = \|f(z)-g(z)\|$$

as the distance between square summable power series $f(z)$ and $g(z)$ with vector coefficients. The space of square summable power series with vector coefficients is a Hilbert space since it is complete in the metric topology.

The multiplicative structure of the complex numbers is lacking in other coefficient spaces. Multiplicative structure is restored by introducing operators on the coefficient space. An operator is a continuous linear transformation of the coefficient space into itself. A complex number is treated as an operator since multiplication by a number is a continuous linear transformation. Since continuity is taken in the Dedekind topology, a continuous transformation is a constant multiple of a contractive transformation. The bound $|\gamma|$ of an operator γ is the least nonnegative number such that the operator is the product of $|\gamma|$ and a contractive transformation. When γ is a number, the bound is the absolute value of γ.

If $W(z)$ is a power series with operator coefficients and if $f(z)$ is a power series with vector coefficients, the product

$$W(z)f(z)$$

is a power series with vector coefficients. Multiplication by $W(z)$ acts as a partially isometric transformation of the space $\mathcal{C}(z)$ of square summable power series with vector coefficients onto a Hilbert space whose elements are power series with vector coefficients. The partially isometric character of the transformation refers to its isometric character on the orthogonal complement of its kernel. The space obtained is contained contractively in $\mathcal{C}(z)$, if, and only if, multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. Multiplication by $W(z)$ is a contractive transformation of the space into itself.

If a Hilbert space $\mathcal{P}$ is contained contractively in a Hilbert space $\mathcal{H}$, a unique Hilbert space $\mathcal{Q}$ exists, which is contained contractively in $\mathcal{H}$, such that the inequality

$$\|c\|_{\mathcal{H}}^2 \leq \|a\|_{\mathcal{P}}^2 + \|b\|_{\mathcal{Q}}^2$$

holds whenever

$$c = a + b$$

is the sum of an element a of $\mathcal{P}$ and an element b of $\mathcal{Q}$, and such that every element c of $\mathcal{H}$ is a sum for which equality holds. The space $\mathcal{Q}$ is called the complementary space to $\mathcal{P}$ in $\mathcal{H}$.

The construction of the space $\mathcal{Q}$ and the verification of its properties are applications of convexity. The space $\mathcal{Q}$ is defined as the set of elements b of $\mathcal{H}$ for which a finite least upper bound

$$\|b\|_{\mathcal{Q}}^2 = \sup[\|a + b\|_{\mathcal{H}}^2 - \|a\|_{\mathcal{P}}^2]$$

is obtained over all elements a of $\mathcal{P}$. If b is an element of $\mathcal{Q}$ and if w is a complex number, then wb is an element of $\mathcal{Q}$ which satisfies the identity

$$\|wb\|_{\mathcal{Q}} = |w|\|b\|_{\mathcal{Q}}.$$

A convex combination

$$(1 - t)a + tb$$

of elements of $\mathcal{Q}$ is an element of $\mathcal{Q}$ which satisfies the convexity identity

$$\|(1-t)a+tb\|_{\mathcal{Q}}^2 + t(1-t)\|b-a\|_{\mathcal{Q}}^2 = (1-t)\|a\|_{\mathcal{Q}}^2 + t\|b\|_{\mathcal{Q}}^2$$

with t in the interval $[0,1]$.

The space $\mathcal{Q}$ is a vector space over the complex numbers. A scalar product is defined in the space by the identity

$$4\langle a,b\rangle_{\mathcal{Q}} = \|a+b\|_{\mathcal{Q}}^2 - \|a-b\|_{\mathcal{Q}}^2 + i\|a+ib\|_{\mathcal{Q}}^2 - i\|a-ib\|_{\mathcal{Q}}^2.$$

Immediate consequences of the definition are symmetry

$$\langle b,a\rangle_{\mathcal{Q}} = \langle a,b\rangle_{\mathcal{Q}}^-$$

and the identity

$$\langle \omega a,b\rangle = \omega\langle a,b\rangle$$

when ω is a fourth root of unity if a and b are elements of $\mathcal{Q}$. An application of the convexity identity gives the identity

$$\|(1-t)a+tb\|_{\mathcal{Q}}^2 - \|(1-t)a-tb\|_{\mathcal{Q}}^2 = t(1-t)\|a+b\|_{\mathcal{Q}}^2 - t(1-t)\|a-b\|_{\mathcal{Q}}^2$$

which implies the identity

$$\langle (1-t)a, tb\rangle_{\mathcal{Q}} = t(1-t)\langle a,b\rangle_{\mathcal{Q}}$$

for all elements a and b of $\mathcal{Q}$ when t is in the interval $[0,1]$. Another application of the convexity identity gives the identity

$$\begin{aligned}\|(1-t)(a+c)+t(b+c)\|_{\mathcal{Q}}^2 - \|(1-t)(a-c)+t(b-c)\|_{\mathcal{Q}}^2 \\ = (1-t)\|a+c\|_{\mathcal{Q}}^2 - (1-t)\|a-c\|_{\mathcal{Q}}^2 + t\|b+c\|_{\mathcal{Q}}^2 - t\|b-c\|_{\mathcal{Q}}^2\end{aligned}$$

which implies the identity

$$\langle (1-t)a+tb, c\rangle_{\mathcal{Q}} = (1-t)\langle a,c\rangle_{\mathcal{Q}} + t\langle b,c\rangle_{\mathcal{Q}}$$

for all elements a, b, and c of $\mathcal{Q}$ when t belongs to the interval $[0,1]$. Linearity of a scalar product follows.

Positivity of a scalar product is immediate from the definition. The inequality

$$\|b\|_{\mathcal{H}} \le \|b\|_{\mathcal{Q}}$$

for elements b of $\mathcal{Q}$ states that the space $\mathcal{Q}$ is contained contractively in the space $\mathcal{H}$. The inequality is used to prove that the space $\mathcal{Q}$ is complete in the metric topology defined by the scalar product of $\mathcal{Q}$. A Cauchy sequence of elements b_n of $\mathcal{Q}$ is a Cauchy sequence of elements of $\mathcal{H}$ since the inequality

$$\|b_m - b_n\|_{\mathcal{H}} \leq \|b_m - b_n\|_{\mathcal{Q}}$$

holds for all nonnegative integers m and n. Since $\mathcal{H}$ is a Hilbert space, an element b of $\mathcal{H}$ exists which is the limit of the sequence in the metric topology of $\mathcal{H}$. Since the inequality

$$\|b_n\|_{\mathcal{Q}} \leq \|b_m\|_{\mathcal{Q}} + \|b_m - b_n\|_{\mathcal{Q}}$$

holds for all nonnegative integers m and n, the numbers $\|b_n\|$ form a Cauchy sequence. A nonnegative number exists which is the limit of the sequence. If a is an element of $\mathcal{P}$, the element $a + b$ of $\mathcal{H}$ is the limit of the elements $a + b_n$ in the metric topology of $\mathcal{H}$. Since the inequality

$$\|a + b_n\|_{\mathcal{H}}^2 \leq \|a\|_{\mathcal{P}}^2 + \|b_n\|_{\mathcal{Q}}^2$$

holds for every nonnegative integer n, the inequality

$$\|a + b\|_{\mathcal{H}}^2 \leq \|a\|_{\mathcal{P}}^2 + \lim \|b_n\|_{\mathcal{Q}}^2$$

is satisfied. Since the inequality

$$\|b\|_{\mathcal{Q}} \leq \lim \|b_n\|_{\mathcal{Q}}$$

follows, b belongs to $\mathcal{Q}$. Since the inequality

$$\|b - b_m\|_{\mathcal{Q}} \leq \lim \|b_n - b_m\|_{\mathcal{Q}}$$

holds for every nonnegative integer m, b is the limit of the elements b_n in the metric topology of $\mathcal{Q}$.

The inequality

$$\|c\|_{\mathcal{H}}^2 \leq \|a\|_{\mathcal{P}}^2 + \|b\|_{\mathcal{Q}}^2$$

holds by the definition of $\mathcal{Q}$ whenever

$$c = a + b$$

is the sum of an element a of $\mathcal{P}$ and an element b of $\mathcal{Q}$. It will be shown that every element c of $\mathcal{H}$ is the sum of an element a of $\mathcal{P}$ and an element b of $\mathcal{Q}$ for which equality holds. When c is given, a continuous linear functional on $\mathcal{P}$ is defined by taking u into

$$\langle u, c \rangle_{\mathcal{H}}$$

since the inclusion of $\mathcal{P}$ in $\mathcal{H}$ is continuous.

The element a of $\mathcal{P}$ which satisfies the identity

$$\langle u, c \rangle_{\mathcal{H}} = \langle u, a \rangle_{\mathcal{P}}$$

is obtained from c under the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$. It will be shown that the element

$$b = c - a$$

of $\mathcal{H}$ belongs to $\mathcal{Q}$. Since the identity

$$\|u + b\|^2_{\mathcal{H}} = \|c\|^2_{\mathcal{H}} + \langle a, u - a\rangle_{\mathcal{P}} + \langle u - a, a\rangle_{\mathcal{P}} + \|u - a\|^2_{\mathcal{H}}$$

holds for every element u of $\mathcal{P}$ and since the inequality

$$\|u - a\|_{\mathcal{H}} \leq \|u - a\|_{\mathcal{P}}$$

is satisfied, the inequality

$$\|u + b\|^2_{\mathcal{H}} \leq \|c\|^2_{\mathcal{H}} - \|a\|^2_{\mathcal{P}} + \|u\|^2_{\mathcal{P}}$$

holds for every element u of $\mathcal{P}$. Since the inequality

$$\|b\|^2_{\mathcal{Q}} \leq \|c\|^2_{\mathcal{H}} - \|a\|^2_{\mathcal{P}}$$

holds by the definition of $\mathcal{Q}$, b is an element of $\mathcal{Q}$. Equality holds since the reverse inequality holds by the definition of $\mathcal{Q}$.

The intersection of $\mathcal{P}$ and $\mathcal{Q}$ is a Hilbert space $\mathcal{P} \wedge \mathcal{Q}$ with scalar product

$$\langle a, b\rangle_{\mathcal{P}\wedge\mathcal{Q}} = \langle a, b\rangle_{\mathcal{P}} + \langle a, b\rangle_{\mathcal{Q}}.$$

The inclusions of $\mathcal{P} \wedge \mathcal{Q}$ and in $\mathcal{P}$ and in $\mathcal{Q}$ are continuous. If an element

$$c = a + b$$

of $\mathcal{H}$ is the sum of an element a of $\mathcal{P}$ and an element b of $\mathcal{Q}$ such that equality holds in the inequality

$$\|c\|^2_{\mathcal{H}} \leq \|a\|^2_{\mathcal{P}} + \|b\|^2_{\mathcal{Q}},$$

then the inequality

$$\|a\|^2_{\mathcal{P}} + \|b\|^2_{\mathcal{Q}} \leq \|a + u\|^2_{\mathcal{P}} + \|b - u\|^2_{\mathcal{Q}}$$

holds for every element u of the intersection of $\mathcal{P}$ and $\mathcal{Q}$. Since u can be replaced by wu for every complex number w, the identity

$$\langle a, u\rangle_{\mathcal{P}} = \langle b, u\rangle_{\mathcal{Q}}$$

holds for every element u of $\mathcal{P} \wedge \mathcal{Q}$. The identity implies uniqueness of the elements a and b in the minimal decomposition of an element c of $\mathcal{H}$. The element a of $\mathcal{P}$ is obtained from c under the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$. The element b of $\mathcal{Q}$ is obtained from c under the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$. The identity

$$\langle c, c'\rangle_{\mathcal{H}} = \langle a, a'\rangle_{\mathcal{P}} + \langle b, b'\rangle_{\mathcal{Q}}$$

holds whenever an element

$$c' = a' + b'$$

of $\mathcal{H}$ is the sum of an element a' of $\mathcal{P}$ and an element b' of $\mathcal{Q}$.

Uniqueness of the complementary space $\mathcal{Q}$ to $\mathcal{P}$ in $\mathcal{H}$ is a consequence of properties of minimal decompositions. The adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$ is the same for every complementary space. The elements of a complementary space $\mathcal{Q}$ obtained from the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$ are dense in $\mathcal{Q}$ and have scalar self–products in $\mathcal{Q}$ which are independent of the choice of complementary space $\mathcal{Q}$. These properties imply uniqueness of the complementary space to $\mathcal{P}$ in $\mathcal{H}$. The space $\mathcal{P}$ is the complementary space to $\mathcal{Q}$ in $\mathcal{H}$. The Hilbert space

$$\mathcal{H} = \mathcal{P} \vee \mathcal{Q}$$

is said to be the complementary sum of $\mathcal{P}$ and $\mathcal{Q}$. The complementary sum is an orthogonal sum if, and only if, the intersection space $\mathcal{P} \wedge \mathcal{Q}$ of $\mathcal{P}$ and $\mathcal{Q}$ contains no nonzero element.

Complementation is preserved under surjective partially isometric transformations. If Hilbert spaces $\mathcal{P}$ and $\mathcal{Q}$ are contained contractively as complementary subspaces of a Hilbert space $\mathcal{P} \vee \mathcal{Q}$ and if T is a partially isometric transformation of $\mathcal{P} \vee \mathcal{Q}$ onto a Hilbert space $\mathcal{H}$, then T acts as a partially isometric transformation of $\mathcal{P}$ onto a Hilbert space $\mathcal{P}'$ which is contained contractively in $\mathcal{H}$, T acts as a partially isometric transformation of $\mathcal{Q}$ onto a Hilbert space $\mathcal{Q}'$ which is contained contractively in $\mathcal{H}$, and the space $\mathcal{P}'$ and $\mathcal{Q}'$ are complementary subspaces of

$$\mathcal{H} = \mathcal{P}' \vee \mathcal{Q}'.$$

The extension space ext $\mathcal{C}(z)$ of $\mathcal{C}(z)$ is the Hilbert space of square summable Laurent series with vector coefficients. The space $\mathcal{C}(z)$ is contained isometrically in ext $\mathcal{C}(z)$. An isometric transformation of $\mathcal{C}(z)$ onto its orthonormal complement in ext $\mathcal{C}(z)$ is defined by taking $f(z)$ into $z^{-1}f(z^{-1})$.

Multiplication by z and division by z are isometric transformations of ext $\mathcal{C}(z)$ into itself. If a Hilbert space $\mathcal{M}$ is contained contractively in $\mathcal{C}(z)$, multiplication by z is a contractive transformation of $\mathcal{M}$ into itself if, and only if, division by z is a contractive transformation of the complementary space to $\mathcal{M}$ in ext $\mathcal{C}(z)$ into itself.

A Hilbert space $\mathcal{H}$ whose elements are power series with vector coefficients is said to satisfy the inequality for difference quotients if $[f(z)-f(0)]/z$ belongs to the space whenever $f(z)$ belongs to the space and if the inequality

$$\|[f(z) - f(0)]/z\|^2_{\mathcal{H}} \leq \|f(z)\|^2_{\mathcal{H}} - |f(0)|^2$$

is satisfied. A Hilbert space whose elements are power series with vector coefficients is said to satisfy the identity for difference quotients if it satisfies the inequality for difference quotients and if equality always holds.

A Hilbert space of power series with vector coefficients which satisfies the inequality for difference quotients is contained contractively in $\mathcal{C}(z)$. The extension space of a Hilbert space $\mathcal{H}$ which satisfies the inequality for difference quotients is the Hilbert space ext $\mathcal{H}$

which is contained contractively in ext $\mathcal{C}(z)$ such that the complementary space to ext $\mathcal{H}$ in ext $\mathcal{C}(z)$ is isometrically equal to the complementary space to $\mathcal{H}$ in $\mathcal{C}(z)$. Division by z is a contractive transformation of ext $\mathcal{H}$ into itself.

A Hilbert space $\mathcal{H}$ which is contained contractively in $\mathcal{C}(z)$ satisfies the inequality for difference quotients if, and only if, multiplication by z is a contractive transformation of the complementary space to $\mathcal{H}$ in $\mathcal{C}(z)$ into itself.

A theorem which is due to Beurling [1] when the coefficient space is the complex numbers applies to spaces which satisfy the inequality for difference quotients and are contained isometrically in $\mathcal{C}(z)$. An isometric inclusion in $\mathcal{C}(z)$ is unnecessary for the conclusion of the theorem as is the restriction on the coefficient space. A weaker hypothesis is sufficient.

Theorem 1. *A Hilbert space $\mathcal{H}$ of power series with vector coefficients which satisfies the inequality for difference quotients is isometrically equal to the space $\mathcal{H}(W)$ for a power series $W(z)$ with operator coefficients such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself if multiplication by z is an isometric transformation of the complementary space to $\mathcal{H}$ in $\mathcal{C}(z)$ into itself.*

Proof of Theorem 1. The complementary space in $\mathcal{C}(z)$ to a Hilbert space $\mathcal{H}$ which satisfies the inequality for difference quotients is a Hilbert space $\mathcal{M}$ such that multiplication by z is a contractive transformation of $\mathcal{M}$ into itself. If multiplication by z is an isometric transformation of $\mathcal{M}$ into itself, the orthogonal complement of the range of multiplication by z is a Hilbert space $\mathcal{B}$ which is contained isometrically in $\mathcal{M}$ and which is mapped isometrically onto orthogonal subspaces of $\mathcal{M}$ under multiplication by z^m and multiplication by z^n for unequal nonnegative integers m and n. The closed span of these subspaces of $\mathcal{M}$ is the full space.

A power series $W(z)$ with operator coefficients acts as a partially isometric multiplication of $\mathcal{C}$ onto $\mathcal{B}$ if, and only if, it acts as a partially isometric multiplication of $\mathcal{C}(z)$ onto $\mathcal{M}$. If an element of $\mathcal{C}(z)$ belongs to the kernel of multiplication by $W(z)$, then all coefficients of the element belong to the kernel of multiplication by $W(z)$.

Such a power series exists if, and only if, the dimension of $\mathcal{B}$ is less than or equal to the dimension of $\mathcal{C}$. The dimension inequality is satisfied when $\mathcal{C}$ has infinite dimension since the dimension of $\mathcal{C}(z)$ is equal to the dimension of $\mathcal{C}$ and since the dimension of $\mathcal{B}$ is less than or equal to the dimension of $\mathcal{C}(z)$. It remains to verify the dimension inequality when $\mathcal{C}$ has finite dimension r.

Argue by contradiction assuming that $\mathcal{B}$ contains an orthonormal set of $r+1$ elements

$$f_0(z), \ldots, f_r(z).$$

If $c_0, \ldots, c_r$ are corresponding vectors, then the square matrix which has entry

$$c_i^- f_j(z)$$

in the i–th row and j–column has vanishing determinant. Expanding the determinant along the i–th row produces the vanishing power series

$$c_i^- f_0(z) g_0(z) + \ldots + c_i^- f_r(z) g_r(z)$$

with

$$(-1)^k g_k(z)$$

the determinant of the matrix obtained by deleting the i–th row and k–th column of the starting matrix. Since c_i is arbitrary and since each power series $g_k(z)$ with complex coefficients is square summable, the element

$$f_0(z)g_0(z) + \ldots + f_r(z)g_r(z)$$

of $\mathcal{M}$ vanishes identically. The power series

$$g_0(z), \ldots, g_r(z)$$

vanish since the elements

$$z^n f_k(z)$$

of $\mathcal{M}$ form an orthonormal set. An inductive argument shows that the determinants of all square submatrices of the starting matrix vanish. A contradiction is obtained since the starting matrix does not vanish identically for all vectors $c_0, \ldots, c_r$.

This completes the proof of the theorem.

The power series constructed in Theorem 1 has a remarkable property since the kernel of multiplication by $W(z)$ as a transformation of $\mathcal{C}(z)$ into itself contains $[f(z) - f(0)]/z$ whenever it contains $f(z)$. A weaker conclusion is obtained when the power series is constructed under a hypothesis which appears in applications to invariant subspaces.

Theorem 2. *A Hilbert space $\mathcal{H}$ of power series with vector coefficients which satisfies the identity for difference quotients is isometrically equal to a space $\mathcal{H}(W)$ for a power series $W(z)$ with operator coefficients such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself.*

Proof of Theorem 2. As in the proof of Theorem 1 the coefficient space can be assumed of dimension r for some positive integer r. The space $\mathcal{H}$ is contained isometrically in the augmented space $\mathcal{H}'$ since $\mathcal{H}$ satisfies the identity for difference quotients. The orthogonal complement of $\mathcal{H}$ in $\mathcal{H}'$ is a Hilbert space $\mathcal{B}$ which is contained contractively in the complementary space $\mathcal{M}$ to $\mathcal{H}$ in $\mathcal{C}(z)$. Multiplication by z is a contractive transformation of $\mathcal{M}$ into itself which acts as an isometric transformation of $\mathcal{M}$ onto the complementary space to $\mathcal{B}$ in $\mathcal{M}$. The Hilbert space $\mathcal{B}(z)$ of square summable power series is defined with coefficients in $\mathcal{B}$. A partially isometric transformation of $\mathcal{B}(z)$ onto $\mathcal{M}$ is defined by taking a power series

$$\sum b_n z^n$$

with coefficients in $\mathcal{B}$ into the element

$$\sum b_n(z) z^n$$

of $\mathcal{C}(z)$.

Since the space $\mathcal{H}$ satisfies the identity for difference quotients, division by z is an isometric transformation of ext $\mathcal{H}$ into itself. Multiplication by z^n is for every positive integer n an isometric transformation of ext $\mathcal{H}$ onto a Hilbert space which is the orthogonal sum of ext $\mathcal{H}$ and the isometric image of the space of polynomial elements of $\mathcal{B}(z)$ of degree less than n.

It is sufficient to show that the space $\mathcal{B}$ has dimension at most r. Argue by contradiction assuming that $\mathcal{B}$ contains an orthonormal set

$$f_0(z), \dots, f_r(z)$$

of $r+1$ elements. If $c_0, \dots, c_r$ are corresponding vectors, then the square matrix which has entry

$$c_i^- f_j(z)$$

in the i–th row and j–th column has vanishing determinant. Expanding the determinant along the i–th row and j–th column produces the vanishing power series

$$c_i^- f_0(z) g_0(z) + \dots + c_i^- f_r(z) g_r(z)$$

with

$$(-1)^k g_k(z)$$

the determinant of the matrix obtained by deleting the i–th row and k–th column. Since the c_i are arbitrary and since each power series $g_k(z)$ with complex coefficients is square summable, the element

$$f_0 g_0(z) + \dots + f_r g_r(z)$$

of $\mathcal{B}(z)$ vanishes. An inductive argument shows that the determinants of all square submatrices of the starting matrix vanish. A contradiction is obtained since the starting matrix does not vanish identically for all vectors $c_0, \dots, c_r$.

This completes the proof of the theorem.

Linear systems are a mechanism for the construction of invariant subspaces of transformations by factorization of analytic functions called transfer functions of linear systems. For transformations which take a Hilbert space contractively into itself, the state space and external spaces of the linear system are Hilbert spaces.

The linear system is a square matrix whose four entries are continuous linear transformations. The matrix acts on a Hilbert space which is the Cartesian product of the state space and the external space. The elements of the Cartesian product are realized as column vectors with upper entry in the state space and lower entry in the external space.

The upper left entry of the matrix is the main transformation, which takes the state space into itself. The upper right entry is the input transformation, which takes the external space into the state space. The lower left entry is the output transformation, which takes the state space into the external space. The lower right entry is the external operator, which takes the external space into itself.

The matrix of the linear system is assumed to have an isometric adjoint. A canonical model of the linear system is constructed in a Hilbert space of power series with vector coefficients for a coefficient space which is isometrically equal to the external space.

When the linear system has matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with isometric adjoint, a power series

$$\sum a_n z^n$$

with vector coefficients is defined by

$$a_n = CA^n f$$

for every element f of the state space. The set of elements of the state space for which the power series vanishes is a closed invariant subspace for the main transformation in which its restriction has an isometric adjoint. Invariant subspaces of transformations with isometric adjoint are constructed without the use the factorization. The subspace is assumed to contain no nonzero element for the construction of the canonical model.

A canonical linear system whose matrix has isometric adjoint is assumed to have as state space a Hilbert space whose elements are power series with vector coefficients. The power series

$$f(z) = \sum a_n z^n$$

associated with an element of the space is assumed to be constructed by iteration of the main transformation and the action of the output transformation. The external operator then takes a power series $f(z)$ into its constant coefficient $f(0)$. The main transformation takes $f(z)$ into

$$[f(z) - f(0)]/z.$$

The state space $\mathcal{H}$ of the linear system is a Hilbert space of power series with vector coefficients whose structure is determined by the isometric property of the adjoint matrix. The augmented space $\mathcal{H}'$ is the Hilbert space of power series $f(z)$ with vector coefficients such that $[f(z) - f(0)]/z$ belongs to $\mathcal{H}$ with scalar product determined by the identity

$$\|[f(z) - f(0)]/z\|^2_{\mathcal{H}} = \|f(z)\|^2_{\mathcal{H}'} - |f(0)|^2.$$

The coefficient space $\mathcal{C}$ is contained isometrically in the argumented space $\mathcal{H}'$. Multiplication by z is an isometric transformation of the space $\mathcal{H}$ onto the orthogonal complement of $\mathcal{C}$ in $\mathcal{H}'$.

The matrix of the linear system is realized as a transformation of the augmented space $\mathcal{H}'$ into itself. The matrix takes an element $f(z)$ of $\mathcal{H}'$ with constant coefficient zero into

$$f(z)/z.$$

The matrix takes an element c of the coefficient space into

$$W(z)c$$

for a power series $W(z)$ with operator coefficients which defines the transfer function of the linear system. The matrix takes an element $f(z)$ of $\mathcal{H}'$ into

$$[f(z) - f(0)]/z + W(z)f(0).$$

Since the matrix has an isometric adjoint, it is a partially isometric transformation of the augmented Hilbert space onto itself. If $f(z)$ is an element of $\mathcal{H}$ and if c is a vector, then

$$g(z) = f(z) + W(z)c$$

is an element of $\mathcal{H}'$. The inequality

$$\|g(z)\|^2_{\mathcal{H}'} \leq \|f(z)\|^2_{\mathcal{H}} + |c|^2$$

is satisfied. Every element $g(z)$ of $\mathcal{H}'$ admits a representation for which equality hold. The space $\mathcal{H}$ satisfies the inequality for difference quotients.

The space $\mathcal{H}$ is contained contractively in the space $\mathcal{H}'$. Multiplication by $W(z)$ is a partially isometric transformation of $\mathcal{C}$ onto the complementary space to $\mathcal{H}$ in $\mathcal{H}'$. Multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. Multiplication by $W(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Hilbert space which is contained contractively in $\mathcal{C}(z)$. The space $\mathcal{H}$ is isometrically equal to the complementary space $\mathcal{H}(W)$ in $\mathcal{C}(z)$ of the range of multiplication by $W(z)$ as it acts on $\mathcal{C}(z)$.

If $W(z)$ is a power series with operator coefficients such that multiplicative by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, then multiplication by $W(z)$ has a unique continuous extension as a contractive transformation of ext $\mathcal{C}(z)$ into itself which commutes with multiplication by z. The conjugate power series

$$W^*(z) = W_0^- + W_1^- z + W_2^- z^2 + \ldots$$

is defined with operator coefficients which are adjoints of the operator coefficients of the power series

$$W(z) = W_0 + W_1 z + W_2 z^2 + \ldots$$

Multiplication by $W^*(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself which has a contractive extension as a transformation of ext $\mathcal{C}(z)$ into itself. Multiplication by $W^*(z^{-1})$ is the contractive transformation of ext $\mathcal{C}(z)$ into itself which takes $f(z)$ into $g(z)$ whenever multiplication by $W^*(z)$ takes $f(z^{-1})$ into $g(z^{-1})$. The adjoint of multiplication by $W(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself is multiplication by $W^*(z^{-1})$. The adjoint of multiplication by $W(z)$ as a transformation of $\mathcal{C}(z)$ into itself takes an element $f(z)$ of $\mathcal{C}(z)$ into the power series which has the same coefficient of z^n as

$$W^*(z^{-1})f(z)$$

for every nonnegative integer n.

A construction of invariant subspaces is due to David Hilbert for contractive transformations of a Hilbert space into itself whose adjoint is isometric. A canonical model of such transformations appears in an expository treatment of the construction by Gustav Herglotz [9].

A Herglotz space is a Hilbert space of power series with vector coefficients such that a contractive transformation of the space into itself with isometric adjoint is defined by taking $f(z)$ into $[f(z)-f(0)]/z$ and such that a continuous transformation of the space into the coefficient space is defined by taking a power series $f(z)$ into its constant coefficient $f(0)$.

A transformation T with domain and range in a Hilbert space $\mathcal{H}$ is said to be dissipative if

$$\langle Tf, f\rangle_{\mathcal{H}} + \langle f, Tf\rangle_{\mathcal{H}}$$

is nonnegative for every element f of the domain of T. The transformation is said to be maximal dissipative if every element of the Hilbert space is a sum

$$f + Tf$$

with f in the domain of T.

If $\phi(z)$ is a power series with operator coefficients such that a maximal dissipative transformation in $\mathcal{C}(z)$ is defined by taking $f(z)$ into

$$\phi(z)f(z)$$

whenever $f(z)$ and $\phi(z)f(z)$ belong to $\mathcal{C}(z)$, then a power series

$$W(z) = [1 - \phi(z)]/[1 + \phi(z)]$$

with operator coefficients is defined such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. The Herglotz space $\mathcal{L}(\phi)$ is a Hilbert space of power series with vector coefficients such that an isometric transformation of the space onto the space $\mathcal{H}(W)$ is defined by taking $f(z)$ into

$$[1 + W(z)]f(z).$$

A contractive transformation of the space $\mathcal{L}(\phi)$ into itself which has an isometric adjoint is defined by taking $f(z)$ into $[f(z) - f(0)]/z$. A continuous transformation of the space $\mathcal{L}(\phi)$ into the coefficient space is defined by taking a power series $f(z)$ into its constant coefficient $f(0)$. The space $\mathcal{L}(\phi)$ is a Herglotz space. A Herglotz space is isometrically equal to a space $\mathcal{L}(\phi)$ for some Herglotz function $\phi(z)$. The Herglotz spaces associated with two Herglotz functions are isometrically equal if, and only if, the Herglotz functions differ by a constant which is a skew–conjugate operator.

The extension space $\mathcal{E}(\phi)$ of a Herglotz space $\mathcal{L}(\phi)$ is a Hilbert space whose elements are Laurent series with vector coefficient such that multiplication by z is an isometric transformation of the space onto itself and such that a partially isometric transformation of $\mathcal{E}(\phi)$ onto $\mathcal{L}(\phi)$ is defined by taking a Laurent series into the power series which has the same coefficient of z^n for every nonnegative integer n.

The Herglotz space $\mathcal{L}(\phi)$ is isometrically equal to $\mathcal{C}(z)$ when $\phi(z)$ is the constant one. The extension space of the space $\mathcal{L}(\phi)$ is then isometrically equal to ext $\mathcal{C}(z)$. A Hilbert space whose elements are Laurent series is isometrically equal to the extension space of a Herglotz space if multiplication by z is an isometric transformation of the space onto itself and if a continuous transformation of the space into the coefficient space is defined by taking a Laurent series into its constant coefficient.

A convex structure applies to the Hilbert spaces which are contained contractively in a given Hilbert space. If Hilbert spaces $\mathcal{P}$ and $\mathcal{Q}$ are contained contractively in the given Hilbert space, and if t is in the interval $[0,1]$, a unique Hilbert space

$$\mathcal{H} = (1-t)\mathcal{P} + t\mathcal{Q}$$

exists, which is contained contractively in the given Hilbert space, such that the convex combination

$$c = (1-t)a + tb$$

belongs to $\mathcal{H}$ whenever a belongs to $\mathcal{P}$ and b belongs to $\mathcal{Q}$, such that the inequality

$$\|c\|_{\mathcal{H}}^2 \leq (1-t)\|a\|_{\mathcal{P}}^2 + t\|b\|_{\mathcal{Q}}^2$$

is satisfied, and such that every element c of $\mathcal{H}$ is a convex combination for which equality holds.

If $\mathcal{E}(\phi)$ and $\mathcal{E}(\psi)$ are extension spaces of Herglotz spaces $\mathcal{L}(\phi)$ and $\mathcal{L}(\psi)$, then a Herglotz space

$$\mathcal{L}(\phi+\psi) = \mathcal{L}(\phi) \vee \mathcal{L}(\psi)$$

exists in which the Herglotz spaces $\mathcal{L}(\phi)$ and $\mathcal{L}(\psi)$ are contained contractively as complementary spaces. The extension spaces $\mathcal{E}(\phi)$ and $\mathcal{E}(\psi)$ are contained contractively as complementary space in the extension space

$$\mathcal{E}(\phi+\psi) = \mathcal{E}(\phi) \vee \mathcal{E}(\psi).$$

A Herglotz space $\mathcal{L}(\theta)$ exists such that the extension space

$$\mathcal{E}(\theta) = \mathcal{E}(\phi) \wedge \mathcal{E}(\psi)$$

is the intersection spaces of the extension spaces $\mathcal{E}(\phi)$ and $\mathcal{E}(\psi)$.

A Herglotz space $\mathcal{L}(\phi-\theta)$ exists such that the extension space $\mathcal{E}(\phi-\theta)$ is the complementary space in the extension space $\mathcal{E}(\phi)$ of the extension space $\mathcal{E}(\phi)$. A Herglotz space $\mathcal{L}(\psi-\theta)$ exists such that the extension space $\mathcal{E}(\psi-\theta)$ is the complementary space in the extension space $\mathcal{E}(\psi)$ of the extension space $\mathcal{E}(\theta)$. A Herglotz space $\mathcal{L}(\phi+\theta)$ exists such that the extension space $\mathcal{E}(\phi+\theta)$ is the complementary space in the extension space $\mathcal{E}(\phi+\psi)$ of the extension space $\mathcal{E}(\psi-\theta)$. A Herglotz space $\mathcal{L}(\psi+\theta)$ exists such that the extension space $\mathcal{E}(\psi+\phi)$ is the complementary space in the extension space $\mathcal{E}(\phi+\psi)$ of the extension space $\mathcal{E}(\phi-\theta)$. The extension space

$$\mathcal{E}(\phi) = (1-t)\mathcal{E}(\phi+\theta) + t\mathcal{E}(\psi-\theta)$$

is a convex combination of the extension spaces $\mathcal{E}(\phi+\theta)$ and $\mathcal{E}(\phi-\theta)$ with t and $1-t$ equal. The extension space

$$\mathcal{E}(\psi) = (1-t)\mathcal{E}(\psi+\theta) + t\mathcal{E}(\psi-\theta)$$

is a convex combination of the extension spaces $\mathcal{E}(\psi+\theta)$ and $\mathcal{E}(\psi-\theta)$ with t and $1-t$ equal.

The Herglotz space $\mathcal{L}(\theta)$ is contained contractively in the Herglotz spaces $\mathcal{L}(\phi)$ and $\mathcal{L}(\psi)$. The Herglotz space $\mathcal{L}(\phi-\theta)$ is the complementary space in the Herglotz space $\mathcal{L}(\phi)$ of the Herglotz space $\mathcal{L}(\theta)$. The Herglotz space $\mathcal{L}(\psi-\theta)$ is the complementary space in the Herglotz space $\mathcal{L}(\psi)$ of the Herglotz space $\mathcal{L}(\theta)$. The Herglotz space $\mathcal{L}(\phi+\theta)$ is the complementary space in the Herglotz space $\mathcal{L}(\phi+\psi)$ of the Herglotz space $\mathcal{L}(\psi-\theta)$. The Herglotz space $\mathcal{L}(\psi+\theta)$ is the complementary space in the Herglotz space $\mathcal{L}(\phi+\psi)$ of the Herglotz space $\mathcal{L}(\phi-\theta)$. The Herglotz space

$$\mathcal{L}(\phi) = (1-t)\mathcal{L}(\phi+\theta) + t\mathcal{L}(\phi-\theta)$$

is a convex combination of the Herglotz spaces $\mathcal{L}(\phi+\theta)$ and $\mathcal{L}(\phi-\theta)$ with t and $1-t$ equal. The Herglotz space

$$\mathcal{L}(\psi) = (1-t)\mathcal{L}(\psi+\theta) + t\mathcal{L}(\psi-\theta)$$

is a convex combination of the Herglotz spaces $\mathcal{L}(\psi+\theta)$ and $\mathcal{L}(\psi-\theta)$ with t and $1-t$ equal.

An extreme point of a convex set is an element of the set which is not a convex combination

$$(1-t)a + tb$$

of distinct elements of the set with t and $1-t$ positive. An element of the convex set of Herglotz spaces which are contained contractively in a given Herglotz space $\mathcal{L}(\psi)$ is an extreme point if, and only if, it is a Herglotz space whose extension space is contained isometrically in the extension space $\mathcal{E}(\psi)$.

A convex combination of Hilbert spaces of power series with vector coefficients which satisfy the inequality for difference quotients is a Hilbert space of power series with vector coefficients which satisfies the inequality for difference quotients. A Hilbert space of power series with vector coefficients is said to satisfy the identity for difference quotients if it satisfies the inequality for difference quotients and if equality always holds.

A Hilbert space $\mathcal{H}$ of power series with vector coefficients which satisfies the identity for difference quotients is an extreme point of the convex set of Hilbert spaces of power series with vector coefficients which satisfy the inequality for difference quotients. If

$$\mathcal{H} = (1-t)\mathcal{P} + t\mathcal{Q}$$

is a convex combination with t and $1-t$ positive of Hilbert space $\mathcal{P}$ and $\mathcal{Q}$ of power series with vector coefficients which satisfy the inequality for difference quotients, then every element

$$h(z) = (1-t)f(z) + tg(z)$$

of $\mathcal{H}$ is a convex combination of elements $f(z)$ of $\mathcal{P}$ and $g(z)$ of $\mathcal{Q}$ such that equality holds in the inequality

$$\|h(z)\|_{\mathcal{H}}^2 \leq (1-t)\|f(z)\|_{\mathcal{P}}^2 + t\|g(z)\|_{\mathcal{Q}}^2.$$

Since the inequality

$$\|[h(z)-h(0)]/z\|_{\mathcal{H}}^2 \leq (1-t)\|[f(z)-f(0)]/z\|_{\mathcal{P}}^2 + t\|[g(z)-g(0)]/z\|_{\mathcal{Q}}^2$$

holds with

$$\|[f(z)-f(0)]/z\|_{\mathcal{P}}^2 \leq \|f(z)\|_{\mathcal{P}}^2 - |f(0)|^2$$

and

$$\|[g(z)-g(0)]/z\|_{\mathcal{Q}}^2 \leq \|g(z)\|_{\mathcal{Q}}^2 - |g(0)|^2,$$

the inequality

$$\|[h(z)-h(0)]/z\|_{\mathcal{H}}^2 \leq \|h(z)\|_{\mathcal{H}}^2 - (1-t)|f(0)|^2 - t|g(0)|^2$$

is satisfied. The inequality reads

$$\|[h(z)-h(0)]/z\|_{\mathcal{H}}^2 \leq \|h(z)\|_{\mathcal{H}}^2 - h(0)|^2 - t(1-t)|g(0)-f(0)|^2$$

since the convexity identity

$$|(1-t)f(0)+tg(0)|^2 + t(1-t)|g(0)-f(0)|^2 = (1-t)|f(0)|^2 + t|g(0)|^2$$

is satisfied. Since the space $\mathcal{H}$ satisfies the identity for difference quotients, the constant coefficients in $f(z), g(z)$, and $h(z)$ are equal. An inductive argument shows that the n–th coefficients of $f(z), g(z)$, and $h(z)$ are equal for every nonnegative integer n. Since $f(z), g(z)$, and $h(z)$ are always equal, the spaces $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{H}$ are isometrically equal.

If a Hilbert space $\mathcal{H}$ of power series with vector coefficients satisfies the inequality for difference quotients, the augmented space is the Hilbert space $\mathcal{H}'$ of power series $f(z)$ with vector coefficients such that $[f(z)-f(0)]/z$ belongs to $\mathcal{H}$ with scalar product determined by the identity

$$\|[f(z)-f(0)]/z\|_{\mathcal{H}}^2 = \|f(z)\|_{\mathcal{H}'}^2 - |f(0)|^2.$$

The space $\mathcal{H}$ is contained contractively in the space $\mathcal{H}'$. The complementary space $\mathcal{B}$ to $\mathcal{H}$ in $\mathcal{H}'$ is a Hilbert space which is contained contractively in the complementary space $\mathcal{M}$ to $\mathcal{H}$ in $\mathcal{C}(z)$. Multiplication by z is an isometric transformation of $\mathcal{M}$ onto the complementary space to $\mathcal{B}$ in $\mathcal{M}$.

If $W(z)$ is a power series with operator coefficients such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, then multiplication by $W(z)$ admits a unique continuous extension as a contractive transformation of ext $\mathcal{C}(z)$ into itself which commutes with multiplication by z.

The conjugate power series

$$W^*(z) = W_0^- + W_1^- z + W_2^- z^2 + \dots$$

is defined with operator coefficients which are adjoints of the operator coefficients of

$$W(z) = W_0 + W_1 z + W_2 z^2 + \dots$$

Multiplication by $W^*(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. The adjoint of multiplication by $W(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself takes $f(z)$ into $g(z)$ when multiplication by $W^*(z)$ takes $z^{-1}f(z^{-1})$ into $z^{-1}g(z^{-1})$. The adjoint of multiplication by $W(z)$ as a transformation of $\mathcal{C}(z)$ into itself takes an element $f(z)$ of $\mathcal{C}(z)$ into the element $g(z)$ of $\mathcal{C}(z)$ which has the same coefficient of z^n for every nonnegative integer n as the element of ext $\mathcal{C}(z)$ obtained from $f(z)$ under the adjoint of multiplication by $W(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself.

If $U(z)$ and $V(z)$ are power series with operator coefficients such that multiplication by $U(z)$ and multiplication by $V(z)$ are contractive transformations of $\mathcal{C}(z)$ into itself, then multiplication by

$$W(z) = U(z)V(z)$$

is a continuous transformation of $\mathcal{C}(z)$ into itself. The space $\mathcal{H}(U)$ is contained contractively in the space $\mathcal{H}(W)$. Multiplication by $U(z)$ is a partially isometric transformation of the space $\mathcal{H}(V)$ onto the complementary space to the space $\mathcal{H}(U)$ in the space $\mathcal{H}(W)$.

A factorization is derived from a contractive inclusion.

Theorem 3. *If $U(z)$ and $W(z)$ are power series with operator coefficients such that multiplication by $U(z)$ and multiplication by $W(z)$ are contractive transformations of $\mathcal{C}(z)$ into itself and if the space $\mathcal{H}(U)$ is contained contractively in the space $\mathcal{H}(W)$, then*

$$W(z) = U(z)V(z)$$

for a power series $V(z)$ with operator coefficients such that multiplication by $V(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself and such that the range of multiplication by $V(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself is orthogonal to the kernel of multiplication by $U(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself.

Proof of Theorem 3. A Hilbert space $\mathcal{C}_r(z)$, which is contained isometrically in ext $\mathcal{C}(z)$, is defined for every nonnegative integer r as the set of elements $f(z)$ such that $z^r f(z)$ belongs to $\mathcal{C}(z)$. Multiplication by $U(z)$ and multiplication by $W(z)$ are contractive transformations of $\mathcal{C}_r(z)$ into itself. The adjoint transformations of $\mathcal{C}_r(z)$ into itself are compositions of the adjoints of multiplication by $U(z)$ and of multiplication by $W(z)$ as transformations of ext $\mathcal{C}(z)$ into itself with the orthogonal projection of ext $\mathcal{C}(z)$ onto $\mathcal{C}_r(z)$.

A space $\mathcal{H}(U_r)$ is defined with

$$U_r(z) = z^r U(z).$$

The set of elements $f(z)$ of $\mathcal{C}_r(z)$ such that $z^r f(z)$ belongs to the space $\mathcal{H}(U_r)$ is a Hilbert space $\mathcal{H}_r(U)$ which is mapped isometrically onto the space $\mathcal{H}(U_r)$ on multiplication by z^r. The space $\mathcal{H}_r(U)$ is contained contractively in $\mathcal{C}_r(z)$. Multiplication $U(z)$ is a partially isometric transformation of $\mathcal{C}_r(z)$ onto the complementary space to the space $\mathcal{H}_r(U)$ in $\mathcal{C}_r(z)$. If $h(z)$ is an element of $\mathcal{C}_r(z)$, a unique element $f(z)$ of $\mathcal{C}(z)$ exists such that

$$h(z) - U(z)f(z)$$

belongs to the space $\mathcal{H}_r(U)$ and such that equality holds in the inequality

$$\|h(z)\|^2 \leq \|h(z) - U(z)f(z)\|^2_{\mathcal{H}_r(U)} + \|f(z)\|^2.$$

The element $f(z)$ of $\mathcal{C}_r(z)$ is obtained from $h(z)$ under the adjoint of multiplication by $U(z)$ as a transformation of $\mathcal{C}(z)$ into itself.

A space $\mathcal{H}(W_r)$ is defined with

$$W_r(z) = z^r W(z).$$

The set of elements $f(z)$ of $\mathcal{C}_r(z)$ such that $z^r f(z)$ belongs to the space $\mathcal{H}(W_r)$ is a Hilbert space $\mathcal{H}_r(W)$ which is mapped isometrically onto the space $\mathcal{H}(W_r)$ on multiplication by z^r. The space $\mathcal{H}_r(W)$ is contained contractively in $\mathcal{C}_r(z)$. Multiplication by $W(z)$ is a partially isometric transformation of $\mathcal{C}_r(z)$ onto the complementary space to the space $\mathcal{H}_r(W)$ in $\mathcal{C}_r(z)$. If $h(z)$ is an element of $\mathcal{C}_r(z)$, a unique element $g(z)$ of $\mathcal{C}_r(z)$ exists such that

$$h(z) - W(z)g(z)$$

belongs to the space $\mathcal{H}_r(W)$ and such that equality holds in the inequality

$$\|h(z)\|^2 \leq \|h(z) - W(z)g(z)\|^2_{\mathcal{H}_r(W)} + \|g(z)\|^2.$$

The element $g(z)$ of $\mathcal{C}_r(z)$ is obtained from $h(z)$ under the adjoint of multiplication by $W(z)$ as a transformation of $\mathcal{C}_r(z)$ into itself.

The space $\mathcal{H}_r(U)$ is contained contractively in the space $\mathcal{H}_r(W)$ since the space $\mathcal{H}(U_r)$ is contained contractively in the space $\mathcal{H}(W_r)$. Since the element

$$h(z) - U(z)f(z)$$

of the space $\mathcal{H}_r(U)$ is obtained from $h(z)$ under the adjoint of the inclusion of the space in $\mathcal{C}_r(z)$ and since the element

$$h(z) - W(z)g(z)$$

of the space $\mathcal{H}_r(W)$ is obtained from $h(z)$ under the adjoint of the inclusion of the space in $\mathcal{C}_r(z)$, the element

$$h(z) - U(z)f(z)$$

of the space $\mathcal{H}_r(U)$ is obtained from the element

$$h(z) - W(z)g(z)$$

of the space $\mathcal{H}_r(W)$ under the adjoint of the inclusion of the space $\mathcal{H}_r(U)$ in the space $\mathcal{H}_r(W)$. The inequality

$$\|g(z)\| \leq \|f(z)\|$$

follows from the inequality

$$\|h(z) - U(z)f(z)\|_{\mathcal{H}_r(U)} \leq \|h(z) - W(z)g(z)\|_{\mathcal{H}_r(W)}.$$

If the adjoint of multiplication by $U(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself takes $h(z)$ into $f(z)$ and if the coefficient of z^n in $h(z)$ vanishes when n is less than $-r$, then the coefficient of z^n in $f(z)$ vanishes when n is less than $-r$. If the adjoint of multiplication by $W(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself takes $h(z)$ into $g(z)$ and if the coefficient of z^n in $h(z)$ vanishes when n is less than $-r$, then the coefficient of z^n in $g(z)$ vanishes when n is less than $-r$.

If $h(z)$ is an element of ext $\mathcal{C}(z)$, a unique element $h_r(z)$ of $\mathcal{C}_r(z)$ exists such that the coefficient of z^n in

$$h(z) - h_r(z)$$

vanishes when n is not less than $-r$. If the adjoint of multiplication by $U(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself takes $h(z)$ into $f(z)$, an element $f_r(z)$ of $\mathcal{C}_r(z)$ exists such that the coefficient of z^n in

$$f(z) - f_r(z)$$

vanishes when n is not less than $-r$. If the adjoint of multiplication by $W(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself takes $h(z)$ into $g(z)$, an element $g_r(z)$ of $\mathcal{C}_r(z)$ exists such that the coefficient of z^n in

$$g(z) - g_r(z)$$

vanishes when n is not less than $-r$. The element

$$h_r(z) - U(z)f_r(z)$$

of $\mathcal{C}_r(z)$ belongs to the space $\mathcal{H}_r(U)$ and equality holds in the inequality

$$\|h_r(z)\|^2 \leq \|h_r(z) - U(z)f_r(z)\|^2_{\mathcal{H}_r(U)} + \|f_r(z)\|^2.$$

The element

$$h_r(z) - W(z)g_r(z)$$

of $\mathcal{C}_r(z)$ belongs to the space $\mathcal{H}_r(W)$ and equality holds in the inequality

$$\|h_r(z)\|^2 \leq \|h_r(z) - W(z)g_r(z)\|^2_{\mathcal{H}_r(W)} + \|g_r(z)\|^2.$$

The inequality

$$\|g(z)\| \leq \|f(z)\|$$

holds since the inequality

$$\|g_r(z)\| \leq \|f_r(z)\|$$

holds for every nonnegative integer r.

The transformation which takes $f(z)$ into $g(z)$ admits a unique continuous extension as a contractive transformation of ext $\mathcal{C}(z)$ into itself which annihilates the kernel of multiplication by $U(z)$. The transformation commutes with multiplication by z and takes elements of ext $\mathcal{C}(z)$ whose coefficient of z^n vanishes when n is not less than $-r$ into elements of ext $\mathcal{C}(z)$ whose coefficient of z^n vanishes when n is not less than $-r$.

A unique power series $V(z)$ with operator coefficients exists such that multiplication by $V(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself and such that the adjoint of multiplication by $V(z)$ takes $f(z)$ into $g(z)$ for every element $f(z)$ of ext $\mathcal{C}(z)$. The identity

$$W(z) = U(z)V(z)$$

is satisfied since

$$W^*(z) = V^*(z)U^*(z).$$

This completes the proof of the theorem.

If $W(z)$ is a power series with operator coefficiently such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, then multiplication by $W(z)$ is an isometric transformation of $\mathcal{C}(z)$ into itself when, and only when,

$$W^*(z)W(z^{-1}) = 1.$$

Multiplication by $W^*(z)$ is an isometric transformation of $\mathcal{C}(z)$ into itself when, and only when,

$$W(z^{-1})W^*(z) = 1.$$

If $W(z)$ is a power series with operator coefficients such that multiplication by $W(z)$ and multiplication by $W^*(z)$ are isometric transformations of $\mathcal{C}(z)$ into itself, then an isometric transformation of ext $\mathcal{C}(z)$ onto itself is defined by taking $f(z)$ into

$$W(z)z^{-1}f(z^{-1}).$$

The inverse transformation takes $f(z)$ into

$$W^*(z)z^{-1}f(z^{-1}).$$

An isometric transformation of the space $\mathcal{H}(W)$ onto the space $\mathcal{H}(W^*)$ is defined by taking $f(z)$ into

$$W^*(z)z^{-1}f(z^{-1}).$$

The inverse transformation takes $f(z)$ into

$$W(z)z^{-1}f(z^{-1}).$$

The adjoint of the transformation of the space $\mathcal{H}(W)$ into itself which takes $f(z)$ into $[f(z) - f(0)]/z$ is unitarily equivalent to the transformation of the space $\mathcal{H}(W^*)$ into itself which takes $f(z)$ into $[f(z)-f(0)]/z$. The adjoint of the transformation of the space $\mathcal{H}(W^*)$ into itself which takes $f(z)$ into $[f(z)-f(0)]/z$ is unitarily equivalent to the transformation of the space $\mathcal{H}(W)$ into itself which takes $f(z)$ into $[f(z) - f(0)]/z$.

When $W(z)$ is a power series with operator coefficients such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, the isometric transformation of the space $\mathcal{H}(W)$ onto the space $\mathcal{H}(W^*)$ is replaced by a relation whose graph is a Hilbert space $\mathcal{D}(W)$ of pairs $(f(z), g(z))$ of elements $f(z)$ of the space $\mathcal{H}(W)$ and $g(z)$ of the space $\mathcal{H}(W^*)$.

Theorem 4. *If $W(z)$ is a power series with operator coefficients such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, then a unique Hilbert space $\mathcal{D}(W)$ exists whose elements are pairs $(f(z), g(z))$ of power series $f(z)$ and $g(z)$ with vector coefficients such that*

$$([f(z) - f(0)]/z,\ zg(z) - W^*(z)f(0))$$

and

$$(zf(z) - W(z)g(0), [g(z) - g(0)]/z)$$

belong to the space whenever $(f(z), g(z))$ belongs to the space and such that the identities

$$\|([f(z) - f(0)]/z, zg(z) - W^*(z)f(0))\|^2_{\mathcal{D}(W)} = \|(f(z), g(z))\|^2_{\mathcal{D}(W)} - |f(0)|^2$$

and

$$\|(zf(z) - W(z)g(0),\ [g(z) - g(0)]/z)\|^2_{\mathcal{D}(W)} = \|(f(z), g(z))\|^2_{\mathcal{D}(W)} - |g(0)|^2$$

are satisfied. A partially isometric transformation of the space $\mathcal{D}(W)$ onto the space $\mathcal{H}(W)$ is defined by taking $(f(z), g(z))$ into $f(z)$. A partially isometric transformation of the space $\mathcal{D}(W)$ onto the space $\mathcal{H}(W^*)$ is defined by taking $(f(z), g(z))$ into $g(z)$. An isometric transformation of the space $\mathcal{D}(W)$ onto the space $\mathcal{D}(W^*)$ is defined by taking $(f(z), g(z))$ into $(g(z), g(z))$.

Proof of Theorem 4. The construction of the space $\mathcal{D}(W)$ applies properties of the space $\mathcal{H}(W_r)$ defined by

$$W_r(z) = z^r W(z)$$

for every nonnegative integer r. Multiplication by z^r is an isometric transformation of the space $\mathcal{H}(W)$ onto the space $\mathcal{H}(W_r)$. The orthogonal complement of the image of the space $\mathcal{H}(W)$ in the space $\mathcal{H}(W_r)$ is contained isometrically in $\mathcal{C}(z)$ and consists of the elements of $\mathcal{C}(z)$ which are polynomials of degree less than r. The space $\mathcal{H}(W)$ is contained contractively in the space $\mathcal{H}(W_r)$. Multiplication by $W(z)$ is a partially isometric transformation of the Hilbert space of polynomial elements of $\mathcal{C}(z)$ of degree less than r onto the complementary space to the space $\mathcal{H}(W)$ in the space $\mathcal{H}(W_r)$.

A Hilbert space $\mathcal{D}_r(W)$ is defined as the set of pairs $(f(z), g(z))$ of elements $f(z)$ of the space $\mathcal{H}(W)$ and polynomial elements

$$g(z) = a_0 + a_1 z + \ldots + a_{r-1} z^{r-1}$$

of $\mathcal{C}(z)$ of degree less than r such that the element

$$z^r f(z) - W(z)(a_0 z^{r-1} + \ldots + a_{r-1})$$

of the space $\mathcal{H}(W_r)$ belongs to the space $\mathcal{H}(W)$. The scalar product in the space $\mathcal{D}(W)$ is determined by the identity

$$\begin{aligned} &\|z^r f(z) - W(z)(a_0 z^{r-1} + \ldots + a_{r-1})\|^2_{\mathcal{H}(W)} \\ &= \|(f(z), g(z))\|^2_{\mathcal{D}_r(W)} - |a_0|^2 - \ldots - |a_{r-1}|^2. \end{aligned}$$

If $f(z)$ is an element of the space $\mathcal{H}(W)$, a unique polynomial element $g(z)$ of $\mathcal{C}(z)$ of degree less than r exists such that

$$\|f(z)\|_{\mathcal{H}(W)} = \|(f(z), g(z))\|_{\mathcal{D}_r(W)}.$$

A partially isometric transformation of the space $\mathcal{D}_{r+1}(W)$ onto the space $\mathcal{D}_r(W)$ is defined by taking $(f(z), g(z))$ into $(f(z), h(z))$ with $h(z)$ the polynomial of degree less than r whose coefficient of z^n is equal to the coefficient of z^n in $g(z)$ when n is less than r.

The pair

$$(zf(z) - W(z)g(0),\ [g(z) - g(0)]/z)$$

belongs to the space $\mathcal{D}_r(W)$ whenever $(f(z), g(z))$ belongs to the space $\mathcal{D}_{r+1}(W)$ and the identity

$$\begin{aligned} &\|(zf(z) - W(z)g(0),\ [g(z) - g(0)]/z)\|^2_{\mathcal{D}_r(W)} \\ &= \|(f(z), g(z))\|^2_{\mathcal{D}_{r+1}(W)} - |g(0)|^2 \end{aligned}$$

is satisfied.

The adjoint of the transformation of the space $\mathcal{H}(W)$ into $\mathcal{C}$ which takes

$$f(z) = \sum a_n z^n$$

into

$$a_r$$

for a nonnegative integer r is the transformation which takes a vector c into the element

$$z^r c - W(z)(W_0^- z^r + \ldots + W_r^-)c$$

of the space $\mathcal{H}(W)$.

The adjoint of the transformation of the space $\mathcal{D}_{r+1}(W)$ into $\mathcal{C}$ which takes $(f(z), g(z))$ into $f(0)$ is the transformation which takes a vector c into the element

$$([1 - W(z)W(0)^-]c,\ (W_1^- + W_2^- z + \ldots + W_r^- z^r)c)$$

of the space $\mathcal{D}_{r+1}(W)$.

The space $\mathcal{D}(W)$ is defined as the set of pairs $(f(z), g(z))$ of elements $f(z)$ of the space $\mathcal{H}(W)$ and power series $g(z)$ with vector coefficients such that an element $(f(z), g_r(z))$ of the space $\mathcal{D}_r(W)$ is defined for every nonnegative integer r with $g_r(z)$ the polynomial of degree less than r whose coefficient of z^n is equal to the coefficient of z^n in $g(z)$ when n is less than r, and such that the sequence of numbers

$$\|(f(z), g_r(z))\|^2_{\mathcal{D}_r(W)}$$

is bounded. A limit exists since the sequence is nondecreasing. The scalar product of the space $\mathcal{D}(W)$ is determined by the requirement that the limit is equal to

$$\|(f(z), g(z))\|^2_{\mathcal{D}(W)}.$$

A contractive transformation of the space $\mathcal{D}(W)$ into itself is defined by taking $(f(z), g(z))$ into

$$(zf(z) - W(z)g(0),\ [g(z) - g(0)]/z).$$

The identity

$$\|(zf(z) - W(z)g(0),\ [g(z) - g(0)]/z)\|^2_{\mathcal{D}(W)} = \|(f(z), g(z))\|^2_{\mathcal{D}(W)} - |g(0)|^2.$$

The adjoint of the transformation of the space $\mathcal{D}(W)$ into $\mathcal{C}$ which takes $f(z), g(z))$ into $f(0)$ is the transformation which takes a vector c into the element

$$([1 - W(z)W(0)^-]c,\ [W^*(z) - W(0)^-]c/z)$$

of the space $\mathcal{D}(W)$.

The adjoint of the transformation of the space $\mathcal{D}(W)$ into itself which takes $(f(z), g(z))$ into

$$(zf(z) - W(z)g(0),\ [g(z) - g(0)]/z)$$

is the transformation which takes $(f(z), g(z))$ into

$$([f(z) - f(0)]/z, zg(z) - W^*(z)f(0)).$$

The identity

$$\|([f(z) - f(0)]/z, zg(z) - W^*(z)f(0))\|^2_{\mathcal{D}(W)} = \|(f(z), g(z))\|^2_{\mathcal{D}(W)} - |f(0)|^2$$

is satisfied.

Assume that a Hilbert space $\mathcal{D}$ whose elements are pairs $(f(z), g(z))$ of power series $f(z)$ and $g(z)$ with vector coefficients has the required properties: The pairs

$$([f(z) - f(0)]/z, \ zg(z) - W^*(z)f(0))$$

and

$$([zf(z) - W(z)g(0), \ [g(z) - g(0)]/z)$$

belong to the space whenever $(f(z), g(z))$ belongs to the space and the identities

$$\|([f(z) - f(0)]/z, zg(z) - W^*(z)f(0))\|^2_{\mathcal{D}} = \|(f(z), g(z))\|^2_{\mathcal{D}} - |f(0)|^2$$

and

$$\|(zf(z) - W(z)g(0), [g(z) - g(0)]/z)\|^2_{\mathcal{D}} = \|(f(z), g(z))\|^2_{\mathcal{D}} - |g(0)|^2$$

are satisfied.

A partially isometric transformation of $\mathcal{D}$ onto a Hilbert space $\mathcal{H}$ of power series with vector coefficients which satisfies the inequality for difference quotients is defined by taking $(f(z), g(z))$ into $f(z)$. The space $\mathcal{H}$ is isometrically equal to the space $\mathcal{H}(W)$ since it is the state space of a canonical linear system which is conjugate isometric and has transfer function $W(z)$. An inductive argument constructs a partially isometric transformation of the space $\mathcal{D}$ onto the space $\mathcal{D}_r(W)$ for every nonnegative integer r. The transformation takes $(f(z), g(z))$ into $(f(z), g_r(z))$ with $g_r(z)$ the polynomial of degree less than r whose coefficient of z^n is equal to the coefficient of z^n in $g(z)$ when n is less than r. The space $\mathcal{D}$ is isometrically equal to the space $\mathcal{D}(W)$ by the construction of the space $\mathcal{D}(W)$.

The isometric transformation of the space $\mathcal{D}(W)$ onto the space $\mathcal{D}(W^*)$ which takes $(f(z), g(z))$ into $(g(z), f(z))$ is a consequence of the characterization of the spaces.

This completes the proof of the theorem.

The relationship between spaces $\mathcal{H}(W)$ and $\mathcal{H}(W^*)$ is well behaved in factorization. Assume that $U(z)$ and $V(z)$ are power series with operator coefficients such that multiplication by $U(z)$ and multiplication by $V(z)$ are contractive transformations of $\mathcal{C}(z)$ into itself. Then

$$W(z) = U(z)V(z)$$

is a power series with operator coefficients such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. A partially isometric transformation of the space $\mathcal{D}(U)$

onto a Hilbert space which is contained contractively in the space $\mathcal{D}(W)$ is defined by taking $(f(z), g(z))$ into

$$(f(z), V^*(z)g(z)).$$

A partially isometric transformation of the space $\mathcal{D}(V)$ onto a Hilbert space which is contained contractively in the space $\mathcal{D}(W)$ is defined by taking $(f(z), g(z))$ into

$$(U(z)f(z), g(z)).$$

The image of the space $\mathcal{D}(U)$ and the image of the space $\mathcal{D}(V)$ are complementary subspaces of the space $\mathcal{D}(W)$.

A Herglotz space is associated with a power series $W(z)$ with operator coefficients such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. Since the adjoint of multiplication by $W(z)$ as a transformation of $\mathcal{C}(z)$ into itself takes $[f(z)-f(0)]/z$ into $[g(z)-g(0)]/z$ whenever it takes $f(z)$ into $g(z)$, the adjoint acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\phi)$ which is contained contractively in $\mathcal{C}(z)$. The adjoint of multiplication by $W(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of ext $\mathcal{C}(z)$ onto ext $\mathcal{L}(\phi)$. The complementary space to the space $\mathcal{L}(\phi)$ in $\mathcal{C}(z)$ is the Herglotz space $\mathcal{L}(1-\phi)$. The complementary space to ext $\mathcal{L}(\phi)$ in ext $\mathcal{C}(z)$ is ext $\mathcal{L}(1-\phi)$.

An element $f(z)$ of $\mathcal{C}(z)$ belongs to the Herglotz space $\mathcal{L}(1-\phi)$ if, and only if, $W(z)f(z)$ belongs to the space $\mathcal{H}(W)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\phi)} = \|f(z)\|^2 + \|W(z)f(z)\|^2_{\mathcal{H}(W)}$$

is satisfied. An element

$$f(z) + z^{-1}g(z^{-1})$$

of ext $\mathcal{C}(z)$ belongs to ext $\mathcal{L}(1-\phi)$ if, and only if, $f(z)$ and $g(z)$ are elements of $\mathcal{C}(z)$ such that

$$(W(z)f(z), -g(z))$$

belongs to the space $\mathcal{D}(W)$. The identity

$$\|f(z) + z^{-1}g(z^{-1})\|^2_{\text{ext } \mathcal{L}(1-\phi)} = \|f(z) + z^{-1}g(z^{-1})\|^2 + \|(W(z)f(z), -g(z))\|^2_{\mathcal{D}(W)}$$

is satisfied.

An element $f(z)$ of $\mathcal{C}(z)$ belongs to the space $\mathcal{H}(W)$ if, and only if, the adjoint of multiplication by $W(z)$ as a transformation of $\mathcal{C}(z)$ into itself takes $f(z)$ into an element $h(z)$ of the Herglotz space $\mathcal{L}(1-\phi)$. The identity

$$\|f(z)\|^2_{\mathcal{H}(W)} = \|f(z)\|^2 + \|h(z)\|^2_{\mathcal{L}(1-\phi)}$$

is satisfied.

If $U(z), V(z)$, and

$$W(z) = U(z)V(z)$$

are power series with operator coefficients such that multiplication by $U(z), V(z)$, and $W(z)$ are contractive transformations of $\mathcal{C}(z)$ into itself and if the adjoint of multiplication by $U(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\phi)$, the adjoint of multiplication by $V(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\theta)$, and the adjoint of multiplication by $W(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\psi)$, then the space $\mathcal{L}(\psi)$ is contained contractively in the space $\mathcal{L}(\theta)$ and the adjoint of multiplication by $V(z)$ acts as a partially isometric transformation of the space $\mathcal{L}(\phi)$ onto the complementary space to the space $\mathcal{L}(\psi)$ in the space $\mathcal{L}(\theta)$. The space $\mathcal{L}(1-\theta)$ is contained contractively in the space $\mathcal{L}(1-\psi)$. The adjoint of multiplication by $V(z)$ acts as a partially isometric transformation of the space $\mathcal{L}(1-\phi)$ onto the complementary space to the space $\mathcal{L}(1-\theta)$ in the space $\mathcal{L}(1-\psi)$.

A canonical factorization of a power series $W(z)$ with operator coefficients applies when multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself but the space $\mathcal{H}(W)$ does not satisfy the identity for difference quotients.

Theorem 5. *If multiplication by a power series $W(z)$ with operator coefficients is a contractive transformation of $\mathcal{C}(z)$ into itself, then*

$$W(z) = U(z)V(z)$$

for power series $U(z)$ and $V(z)$ with operator coefficients such that multiplication by $U(z)$ and multiplication by $V(z)$ are contractive transformations of $\mathcal{C}(z)$ into itself, such that the space $\mathcal{H}(U)$ is contained isometrically in the space $\mathcal{H}(W)$ and satisfies the identity for difference quotients, such that multiplication by $U(z)$ is a partially isometric transformation of the space $\mathcal{H}(V)$ onto the orthogonal complement of the space $\mathcal{H}(U)$ in the space $\mathcal{H}(W)$ whose kernel contains $[f(z)-f(0)]/z$ whenever it contains $f(z)$, and such that the orthogonal complement of the kernel in the space $\mathcal{H}(V)$ is the closure of the set of products $V(z)f(z)$ with $f(z)$ a polynomial element of the space $\mathcal{L}(1-\theta)$.

Proof of Theorem 5. The adjoint of multiplication by $W(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\psi)$ which is contained contractively in $\mathcal{C}(z)$. The complementary space to the space $\mathcal{L}(\psi)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\psi)$ whose elements are the elements $f(z)$ of $\mathcal{C}(z)$ such that $W(z)f(z)$ belongs to the space $\mathcal{H}(W)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\psi)} = \|f(z)\|^2 + \|W(z)f(z)\|^2_{\mathcal{H}(W)}$$

is satisfied.

A Herglotz space $\mathcal{L}(\eta)$ is defined which is contained contractively in $\mathcal{C}(z)$ and whose complementary space in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\eta)$ which is contained isometrically in the space $\mathcal{L}(1-\psi)$ and is the closure of the polynomial elements of the space. The space $\mathcal{L}(\psi)$ is the closure of its polynomial elements and is contained contractively in the space $\mathcal{L}(\eta)$. A Herglotz space $\mathcal{L}(\theta)$ is defined which is contained isometrically in the space $\mathcal{L}(\eta)$ and is the closure of the polynomial elements of the space. The space $\mathcal{L}(\psi)$ is contained contractively in the space $\mathcal{L}(\theta)$.

The orthogonal complement of the space $\mathcal{L}(\theta)$ in the space $\mathcal{L}(\eta)$ is a Herglotz space $\mathcal{L}(\eta-\theta)$ which contains no nonzero polynomial. The space $\mathcal{L}(1-\eta)$ is contained contractively in the space $\mathcal{L}(1-\theta)$. Since the space $\mathcal{L}(\eta-\theta)$ is contained isometrically in the space $\mathcal{L}(\eta)$ and is contained contractively in the space $\mathcal{L}(\eta-\psi)$ which is contained contractively in the space $\mathcal{L}(\eta)$, the space $\mathcal{L}(\eta-\theta)$ is contained isometrically in the space $\mathcal{L}(\eta-\psi)$. The orthogonal complement of the space $\mathcal{L}(\eta-\theta)$ in the space $\mathcal{L}(\eta-\psi)$ is the Herglotz space $\mathcal{L}(\theta-\psi)$.

Since the space $\mathcal{L}(1-\eta)$ is contained isometrically in the space $\mathcal{L}(1-\psi)$, the space $\mathcal{L}(\eta-\psi)$ is contained isometrically in the space $\mathcal{L}(1-\psi)$. The space $\mathcal{L}(\eta-\theta)$ is contained isometrically in the space $\mathcal{L}(1-\psi)$ since it is contained isometrically in the space $\mathcal{L}(\eta-\psi)$. Since the orthogonal complement of the space $\mathcal{L}(\eta-\theta)$ in the space $\mathcal{L}(1-\theta)$ is the space $\mathcal{L}(1-\eta)$ which is contained isometrically in the space $\mathcal{L}(1-\theta)$, the space $\mathcal{L}(1-\theta)$ is contained isometrically in the space $\mathcal{L}(1-\psi)$.

Since the space $\mathcal{L}(\theta)$ is contained contractively in $\mathcal{C}(z)$ and is the closure of its polynomial elements, a power series $V(z)$ with operator coefficients exists such that multiplication by $V(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that the adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the space $\mathcal{L}(\theta)$, and such that the kernel of the transformation contains $zf(z)$ whenever it contains $f(z)$. The complementary space to the space $\mathcal{L}(\theta)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\theta)$ whose elements are the elements $f(z)$ of $\mathcal{C}(z)$ such that $V(z)f(z)$ belongs to the space $\mathcal{H}(V)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\theta)} = \|f(z)\|^2 + \|V(z)f(z)\|^2_{\mathcal{H}(V)}.$$

The kernel of the adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself is contained isometrically in the space $\mathcal{H}(V)$ and is the orthogonal complement of the image of the space $\mathcal{L}(1-\theta)$ in the space $\mathcal{H}(V)$. A partially isometric transformation of the space $\mathcal{H}(V)$ into the space $\mathcal{H}(W)$ exists which takes $V(z)f(z)$ into $W(z)f(z)$ for every element $f(z)$ of the space $\mathcal{L}(1-\theta)$ and whose kernel is the kernel of the adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself.

An element

$$f(z)+z^{-1}g(z^{-1})$$

of ext $\mathcal{C}(z)$ belongs to ext $\mathcal{L}(1-\psi)$ if, and only if, $f(z)$ and $g(z)$ are elements of $\mathcal{C}(z)$ such that $(W(z)f(z),-g(z))$ belongs to the space $\mathcal{D}(W)$. The identity

$$\|f(z)+z^{-1}g(z^{-1})\|^2_{\text{ext }\mathcal{L}(1-\psi)} = \|f(z)+z^{-1}g(z^{-1})\|^2 + \|(W(z)f(z),-g(z))\|^2_{\mathcal{D}(W)}$$

is satisfied.

An element

$$f(z)+z^{-1}g(z^{-1})$$

of ext $\mathcal{C}(z)$ belongs to ext $\mathcal{L}(1-\theta)$ if, and only if, $f(z)$ and $g(z)$ are elements of $\mathcal{C}(z)$ such that $(V(z)f(z),-g(z))$ belongs to the space $\mathcal{D}(V)$. The identity

$$\|f(z)+z^{-1}g(z^{-1})\|^2_{\text{ext }\mathcal{L}(1-\theta)} = \|f(z)+z^{-1}g(z^{-1})\|^2 + \|(V(z)f(z),-g(z))\|^2_{\mathcal{D}(V)}$$

is satisfied. An isometric transformation of the kernel of the adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself into the space $\mathcal{D}(V)$ is defined by taking $f(z)$ into $(f(z), 0)$. The image of the kernel is the orthogonal complement in the space $\mathcal{D}(V)$ of the image of ext $\mathcal{L}(1-\theta)$.

Since the space ext $\mathcal{L}(1-\theta)$ is contained isometrically in the space ext $\mathcal{L}(1-\psi)$, a partially isometric transformation of the space $\mathcal{D}(V)$ into the space $\mathcal{D}(W)$ exists which takes $(V(z)f(z), -g(z))$ into $(W(z)f(z), -g(z))$ whenever $f(z)$ and $g(z)$ are elements of $\mathcal{C}(z)$ such that

$$f(z) + z^{-1}g(z^{-1})$$

belongs to the space ext $\mathcal{L}(1-\theta)$ and whose kernel is the set of elements $(f(z), 0)$ with $f(z)$ in the kernel of the adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself.

The space $\mathcal{H}(V^*)$ is continued isometrically in the space $\mathcal{H}(W^*)$. A power series $U(z)$ with operator coefficients exists by Theorem 3 such that multiplication by $U(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that

$$W(z) = U(z)V(z),$$

and such that the range of multiplication by $V(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself is orthogonal to the kernel of multiplication by $U(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself.

Since the kernel of multiplication by $V^*(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself is orthogonal to the range of multiplication by $U^*(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself, the kernel of the adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself is orthogonal to the range of multiplication by $U^*(z)$ as a transformation of $\mathcal{C}(z)$ into itself and is contained isometrically in the space $\mathcal{H}(U^*)$. Multiplication by $U(z)$ annihilates the kernel of the adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself. The space $\mathcal{H}(U)$ is contained isometrically in the space $\mathcal{H}(W)$ since multiplication by $U(z)$ is a partially isometric transformation of the space $\mathcal{H}(V)$ into the space $\mathcal{H}(W)$.

The adjoint of multiplication by $U(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\phi)$ which is contained contractively in $\mathcal{C}(z)$. The complementary space to the space $\mathcal{L}(\phi)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\phi)$ whose elements are the elements $f(z)$ of $\mathcal{C}(z)$ such that $U(z)f(z)$ belongs to the space $\mathcal{H}(U)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\phi)} = \|f(z)\|^2 + \|U(z)f(z)\|^2_{\mathcal{H}(U)}$$

is satisfied.

The adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of the space $\mathcal{L}(1-\phi)$ onto the orthogonal complement of the space $\mathcal{L}(1-\theta)$ in the space $\mathcal{L}(1-\psi)$. The kernel of the adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself is contained isometrically in the space

$\mathcal{L}(1-\phi)$. Since the orthogonal complement of the kernel is mapped isometrically onto the orthogonal complement of the space $\mathcal{L}(1-\theta)$ in the space $\mathcal{L}(1-\psi)$, since the transformation takes $[f(z)-f(0)]/z$ into $[g(z)-g(0)]/z$ whenever it takes $f(z)$ into $g(z)$, and since the orthogonal complement of the space $\mathcal{L}(1-\theta)$ in the space $\mathcal{L}(1-\psi)$ contains no nonzero polynomial, the polynomial elements of the space $\mathcal{L}(1-\phi)$ belong to the kernel of the adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself. Since the kernel is contained isometrically in the space $\mathcal{L}(1-\phi)$, the space $\mathcal{H}(U)$ satisfies the identity for difference quotients.

This completes the proof of the theorem.

An application of commutant lifting [6] is made to the structure of a contractive transformation of a space $\mathcal{H}(A)$ into a space $\mathcal{H}(B)$ which takes $[f(z)-f(0)]/z$ into $[g(z)-g(0)]/z$ whenever it takes $f(z)$ into $g(z)$.

Theorem 6. *If a contractive transformation T of a space $\mathcal{H}(A)$ which satisfies the identity for difference quotients into a space $\mathcal{H}(B)$ takes $[f(z)-f(0)]/z$ into $[g(z)-g(0)]/z$ whenever it takes $f(z)$ into $g(z)$ for power series $A(z)$ and $B(z)$ with operator coefficients such that multiplication by $A(z)$ and multiplication by $B(z)$ are contractive transformations of $\mathcal{C}(z)$ into itself, then a contractive transformation T' of the augmented space $\mathcal{H}(A')$,*

$$A'(z) = zA(z)$$

into the augmented space $\mathcal{H}(B')$,

$$B'(z) = zB(z),$$

exists which extends T and takes $[f(z)-f(0)]/z$ into $[g(z)-g(0)]/z$ whenever it takes $f(z)$ into $g(z)$.

Proof of Theorem 6. The construction of T' is an application of complementation. The coefficient space $\mathcal{C}$ is contained isometrically in the spaces $\mathcal{H}(A')$ and $\mathcal{H}(B')$. Multiplication by z is an isometric transformation of the space $\mathcal{H}(A)$ onto the orthogonal complement of $\mathcal{C}$ in the space $\mathcal{H}(A')$ and of the space $\mathcal{H}(B)$ onto the orthogonal complement of $\mathcal{C}$ in $\mathcal{H}(B')$. The space $\mathcal{H}(A)$ is contained isometrically in the space $\mathcal{H}(A')$ since the space $\mathcal{H}(A)$ satisfies the identity for difference quotients. The space $\mathcal{H}(B)$ is contained contractively in the space $\mathcal{H}(B')$.

Since T' is defined to agree with T on the space $\mathcal{H}(A)$, it remains to define T' on the orthogonal complement of the space $\mathcal{H}(A)$ in the space $\mathcal{H}(A')$. The action of T' is determined within a constant since the transformation is required to take $[f(z)-f(0)]/z$ into $[g(z)-g(0)]/z$ whenever it takes $f(z)$ into $g(z)$. A contractive transformation S is defined of the space $\mathcal{H}(A')$ onto the orthogonal complement of $\mathcal{C}$ in the space $\mathcal{H}(B')$. The transformation S is defined on the space $\mathcal{H}(A)$ to take $f(z)$ into $g(z)-g(0)$ when T takes $f(z)$ into $g(z)$. The transformation S is defined on the orthogonal complement of $\mathcal{C}$ to take $f(z)$ into $g(z)$ with $g(0)$ equal to zero when T takes $[f(z)-f(0)]/z$ into $[g(z)-g(0)]/z$. The definition of S on $\mathcal{H}(A)$ is consistent with the definition of S on the orthogonal complement of $\mathcal{C}$ since T takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$.

The transformation S acts as a partially isometric transformation of the space $\mathcal{H}(A)$ onto a Hilbert space $\mathcal{P}$ which is contained contractively in the orthogonal complement of $\mathcal{C}$ in the space $\mathcal{H}(B')$. The transformation acts as a partially isometric transformation of the space $\mathcal{H}(A')$ onto a Hilbert space $\mathcal{M}$ which is contained contractively in the orthogonal complement of $\mathcal{C}$ in the space $\mathcal{H}(B')$. The space $\mathcal{P}$ is contained contractively in $\mathcal{M}$. The transformation acts as a partially isometric transformation of the orthogonal complement of the space $\mathcal{H}(A)$ in the space $\mathcal{H}(A')$ onto the complementary space $\mathcal{Q}$ to $\mathcal{P}$ in $\mathcal{M}$.

A Hilbert space $\mathcal{M}'$ is defined as the set of elements $f(z)$ of the space $\mathcal{H}(B')$ such that $f(z) - f(0)$ belongs to $\mathcal{M}$ with scalar product determined by the identity

$$\|f(z)\|^2_{\mathcal{M}'} = \|f(z) - f(0)\|^2_{\mathcal{M}} + |f(0)|^2.$$

Since T acts as partially isometric transformation of the space $\mathcal{H}(A)$ onto a Hilbert space $\mathcal{P}'$ which is contained contractively in the space $\mathcal{H}(B')$ and since S takes $f(z)$ into $g(z) - g(0)$ whenever T takes $f(z)$ into $g(z)$, a partially isometric transformation of $\mathcal{P}'$ onto $\mathcal{P}$ is defined by taking $g(z)$ into $g(z) - g(0)$. The space $\mathcal{P}'$ is contained contractively in $\mathcal{M}'$. The complementary space $\mathcal{Q}'$ to $\mathcal{P}'$ in $\mathcal{M}'$ is a Hilbert space which is contained contractively in $\mathcal{M}'$. A partially isometric transformation of $\mathcal{Q}'$ into $\mathcal{Q}$ is defined by taking $g(z)$ into $g(z) - g(0)$. The transformation T' is defined on the orthogonal complement of the space $\mathcal{H}(A)$ in the space $\mathcal{H}(A')$ to take $f(z)$ into $g(z)$ when $g(z)$ is the element of $\mathcal{Q}'$ of least norm such that S takes $f(z)$ into $g(z) - g(0)$.

This completes the proof of the theorem.

If $A(z)$ and $B(z)$ are power series with operator coefficients such that multiplication by $A(z)$ and multiplication by $B(z)$ are contractive transformations of $\mathcal{C}(z)$ into itself and if

$$A(z)C(z) = D(z)B(z)$$

for power series $C(z)$ and $D(z)$ with operator coefficients such that multiplication by $C(z)$ and multiplication by $D(z)$ are contractive transformations of $\mathcal{C}(z)$ into itself, then the space $\mathcal{H}(A)$ is contained contractively in the space $\mathcal{H}(DB)$ and a contractive transformation T of the space $\mathcal{H}(A)$ into the space $\mathcal{H}(B)$ which takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$ is defined by taking $f(z)$ into $g(z)$ when

$$f(z) - D(z)g(z)$$

belongs to the space $\mathcal{H}(D)$ and equality holds in the inequality

$$\|f(z)\|^2_{\mathcal{H}(DB)} \leq \|f(z) - D(z)g(z)\|^2_{\mathcal{H}(D)} + \|g(z)\|^2_{\mathcal{H}(B)}.$$

Every contractive transformation T of the space $\mathcal{H}(A)$ into the space $\mathcal{H}(B)$ which takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$ is obtained when the space $\mathcal{H}(A)$ satisfies the identity for difference quotients. The construction of the power series $D(z)$ is an application of commutant lifting.

Spaces $\mathcal{H}(A_r)$ and $\mathcal{H}(B_r)$ are defined by

$$A_r(z) = z^r A(z)$$

and

$$B_r(z) = z^r B(z)$$

for every nonnegative integer r. The spaces $\mathcal{H}(A_r)$ satisfy the identity for difference quotients since the space $\mathcal{H}(A)$ satisfies the identity for difference quotients. A contractive transformation T_r of the space $\mathcal{H}(A_r)$ into the space $\mathcal{H}(B_r)$ which takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$ is defined inductively for nonnegative integers r. The transformation T_0 of the space $\mathcal{H}(A_0)$ into the space $\mathcal{H}(B_0)$ is the transformation T of the space $\mathcal{H}(A)$ into the space $\mathcal{H}(B)$. When T_r is defined with the required properties, the transformation T_{r+1} is defined with the required properties by Theorem 5 so as to extend the transformation T_r.

A transformation of the union of the spaces $\mathcal{H}(A_r)$ into the union of the spaces $\mathcal{H}(B_r)$ is defined with agrees with T_r on the space $\mathcal{H}(A_r)$ for every nonnegative integer r. The space of polynomial elements of $\mathcal{C}(z)$ of degree less than r is contained isometrically in the space $\mathcal{H}(A_r)$ and in the space $\mathcal{H}(B_r)$ for every nonnegative integer r. The transformation T_r of the space $\mathcal{H}(A_r)$ into the space $\mathcal{H}(B_r)$ takes polynomials of degree less than r into polynomials of degree less than r since it takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$. Since the polynomial elements of $\mathcal{C}(z)$ are dense in $\mathcal{C}(z)$, the transformation has a unique continuous extension as a contractive transformation of $\mathcal{C}(z)$ into itself which takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$. The transformation is the adjoint of multiplication by $D(z)$ as a transformation of $\mathcal{C}(z)$ into itself for a power series $D(z)$ with operator coefficients such that multiplication by $D(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself.

The transformation of the space $\mathcal{H}(A)$ into the space $\mathcal{H}(B)$ determines a factorization by the choice of $D(z)$. The identity for difference quotients is not required for the space $\mathcal{H}(A)$.

Theorem 7. *If $A(z), B(z)$, and $D(z)$ are power series with operator coefficients such that multiplication by $A(z)$, multiplication by $B(z)$, and multiplication by $D(z)$ are contractive transformations of $\mathcal{C}(z)$ into itself and if the adjoint of multiplication by $D(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a contractive transformation of the space $\mathcal{H}(A)$ into the space $\mathcal{H}(B)$, then a power series $C(z)$ with operator coefficients exists such that multiplication by $C(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that*

$$A(z)C(z) = D(z)B(z),$$

and such that the range of multiplication by $C(z)$ as a transformation of $\operatorname{ext} \mathcal{C}(z)$ into itself is orthogonal to the kernel of multiplication by $A(z)$ as a transformation of $\operatorname{ext} \mathcal{C}(z)$ into itself.

Proof of Theorem 7. Since the adjoint of multiplication by $D(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a contractive transformation of the space $\mathcal{H}(A)$ into the space $\mathcal{H}(B)$, the adjoint of multiplication by $D(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself acts as a contractive transformation of the space ext $\mathcal{H}(A)$ into the space ext $\mathcal{H}(B)$. Multiplication by $D(z)$ acts as a contractive transformation of the complementary space to the space $\mathcal{H}(B)$ in $\mathcal{C}(z)$ into the complementary space to the space $\mathcal{H}(A)$ in $\mathcal{C}(z)$. Since multiplication by $D(z)$ acts as a partially isometric transformation of the complementary space to the space $\mathcal{H}(B)$ in $\mathcal{C}(z)$ onto the complementary space to the space $\mathcal{H}(DB)$ in $\mathcal{C}(z)$, the complementary space to the space $\mathcal{H}(DB)$ in $\mathcal{C}(z)$ is contained contractively in the complementary space to the space $\mathcal{H}(A)$ in $\mathcal{C}(z)$. The space $\mathcal{H}(A)$ is contained contractively in the space $\mathcal{H}(DB)$.

A power series $C(z)$ with operator coefficients exists by Theorem 3 such that multiplication by $C(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that

$$A(z)C(z) = D(z)B(z),$$

and such that the range of multiplication by $C(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself is orthogonal to the kernel of multiplication by $A(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself.

This completes the proof of the theorem.

A power series $W(z)$ with operator coefficients such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself admits by Theorem 5 a canonical factorization

$$W(z) = U(z)V(z)$$

into power series $U(z)$ and $V(z)$ with operator coefficients such that multiplication by $U(z)$ and multiplication by $V(z)$ are contractive transformations of $\mathcal{C}(z)$ into itself: The space $\mathcal{H}(U)$ is contained isometrically in the space $\mathcal{H}(W)$ and satisfies the identity for difference quotients. The set of elements of the space $\mathcal{H}(V)$ which are products $V(z)f(z)$ for a polynomial element of $\mathcal{C}(z)$ is dense in the orthogonal complement of the set of elements of the space which belong to the kernel of multiplication by $U(z)$.

The factorization is compatible with contractive transformations of the space $\mathcal{H}(W)$ into itself which take $[f(z)-f(0)]/z$ into $[g(z)-g(0)]/z$ whenever they take $f(z)$ into $g(z)$.

Theorem 8. *If multiplication by a power series $W(z)$ with operator coefficients is a contractive transformation of $\mathcal{C}(z)$ into itself and if in the canonical factorization*

$$W(z) = U(z)V(z)$$

the adjoint of multiplication by $P(z)$ acts as a contractive transformation of the space $\mathcal{H}(W)$ into itself, then the adjoint of multiplication by $W(z)$ acts as a contractive transformation of the space $\mathcal{H}(U)$ into itself.

Proof of Theorem 8. The adjoint of multiplication by $W(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\psi)$

which is contained contractively in $\mathcal{C}(z)$. The complementary space to the space $\mathcal{L}(\psi)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\psi)$ whose elements are the elements $f(z)$ of $\mathcal{C}(z)$ such that $W(z)f(z)$ belongs to the space $\mathcal{H}(W)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\psi)} = \|f(z)\|^2 + \|W(z)f(z)\|^2_{\mathcal{H}(W)}$$

is satisfied.

A power series $Q(z)$ with operator coefficients exists by Theorem 7 such that multiplication by $Q(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that

$$W(z)Q(z) = P(z)W(z),$$

and such that the range of multiplication by $Q(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself is orthogonal to the kernel of multiplication by $W(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself.

An element $f(z)$ of $\mathcal{C}(z)$ belongs to the space $\mathcal{H}(W)$ if, and only if, the adjoint of multiplication by $W(z)$ takes $f(z)$ into an element $g(z)$ of the space $\mathcal{L}(1-\psi)$. The identity

$$\|f(z)\|^2_{\mathcal{H}(W)} = \|f(z)\|^2 + \|g(z)\|^2_{\mathcal{L}(1-\psi)}$$

is satisfied. The adjoint of multiplication by $Q(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a contractive transformation of the space $\mathcal{L}(1-\psi)$ into itself.

The adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\theta)$ which is contained contractively in $\mathcal{C}(z)$. The complementary space to the space $\mathcal{L}(\theta)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\theta)$ whose elements are the elements $f(z)$ of $\mathcal{C}(z)$ such that $V(z)f(z)$ belongs to the space $\mathcal{H}(V)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\theta)} = \|f(z)\|^2 + \|V(z)f(z)\|^2_{\mathcal{H}(V)}$$

is satisfied.

The space $\mathcal{L}(1-\theta)$ is contained isometrically in the space $\mathcal{L}(1-\psi)$. The closure of the polynomial elements of the space $\mathcal{L}(1-\psi)$ is a Herglotz space $\mathcal{L}(1-\eta)$ which is contained isometrically in the space $\mathcal{L}(1-\theta)$. Since the space $\mathcal{L}(1-\eta)$ is contained contractively in $\mathcal{C}(z)$, the complementary space to the space $\mathcal{L}(1-\eta)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(\eta)$ which is contained contractively in $\mathcal{C}(z)$. The space $\mathcal{L}(\theta)$ is contained isometrically in the space $\mathcal{L}(\eta)$ and is the closure of the polynomial elements of the space.

The adjoint of multiplication by $Q(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a contractive transformation of the space $\mathcal{L}(1-\psi)$ into itself since

$$Q^*(z)W^*(z) = W^*(z)P^*(z).$$

The transformation takes polynomial elements of the space $\mathcal{L}(1-\psi)$ into polynomial elements of the space since it takes $[f(z)-f(0)]/z$ into $[g(z)-g(0)]/z$ whenever it takes

$f(z)$ into $g(z)$. Since the space $\mathcal{L}(1-\eta)$ is contained isometrically in the space $\mathcal{L}(1-\psi)$ and is the closure of the polynomial elements of the space, the transformation acts as a contractive transformation of the space $\mathcal{L}(1-\eta)$ into itself.

The adjoint of multiplication by $Q(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself acts as a contractive transformation of ext $\mathcal{L}(1-\eta)$ into itself which takes $zf(z)$ into $zg(z)$ whenever it takes $f(z)$ into $g(z)$. Since multiplication by z is an isometric transformation of ext $\mathcal{L}(1-\eta)$ onto itself, multiplication by $Q(z)$ is a contractive transformation of ext $\mathcal{L}(1-\eta)$ into itself. The adjoint of multiplication by $Q(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself acts as a contractive transformation of ext $\mathcal{L}(\eta)$ into itself since ext $\mathcal{L}(\eta)$ is the complementary space to ext $\mathcal{L}(1-\eta)$ in ext $\mathcal{C}(z)$. The adjoint of multiplication by $Q(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a contractive transformation of the space $\mathcal{L}(\eta)$ into itself.

Since the space $\mathcal{L}(\theta)$ is contained isometrically in the space $\mathcal{L}(\eta)$ and is the closure of the polynomial elements of the space, the adjoint of multiplication by $Q(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a contractive transformation of the space $\mathcal{L}(\theta)$ into itself and of the space $\mathcal{L}(1-\theta)$ into itself. Since the adjoint of multiplication by $Q(z)$ acts as a contractive transformation of the space $\mathcal{L}(1-\psi)$ into itself, it acts as a contractive transformation of the orthogonal complement $\mathcal{L}(\theta-\psi)$ of the space $\mathcal{L}(1-\theta)$ in the space $\mathcal{L}(1-\psi)$ into itself.

The adjoint of multiplication by $U(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\phi)$ which is contained contractively in $\mathcal{C}(z)$. The complementary space to the space $\mathcal{L}(\phi)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\phi)$ whose elements are the elements $f(z)$ of $\mathcal{C}(z)$ such that $U(z)f(z)$ belongs to the space $\mathcal{H}(U)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\phi)} = \|f(z)\|^2 + \|U(z)f(z)\|^2_{\mathcal{H}(U)}$$

is satisfied. An element $f(z)$ of $\mathcal{C}(z)$ belongs to the space $\mathcal{H}(U)$ if, and only if, the adjoint of multiplication by $U(z)$ takes $f(z)$ into an element $g(z)$ of the space $\mathcal{L}(1-\phi)$. The identity

$$\|f(z)\|^2_{\mathcal{H}(U)} = \|f(z)\|^2 + \|g(z)\|^2_{\mathcal{L}(1-\phi)}$$

is satisfied.

The adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of the space $\mathcal{L}(1-\phi)$ onto the space $\mathcal{L}(\theta-\psi)$ whose kernel is the adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself. The adjoint of multiplication by $W(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of the space $\mathcal{L}(1-\phi)$ onto the space $\mathcal{L}(\theta-\psi)$ since

$$W^*(z) = V^*(z)U^*(z).$$

The kernel of the transformation is the kernel of the adjoint of multiplication by $U(z)$ as a transformation of $\mathcal{C}(z)$ into itself.

An element $f(z)$ of $\mathcal{C}(z)$ belongs to the space $\mathcal{H}(U)$ if, and only if, the adjoint of multiplication by $W(z)$ takes $f(z)$ into an element $g(z)$ of the space $\mathcal{L}(\theta-\psi)$. The identity

$$\|f(z)\|^2_{\mathcal{H}(U)} = \|f(z)\|^2 + \|g(z)\|^2_{\mathcal{L}(\theta-\psi)}$$

is satisfied. The adjoint of multiplication by $P(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a contractive transformation of the space $\mathcal{H}(U)$ into itself since the adjoint of multiplication by $Q(z)$ as a transformation of the space $\mathcal{L}(\theta - \psi)$ into itself and since

$$W^*(z)P^*(z) = Q^*(z)W^*(z).$$

This completes the proof of the theorem.

Since the adjoint of multiplication by $P(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a contractive transformation of the space $\mathcal{H}(U)$ into itself a power series $S(z)$ with operator coefficients exists by Theorem 7 such that multiplication by $S(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that

$$U(z)S(z) = P(z)U(z),$$

and such that the range of multiplication by $S(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself is orthogonal to the kernel of multiplication by $U(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself. The adjoint of multiplication by $S(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a contractive transformation of the space $\mathcal{H}(V)$ into itself since

$$V(z)Q(z) = S(z)V(z).$$

A variant of the canonical decomposition has similar invariance properties. Assume that a Herglotz space $\mathcal{L}(\psi)$ is contained contractively in $\mathcal{C}(z)$ and is the closure of its polynomial elements. A power series $W(z)$ with operator coefficients exists such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that the adjoint of multiplication by $W(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the space $\mathcal{L}(\psi)$, and such that the kernel of the transformation contains $zf(z)$ whenever it contains $f(z)$.

The complementary space to the space $\mathcal{L}(\psi)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\psi)$ whose elements are the elements $f(z)$ of $\mathcal{C}(z)$ such that $W(z)f(z)$ belongs to the space $\mathcal{H}(W)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\psi)} = \|f(z)\|^2 + \|W(z)f(z)\|^2_{\mathcal{H}(W)}$$

is satisfied. An element

$$f(z) + z^{-1}g(z^{-1})$$

is ext $\mathcal{C}(z)$ belongs to ext $\mathcal{L}(1-\psi)$ if, and only if, $f(z)$ and $g(z)$ are elements of $\mathcal{C}(z)$ such that $(W(z)f(z), -g(z))$ belongs to the space $\mathcal{D}(W)$. The identity

$$\|f(z) + z^{-1}g(z^{-1})\|^2_{\text{ext } \mathcal{L}(1-\psi)} = \|f(z) + z^{-1}g(z^{-1})\|^2 + \|(W(z)f(z), -g(z))\|^2_{\mathcal{D}(W)}$$

is satisfied. The orthogonal complement in the space $\mathcal{H}(W)$ of the image of the space $\mathcal{L}(1-\psi)$ is the set of elements of $\mathcal{C}(z)$ whose coefficients are annihilated by the adjoint of multiplication by $W(z)$. The orthogonal complement in the space $\mathcal{D}(W)$ of the image of ext $\mathcal{L}(1-\psi)$ is the set of pairs $(f(z), g(z))$ of elements of $\mathcal{C}(z)$ such that $g(z)$ vanishes and the coefficients of $f(z)$ are annihilated by the adjoint of multiplication by $W(z)$.

A Herglotz space $\mathcal{L}(1-\theta)$ is defined which is contained contractively in the space $\mathcal{L}(1-\psi)$ such that ext $\mathcal{L}(1-\theta)$ is the set of elements

$$h(z) = f(z) + z^{-1}g(z^{-1})$$

of ext $\mathcal{L}(1-\psi)$ with scalar product determined by the identity

$$\|h(z)\|^2_{\text{ext } \mathcal{L}(1-\theta)} = \|h(z)\|^2 + \|h(z)\|^2_{\text{ext } \mathcal{L}(1-\psi)}.$$

The space $\mathcal{L}(1-\theta)$ is the complementary space in $\mathcal{C}(z)$ of a Herglotz space $\mathcal{L}(\theta)$ which is contained contractively in $\mathcal{C}(z)$. The space $\mathcal{L}(\psi)$ is contained contractively in the space $\mathcal{L}(\theta)$ since ext $\mathcal{L}(\psi)$ is contained contractively in ext $\mathcal{L}(\theta)$.

The space ext $\mathcal{L}(\psi)$ is dense in the space ext $\mathcal{L}(\theta)$: If a Hilbert space $\mathcal{P}$ is contained contractively in a Hilbert space $\mathcal{H}$ and if a Hilbert space $\mathcal{Q}$ is defined as the set of elements c of $\mathcal{P}$ with scalar product determined by the identity

$$\|c\|^2_{\mathcal{Q}} = \|c\|^2_{\mathcal{H}} + \|c\|^2_{\mathcal{P}},$$

then the complementary space to $\mathcal{P}$ in $\mathcal{H}$ is dense in the complementary space to $\mathcal{Q}$ in $\mathcal{H}$. The proof is given by reduction to the case in which $\mathcal{H}$ is one–dimensional.

The space $\mathcal{L}(\psi)$ is dense in the space $\mathcal{L}(\theta)$ since ext $\mathcal{L}(\psi)$ is dense in ext $\mathcal{L}(\theta)$. The space $\mathcal{L}(\theta)$ is the closure of its polynomial elements since the space $\mathcal{L}(\psi)$ is the closure of its polynomial elements. A power series $V(z)$ with operator coefficients exists such that multiplication by $V(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that the adjoint of multiplication by $V(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the space $\mathcal{L}(\theta)$, and such that the kernel of the adjoint of multiplication by $V(z)$ contains $zf(z)$ whenever it contains $f(z)$.

The complementary space to the space $\mathcal{L}(\theta)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\theta)$ whose elements are the elements $f(z)$ of $\mathcal{C}(z)$ such that $V(z)f(z)$ belongs to the space $\mathcal{H}(V)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\theta)} = \|f(z)\|^2 + \|V(z)f(z)\|^2_{\mathcal{H}(V)}$$

is satisfied. An element

$$f(z) + z^{-1}g(z^{-1})$$

of ext $\mathcal{C}(z)$ belongs to ext $\mathcal{L}(1-\theta)$ if, and only if, $f(z)$ and $g(z)$ are elements of $\mathcal{C}(z)$ such that $(V(z)f(z), -g(z))$ belongs to the space $\mathcal{D}(V)$. The identity

$$\|f(z) + z^{-1}g(z^{-1})\|^2_{\text{ext } \mathcal{L}(1-\theta)} = \|f(z) + z^{-1}g(z^{-1})\|^2 + \|(V(z)f(z), -g(z))\|^2_{\mathcal{D}(V)}$$

is satisfied. The orthogonal complement in the space $\mathcal{H}(V)$ of the image of the space $\mathcal{L}(1-\theta)$ is the set of elements of $\mathcal{C}(z)$ whose coefficients are annihilated by the adjoint of multiplication by $V(z)$. The orthogonal complement in the space $\mathcal{D}(V)$ of the image of ext $\mathcal{L}(1-\theta)$ is the set of pairs $(f(z), g(z))$ of elements of $\mathcal{C}(z)$ such that $g(z)$ vanishes and the coefficients of $f(z)$ are annihilated by the adjoint of multiplication by $V(z)$.

A contractive transformation of the space $\mathcal{D}(V)$ into the space $\mathcal{D}(W)$ is defined which takes $(V(z)f(z), -g(z))$ into $(W(z)f(z), -g(z))$ whenever $f(z)$ and $g(z)$ are elements of $\mathcal{C}(z)$ such that

$$f(z) + z^{-1}g(z^{-1})$$

belongs to ext $\mathcal{L}(1-\theta)$ and which annihilates elements of the space $\mathcal{D}(V)$ which are orthogonal to the image of ext $\mathcal{L}(1-\theta)$.

The space $\mathcal{H}(V^*)$ is contained contractively in the space $\mathcal{H}(W^*)$. A power series $U(z)$ with operator coefficients exists by Theorem 3 such that multiplication by $U(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that

$$W(z) = U(z)V(z),$$

and such that the range of multiplication by $U^*(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself is orthogonal to the kernel of multiplication by $V^*(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself.

A contractive transformation of the space $\mathcal{D}(U)$ into the space $\mathcal{D}(W)$ which take $(f(z), g(z))$ into $(f(z), V(z)g(z))$ acts as a partially isometric transformation of the space $\mathcal{D}(U)$ onto a Hilbert space which is contained contractively in the space $\mathcal{D}(W)$. The contractive transformation of the space $\mathcal{D}(V)$ into the space $\mathcal{D}(W)$ which takes $(f(z), g(z))$ into $(V(z)f(z), g(z))$ acts as a partially isometric transformation of the space $\mathcal{D}(V)$ onto the complementary space to the image of the space $\mathcal{D}(U)$ in the space $\mathcal{D}(W)$.

If a dense set of elements of the space $\mathcal{H}(W)$ are products $W(z)f(z)$ with $f(z)$ a polynomial element of the space $\mathcal{L}(1-\psi)$, a partially isometric transformation of ext $\mathcal{L}(1-\psi)$ onto the image of the space $\mathcal{D}(V)$ in the space $\mathcal{D}(W)$ is defined by taking

$$f(z) + z^{-1}g(z^{-1})$$

for elements $f(z)$ and $g(z)$ of $\mathcal{C}(z)$ into

$$(W(z)f(z), -g(z)).$$

The space $\mathcal{H}(U)$ is contained contractively in the space $\mathcal{H}(W)$. A partially isometric transformation of the space $\mathcal{L}(1-\psi)$ onto the complementary space to the space $\mathcal{H}(U)$ in the space $\mathcal{H}(W)$ is defined by taking $f(z)$ into $W(z)f(z)$. If $P(z)$ is a power series with operator coefficients such that multiplication by $P(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself and such that the adjoint of multiplication by $P(z)$ acts as a contractive transformation of the space $\mathcal{H}(W)$ into itself, then the adjoint of multiplication by $P(z)$ acts as a contractive transformation of the space $\mathcal{H}(U)$ into itself.

A convex decomposition is made of a Hilbert space of power series with vector coefficients which satisfies the inequality for difference quotients but which does not satisfy the identity for difference quotients when the space is suitably confined.

A space $\mathcal{H}$ of power series with vector coefficients which satisfies the inequality for difference quotients is assumed to be contained contractively in a space $\mathcal{H}(W)$ for a power series $W(z)$ with operator coefficients such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself and such that the space $\mathcal{H}(W)$ satisfies the inequality for difference quotients.

The coefficient space can be assumed without less of generality to have infinite dimension. A power series $U(z)$ with operator coefficients exists by the proof of Theorem 1 such that multiplication by $U(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself and such that the space $\mathcal{H}(W)$ is isometrically equal to the space $\mathcal{H}(U)$. A power series $V(z)$ with operator coefficients exists by Theorem 3 such that multiplication by $V(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that

$$W(z) = U(z)V(z),$$

and such that the range of multiplication by $V(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself is orthogonal to the kernel of multiplication by $U(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself.

The adjoint of multiplication by $W(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\psi)$ which is contained contractively in $\mathcal{C}(z)$. The complementary space to the space $\mathcal{L}(\psi)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\psi)$ whose elements are the elements $f(z)$ of $\mathcal{C}(z)$ such that $W(z)f(z)$ belongs to the space $\mathcal{H}(W)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\psi)} = \|f(z)\|^2 + \|W(z)f(z)\|^2_{\mathcal{H}(W)}$$

is satisfied. Multiplication by $W(z)$ annihilates polynomial elements of the space $\mathcal{L}(1-\psi)$.

The adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\theta)$ which is contained contractively in $\mathcal{C}(z)$. The complementary space to the space $\mathcal{L}(\theta)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\theta)$ whose elements are the elements $f(z)$ of $\mathcal{C}(z)$ such that $V(z)f(z)$ belongs to the space $\mathcal{H}(V)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\theta)} = \|f(z)\|^2 + \|V(z)f(z)\|^2_{\mathcal{H}(V)}$$

is satisfied. Multiplication by $V(z)$ annihilates polynomial elements of the space $\mathcal{L}(1-\theta)$. The polynomial elements of the space $\mathcal{L}(1-\theta)$ are identical with the polynomial elements of the space $\mathcal{L}(1-\psi)$. The complementary space to the space $\mathcal{L}(1-\theta)$ in the space $\mathcal{L}(1-\psi)$ is a Herglotz space $\mathcal{L}(\theta-\psi)$ which contains no nonzero polynomial. The space $\mathcal{L}(\theta)$ is the closure of its polynomial elements.

The adjoint of multiplication by $U(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\phi)$ which is contained contractively in $\mathcal{C}(z)$. The complementary space to the space $\mathcal{L}(\phi)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\phi)$ whose elements are the elements $f(z)$ of $\mathcal{C}(z)$ such that $U(z)f(z)$ belongs to the space $\mathcal{H}(U)$. The identity

$$\|f(z)^2_{\mathcal{L}(1-\phi)} = \|f(z)\|^2 + \|U(z)f(z)\|^2_{\mathcal{H}(U)}$$

is satisfied. The adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of the space $\mathcal{L}(1-\phi)$ onto the space $\mathcal{L}(\theta-\psi)$. Since the space $\mathcal{L}(\theta-\psi)$ contains no nonzero polynomial, the adjoint of multiplication by $V(z)$ annihilates the polynomial elements of the space $\mathcal{L}(1-\phi)$.

A Herglotz space $\mathcal{L}(1-\eta)$ is defined which is contained isometrically in the space $\mathcal{L}(1-\phi)$ and which is the closure of the polynomial elements of the space. The space $\mathcal{L}(\phi)$ is the closure of its polynomial elements and is contained contractively in the Herglotz space $\mathcal{L}(\eta)$ which is the complementary space to the space $\mathcal{L}(1-\eta)$ in $\mathcal{C}(z)$. A Herglotz space $\mathcal{L}(\alpha)$ is defined which is contained isometrically in the space $\mathcal{L}(\eta)$ and which is the closure of the polynomial elements of the space. The space $\mathcal{L}(\phi)$ is contained contractively in the space $\mathcal{L}(\alpha)$. The space $\mathcal{L}(1-\eta)$ is contained isometrically in the space $\mathcal{L}(1-\alpha)$. The space $\mathcal{L}(1-\alpha)$ is contained isometrically in the space $\mathcal{L}(1-\phi)$.

Since the adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the space $\mathcal{L}(\theta)$ and since the space $\mathcal{L}(\alpha)$ is contained contractively in $\mathcal{C}(z)$, the adjoint of multiplication by $V(z)$ acts as a partially isometric transformation of the space $\mathcal{L}(\alpha)$ onto a Herglotz space $\mathcal{L}(\beta)$ which is contained contractively in the space $\mathcal{L}(\theta)$. The complementary space to the space $\mathcal{L}(\beta)$ in the space $\mathcal{L}(\theta)$ is a Herglotz space $\mathcal{L}(\theta-\beta)$. The adjoint of multiplication by $V(z)$ acts as a partially isometric transformation of the space $\mathcal{L}(1-\alpha)$ onto the space $\mathcal{L}(\theta-\beta)$. Since the adjoint of multiplication by $V(z)$ acts as a contractive transformation of the space $\mathcal{L}(1-\eta)$ into a space $\mathcal{L}(\theta-\psi)$ which contains no nonzero polynomial and since the space $\mathcal{L}(1-\eta)$ is the closure of its polynomial elements, the adjoint of multiplication by $V(z)$ annihilates the elements of the space $\mathcal{L}(1-\eta)$. The adjoint of multiplication by $V(z)$ acts as a partially isometric transformation of the space $\mathcal{L}(\eta)$ onto the space $\mathcal{L}(\theta)$.

Since the adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of the space $\mathcal{L}(\alpha)$ onto the space $\mathcal{L}(\beta)$, the adjoint of multiplication by $V(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of ext $\mathcal{L}(\alpha)$ onto ext $\mathcal{L}(\beta)$. The adjoint of multiplication by $V(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of ext $\mathcal{L}(\beta)$ onto ext $\mathcal{L}(\eta)$.

Since the space $\mathcal{L}(\alpha)$ is contained isometrically in the space $\mathcal{L}(\eta)$, ext $\mathcal{L}(\alpha)$ is contained isometrically in ext $\mathcal{L}(\eta)$. Since the polynomial elements of the space $\mathcal{L}(\alpha)$ are the polynomial elements of the space $\mathcal{L}(\eta)$, the partially isometric transformations of ext $\mathcal{L}(\alpha)$ onto the space $\mathcal{L}(\alpha)$ and of ext $\mathcal{L}(\beta)$ onto the space $\mathcal{L}(\beta)$ have the same kernel. The space $\mathcal{L}(\beta)$ which is the image of the space $\mathcal{L}(\alpha)$ is contained isometrically in the space $\mathcal{L}(\theta)$ which is the image of the space $\mathcal{L}(\eta)$.

The space $\mathcal{L}(\theta-\beta)$ is the closure of its polynomial elements since it is contained isometrically in the space $\mathcal{L}(\theta)$ which is the closure of its polynomial elements. Since the space $\mathcal{L}(\theta-\beta)$ is contained contractively in a space $\mathcal{L}(\theta-\psi)$ which contains no nonzero polynomial, the space $\mathcal{L}(\theta-\beta)$ contains no nonzero element. The space $\mathcal{L}(\beta)$ is isometrically equal to the space $\mathcal{L}(\theta)$. Since the space $\mathcal{L}(1-\alpha)$ is contained in the kernel of the adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself, it is contained contractively in the set of elements of the space $\mathcal{H}(V)$ which are included isometrically in $\mathcal{C}(z)$.

A convex decomposition of a space $\mathcal{H}$ which satisfies the inequality for difference quotients but does not satisfy the identity of difference quotients is made when the space is contained contractively in a space $\mathcal{H}(W)$ such that the space $\mathcal{D}(W)$ is the graph of an injective transformation of the space $\mathcal{H}(W)$ onto the space $\mathcal{H}(W^*)$.

The space $\mathcal{H}$ can be assumed isometrically equal to a space $\mathcal{H}(U)$ for a power series $U(z)$ with operator coefficients such that multiplication by $U(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. A power series $V(z)$ with operator coefficients exists such that multiplication by $V(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that

$$W(z) = U(z)V(z),$$

and such that the range of multiplication by $V(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself is orthogonal to the kernel of multiplication by $V(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself.

Power series $A(z)$ and $C(z)$ with operator coefficients exist by Theorem 5 such that multiplication by $A(z)$ and multiplication by $C(z)$ are contractive transformations of $\mathcal{C}(z)$ into itself, such that

$$U(z) = C(z)A(z),$$

such that the space $\mathcal{H}(C)$ is contained isometrically in the space $\mathcal{H}(U)$ and satisfies the identity for difference quotients, and such that the orthogonal complement in the space $\mathcal{H}(A)$ of the set of products $A(z)f(z)$ with $f(z)$ a polynomial element of $\mathcal{C}(z)$ is the kernel of multiplication by $C(z)$ as a transformation of the space $\mathcal{H}(A)$ into the space $\mathcal{H}(U)$.

Power series $B(z)$ and $D(z)$ with operator coefficients exist by Theorem 5 such that multiplication by $B(z)$ and multiplication by $D(z)$ are contractive transformations of $\mathcal{C}(z)$ into itself, such that

$$V(z) = B(z)D(z),$$

such that the space $\mathcal{H}(D^*)$ is contained isometrically in the space $\mathcal{H}(V^*)$ and satisfies the identity for difference quotients, and such that the orthogonal complement in the space $\mathcal{H}(B^*)$ of elements which are products $B^*(z)f(z)$ with $f(z)$ a polynomial element of $\mathcal{C}(z)$ is the kernel of multiplication by $D^*(z)$ as a transformation of the space $\mathcal{H}(B^*)$ into the space $\mathcal{H}(V^*)$.

As in the proof of Theorem 5 a partially isometric transformation of the space $\mathcal{D}(A)$ into the space $\mathcal{D}(U)$ is defined by taking $(f(z), g(z))$ into $(C(z)f(z), g(z))$. A partially isometric transformation of the space $\mathcal{D}(C)$ onto the orthogonal complement of the image of the space $\mathcal{D}(A)$ in the space $\mathcal{D}(U)$ is defined by taking $(f(z), g(z))$ into $(f(z), A^*(z)g(z))$. The kernel of the transformation is the set of pairs $(f(z), g(z))$ of elements of $\mathcal{C}(z)$ such that $f(z)$ vanishes and $g(z)$ belongs to the kernel of multiplication by $A^*(z)$.

The space $\mathcal{D}(U^*)$ is the graph of a transformation of the space $\mathcal{H}(U^*)$ into the space $\mathcal{H}(U)$ since the space $\mathcal{D}(W^*)$ is the graph of a transformation of the space $\mathcal{H}(W^*)$ onto the space $\mathcal{H}(W)$. The space $\mathcal{H}(A^*)$ satisfies the identity for difference quotients since it is contained isometrically in the space $\mathcal{H}(U^*)$ which satisfies the identity for difference quotients. Multiplication by $A^*(z)$ annihilates every polynomial element $f(z)$ of $\mathcal{C}(z)$ such that $A^*(z)f(z)$ belongs to the space $\mathcal{H}(A^*)$.

As in the proof of Theorem 5 a partially isometric transformation of the space $\mathcal{D}(B^*)$ into the space $\mathcal{D}(V^*)$ is defined by taking $(f(z), g(z))$ into $(D^*(z)f(z), g(z))$. A partially isometric transformation of the space $\mathcal{D}(D^*)$ onto the orthogonal complement of the image of the space $\mathcal{D}(B^*)$ in the space $\mathcal{D}(V^*)$ is defined by taking $(f(z), g(z))$ into $(f(z), B(z)g(z))$. The kernel of the transformation is the set of pairs $(f(z), g(z))$ of elements of $\mathcal{C}(z)$ such that $f(z)$ vanishes and $g(z)$ belongs to the kernel of multiplication by $B(z)$.

The space $\mathcal{D}(V)$ is the graph of a transformation of the space $\mathcal{H}(V)$ onto the space $\mathcal{H}(V^*)$ since the space $\mathcal{D}(W)$ is the graph of a transformation of the space $\mathcal{H}(W)$ onto the space $\mathcal{H}(W^*)$. The space $\mathcal{H}(B)$ satisfies the identity for difference quotients since it is contained isometrically in the space $\mathcal{H}(V)$ which satisfies the identity for difference quotients. Multiplication by $B(z)$ annihilates every polynomial element $f(z)$ of $\mathcal{C}(z)$ such that $B(z)f(z)$ belongs to the space $\mathcal{H}(B)$.

The adjoint of multiplication by $A(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\theta)$ which is contained contractively in $\mathcal{C}(z)$. The complementary space to the space $\mathcal{L}(\theta)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\theta)$ whose elements are the elements $f(z)$ of $\mathcal{C}(z)$ such that $A(z)f(z)$ belongs to the space $\mathcal{H}(A)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\theta)} = \|f(z)\|^2 + \|A(z)f(z)\|^2_{\mathcal{H}(A)}$$

is satisfied. The orthogonal complement in the space $\mathcal{H}(A)$ of the image of the space $\mathcal{L}(1-\theta)$ is the kernel of multiplication by $C(z)$ as a transformation of the space $\mathcal{H}(A)$ into the space $\mathcal{H}(U)$.

Since the space $\mathcal{D}(W)$ is the graph of a transformation of the space $\mathcal{H}(W)$ into the space $\mathcal{H}(W^*)$, the adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself annihilates the elements of the space $\mathcal{L}(1-\theta)$. The space $\mathcal{L}(1-\theta)$ is contained contractively in the set of elements of the space $\mathcal{H}(V)$ which are included isometrically in $\mathcal{C}(z)$. Since the space $\mathcal{H}(B)$ satisfies the identity for difference quotients, the space $\mathcal{L}(1-\theta)$ is contained contractively in the set of elements of the space $\mathcal{H}(B)$ which are included isometrically in $\mathcal{C}(z)$. The space $\mathcal{L}(1-\theta)$ is the set of elements $f(z)$ of the space $\mathcal{H}(B)$ such that $A(z)f(z)$ belongs to the space $\mathcal{H}(A)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\phi)} = \|f(z)\|^2_{\mathcal{H}(B)} + \|A(z)f(z)\|^2_{\mathcal{H}(A)}$$

is satisfied.

An element

$$f(z) + z^{-1}g(z^{-1})$$

of ext $\mathcal{C}(z)$ belongs to ext $\mathcal{L}(1-\theta)$ if, and only if, $f(z)$ and $g(z)$ are elements of $\mathcal{C}(z)$ such that $(A(z)f(z), -g(z))$ belongs to the space $\mathcal{D}(A)$. The identity

$$\|f(z) + z^{-1}g(z^{-1})\|^2_{\text{ext } \mathcal{L}(1-\theta)} = \|f(z) + z^{-1}g(z^{-1})\|^2 + \|(A(z)f(z), -g(z))\|^2_{\mathcal{D}(A)}$$

is satisfied. Since $(f(z), -B^*(z)g(z))$ is an element of the space $\mathcal{D}(B)$ which has the same norm as the element $f(z) + z^{-1}g(z^{-1})$ of ext $\mathcal{C}(z)$, an element

$$f(z) + z^{-1}g(z^{-1})$$

of ext $\mathcal{C}(z)$ belongs to ext $\mathcal{L}(1-\theta)$ if, and only if, $f(z)$ and $g(z)$ are elements of $\mathcal{C}(z)$ such that $(A(z)f(z), -g(z))$ belongs to the space $\mathcal{D}(A)$ and $(-f(z), B^*(z)g(z))$ belongs to the space $\mathcal{D}(B)$. The identity

$$\|f(z) + z^{-1}g(z^{-1})\|^2_{\text{ext } \mathcal{L}(1-\theta)} = \|(A(z)f(z), -g(z))\|^2_{\mathcal{D}(A)} + \|(-f(z), B^*(z)g(z))\|^2_{\mathcal{D}(B)}$$

is satisfied.

The adjoint of multiplication by $B^*(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\phi)$ which is contained contractively in $\mathcal{C}(z)$. The complementary space to the space $\mathcal{L}(\phi)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\phi)$ whose elements are the elements $f(z)$ of $\mathcal{C}(z)$ such that $B^*(z)f(z)$ belongs to the space $\mathcal{H}(B^*)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\phi)} = \|f(z)\|^2 + \|B^*(z)f(z)\|^2_{\mathcal{H}(B^*)}$$

is satisfied. The orthogonal complement in the space $\mathcal{H}(B^*)$ of the image of the space $\mathcal{L}(1-\phi)$ is the kernel of multiplication by $A^*(z)$ as a transformation of the space $\mathcal{H}(B^*)$ into the space $\mathcal{H}(V^*)$.

Since the space $\mathcal{D}(W^*)$ is the graph of a transformation of the space $\mathcal{H}(W^*)$ into the space $\mathcal{H}(W)$, the adjoint of multiplication by $U^*(z)$ as a transformation of $\mathcal{C}(z)$ into itself annihilates the elements of the space $\mathcal{L}(1-\phi)$. The space $\mathcal{L}(1-\phi)$ is contained contractively in the set of elements of the space $\mathcal{H}(U^*)$ which are included isometrically in $\mathcal{C}(z)$. Since the space $\mathcal{H}(A^*)$ satisfies the identity for difference quotients, the space $\mathcal{L}(1-\phi)$ is contained contractively in the set of elements of the space $\mathcal{H}(A^*)$ which are included isometrically in $\mathcal{C}(z)$. The space $\mathcal{L}(1-\phi)$ is the set of elements $f(z)$ of the space $\mathcal{H}(A^*)$ such that $B^*(z)f(z)$ belongs to the space $\mathcal{H}(B^*)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\phi)} = \|f(z)\|^2_{\mathcal{H}(A^*)} + \|B^*(z)f(z)\|^2_{\mathcal{H}(B^*)}$$

is satisfied.

An element

$$f(z) + z^{-1}g(z^{-1})$$

of ext $\mathcal{C}(z)$ belongs to ext $\mathcal{L}(1-\phi)$ if, and only if, $f(z)$ and $g(z)$ are elements of $\mathcal{C}(z)$ such that $(B^*(z)f(z), -g(z))$ belongs to the space $\mathcal{D}(B^*)$. The identity

$$\|f(z) + z^{-1}g(z^{-1})\|^2_{\text{ext } \mathcal{L}(1-\phi)} = \|f(z) + z^{-1}g(z^{-1})\|^2 + \|(B^*(z)f(z), -g(z))\|^2_{\mathcal{D}(B^*)}$$

is satisfied. Since $(f(z), -A(z)g(z))$ is an element of the space $\mathcal{D}(A^*)$ which has the same norm as the element $f(z) + z^{-1}g(z^{-1})$ of ext $\mathcal{C}(z)$, an element

$$f(z) + z^{-1}g(z^{-1})$$

of ext $\mathcal{C}(z)$ belongs to ext $\mathcal{L}(1-\phi)$ if, and only if, $f(z)$ and $g(z)$ are elements of $\mathcal{C}(z)$ such that $(B^*(z)f(z), -g(z))$ belongs to the space $\mathcal{D}(B^*)$ and $(-f(z), A(z)g(z))$ belongs to the space $\mathcal{D}(A^*)$. The identity

$$\|f(z) + z^{-1}g(z^{-1})\|^2_{\text{ext } \mathcal{L}(1-\phi)} = \|(B^*(z)f(z), -g(z))\|^2_{\mathcal{D}(B^*)} + \|(-f(z), A(z)g(z))\|^2_{\mathcal{D}(A^*)}$$

is satisfied.

An isometric transformation of ext $\mathcal{L}(1-\phi)$ onto ext $\mathcal{L}(1-\theta)$ is defined by taking

$$f(z) + z^{-1}g(z^{-1})$$

into

$$g(z) + z^{-1}f(z^{-1})$$

with $f(z)$ and $g(z)$ elements of $\mathcal{C}(z)$. A Herglotz space $\mathcal{L}(\theta^*)$ exists,

$$\theta^*(z) = \sum \theta_n^- z^n$$

if

$$\theta(z) = \sum \theta_n z^n.$$

The spaces $\mathcal{L}(\phi)$ and $\mathcal{L}(\theta^*)$ are isometrically equal.

Preparations are made for a convex decomposition.

Theorem 9. *If a Hilbert space $\mathcal{H}$ of power series with vector coefficients satisfies the inequality for difference quotients and is contained contractively in a space $\mathcal{H}(W)$ for a power series $W(z)$ with operator coefficients such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself and such that the space $\mathcal{D}(W)$ is the graph of an injective transformation of the space $\mathcal{H}(W)$ onto the space $\mathcal{H}(W^*)$, then a convex decomposition*

$$\mathcal{H} = (1-t)\mathcal{H}_+ + t\mathcal{H}_-$$

applies with t and $1-t$ equal and with $\mathcal{H}_+$ and $\mathcal{H}_-$ Hilbert spaces of power series with vector coefficients which satisfy the inequality for difference quotients and which are not isometrically equal when $\mathcal{H}$ does not satisfy the identity for difference quotients.

Proof of Theorem 9. Previous constructions are assembled in a consistent notation. The space $\mathcal{H}$ can be assumed isometrically equal to the space $\mathcal{H}(U)$ for a power series $U(z)$ with operator coefficients such that multiplication by $U(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that

$$W(z) = U(z)V(z),$$

and such that the range of multiplication by $V(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself is orthogonal to the kernel of multiplication by $U(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself.

A Herglotz space $\mathcal{L}(\psi)$ is constructed which is contained contractively in $\mathcal{C}(z)$ and is the closure of its polynomial elements such that $U(z)f(z)$ belongs to the space $\mathcal{H}(U)$ for every element $f(z)$ of the space $\mathcal{L}(1-\psi)$ and such that the identity

$$\|f(z)\|^2_{\mathcal{L}(1-\psi)} = \|f(z)\|^2 + \|U(z)f(z)\|^2_{\mathcal{H}(U)}$$

is satisfied. The Herglotz space $\mathcal{L}(\psi^*)$ is contained contractively in $\mathcal{C}(z)$ and is the closure of its polynomial elements. If $f(z)$ is an element of the space $\mathcal{L}(1-\psi^*)$, then $V^*(z)f(z)$ is an element of the space $\mathcal{H}(V^*)$ which satisfies the identity

$$\|f(z)\|^2_{\mathcal{L}(1-\psi^*)} = \|f(z)\|^2 + \|V^*(z)f(z)\|^2_{\mathcal{H}(V^*)}.$$

The contractive transformation of the space $\mathcal{D}(U)$ into the space $\mathcal{D}(W)$ which takes $(f(z), g(z))$ into $(f(z), V^*(z)g(z))$ acts as a partially isometric transformation of the space $\mathcal{D}(U)$ onto a Hilbert space which is contained contractively in the space $\mathcal{D}(W)$. The contractive transformation of the space $\mathcal{D}(V)$ into the space $\mathcal{D}(W)$ which takes $(f(z), g(z))$ into $(V(z)f(z), g(z))$ acts as a partially isometric transformation of the space $\mathcal{D}(V)$ onto the complementary space to the image of the space $\mathcal{D}(U)$ in the space $\mathcal{D}(W)$.

A Herglotz space $\mathcal{L}(\theta)$ which is contained contractively in $\mathcal{C}(z)$ and which is the closure of its polynomial elements is defined so that ext $\mathcal{L}(1-\theta)$ is the set of elements $h(z)$ of ext $\mathcal{L}(1-\psi)$ in the scalar product determined by the identity

$$\|h(z)\|^2_{\text{ext } \mathcal{L}(1-\theta)} = \|h(z)\|^2 + \|h(z)\|^2_{\text{ext } \mathcal{L}(1-\psi)}.$$

The space $\mathcal{L}(\theta^*)$ is the closure of its polynomial elements.

A power series $A(z)$ with operator coefficients is defined so that multiplication by $A(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, so that the adjoint of multiplication by $A(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the space $\mathcal{L}(\theta)$, and so that the kernel of the transformation contains $zf(z)$ whenever it contains $f(z)$. The space $\mathcal{L}(1-\theta)$ is the set of elements $f(z)$ of $\mathcal{C}(z)$ such that $A(z)f(z)$ belongs to the space $\mathcal{H}(A)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\theta)} = \|f(z)\|^2 + \|A(z)f(z)\|^2_{\mathcal{H}(A)}$$

is satisfied. The orthogonal complement in the space $\mathcal{H}(A)$ of the image of the space $\mathcal{L}(1-\theta)$ is the set of elements of $\mathcal{C}(z)$ whose coefficients are annihilated by the adjoint of multiplication by $A(z)$. An element

$$f(z) + z^{-1}g(z^{-1})$$

of ext $\mathcal{C}(z)$ belongs to ext $\mathcal{L}(1-\theta)$ if, and only if, $f(z)$ and $g(z)$ are elements of $\mathcal{C}(z)$ such that $(A(z)f(z), -g(z))$ belongs to the space $\mathcal{D}(A)$. The identity

$$\|f(z) + z^{-1}g(z^{-1})\|^2_{\text{ext } \mathcal{L}(1-\theta)} = \|f(z) + z^{-1}g(z^{-1})\|^2 + \|(a(z)f(z), -g(z))\|^2_{\mathcal{D}(A)}$$

is satisfied.

A power series $C(z)$ with operator coefficients exists such that multiplication by $C(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that

$$U(z) = C(z)A(z),$$

and such that the range of multiplication by $A(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself is orthogonal to the kernel of multiplication by $C(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself.

The space $\mathcal{H}(C)$ is contained contractively in the space $\mathcal{H}(U)$. Multiplication by $C(z)$ is a partially isometric transformation of the space $\mathcal{H}(A)$ onto the complementary space to the space $\mathcal{H}(C)$ in the space $\mathcal{H}(U)$. The kernel of the transformation is the set of elements of $\mathcal{C}(z)$ whose coefficients are annihilated by the adjoint of multiplication by $A(z)$. A partially isometric transformation of the space $\mathcal{L}(1-\psi)$ onto the image of the space $\mathcal{H}(A)$ is defined by taking $f(z)$ into $U(z)f(z)$.

The contractive transformation of the space $\mathcal{D}(C)$ into the space $\mathcal{D}(U)$ which takes $(f(z), g(z))$ into $(f(z), A^*(z)g(z))$ acts as a partially isometric transformation of the space $\mathcal{D}(C)$ onto a Hilbert space which is contained contractively in the space $\mathcal{D}(U)$. The contractive transformation of the space $\mathcal{D}(A)$ into the space $\mathcal{D}(U)$ which takes $(f(z), g(z))$ into $(C(z)f(z), g(z))$ acts as a partially isometric transformation of the space $\mathcal{D}(A)$ onto the complementary space to the image of the space $\mathcal{D}(C)$ in the space $\mathcal{D}(U)$. The kernel of the transformation is the set of pairs $(f(z), g(z))$ of elements of $\mathcal{C}(z)$ such that $g(z)$ vanishes and the coefficients of $f(z)$ are annihilated by the adjoint of multiplication by $A(z)$. A partially isometric transformation of ext $\mathcal{L}(1-\psi)$ onto the image of the space $\mathcal{D}(A)$ is defined by taking $f(z) + z^{-1}g(z^{-1})$ into $(U(z)f(z), -g(z))$.

A power series $B(z)$ with operator coefficients is defined so that multiplication by $B(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, so that the adjoint of multiplication by $B^*(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the space $\mathcal{L}(\theta^*)$, and so that the kernel of the transformation contains $zf(z)$ whenever it contains $f(z)$. The space $\mathcal{L}(1-\theta^*)$ is the set of elements $f(z)$ of $\mathcal{C}(z)$ such that $B^*(z)f(z)$ belongs to the space $\mathcal{H}(B^*)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\theta^*)} = \|f(z)\|^2 + \|B^*(z)f(z)\|^2_{\mathcal{H}(B^*)}$$

is satisfied. The orthogonal complement in the space $\mathcal{H}(B^*)$ of the image of the space $\mathcal{L}(1-\theta^*)$ is the set of elements of $\mathcal{C}(z)$ whose coefficients are annihilated by the adjoint of multiplication by $B^*(z)$. An element

$$f(z) + z^{-1}g(z^{-1})$$

of ext $\mathcal{C}(z)$ belongs to ext $\mathcal{L}(1-\theta^*)$ if, and only if, $f(z)$ and $g(z)$ are elements of $\mathcal{C}(z)$ such that $(B^*(z)f(z), -g(z))$ belongs to the space $\mathcal{D}(B^*)$. The identity

$$\|f(z) + z^{-1}g(z^{-1})\|^2_{\text{ext }\mathcal{L}(1-\theta^*)} = \|f(z) + z^{-1}g(z^{-1})\|^2 + \|(B^*(z)f(z), -g(z))\|^2_{\mathcal{D}(B^*)}$$

is satisfied.

A power series $D(z)$ with operator coefficients exists such that multiplication by $D(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that

$$V(z) = B(z)D(z),$$

and such that the range of multiplication by $B^*(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself is orthogonal to the kernel of multiplication by $D^*(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself.

The space $\mathcal{H}(D^*)$ is contained contractively in the space $\mathcal{H}(V^*)$. Multiplication by $D^*(z)$ is a partially isometric transformation of the space $\mathcal{H}(B^*)$ into the complementary space to the space $\mathcal{H}(D^*)$ in the space $\mathcal{H}(V^*)$. The kernel of the transformation is the set of elements of $\mathcal{C}(z)$ whose coefficients are annihilated by the adjoint of multiplication by $B^*(z)$. A partially isometric transformation of the space $\mathcal{L}(1-\psi^*)$ onto the image of the space $\mathcal{H}(B^*)$ is defined by taking $f(z)$ into $V^*(z)f(z)$.

The contractive transformation of the space $\mathcal{D}(D^*)$ into the space $\mathcal{D}(V^*)$ which takes $(f(z), g(z))$ into $(f(z), B(z)g(z))$ acts as a partially isometric transformation of the space $\mathcal{D}(D^*)$ onto a Hilbert space which is contained contractively in the space $\mathcal{D}(V^*)$. The contractive transformation of the space $\mathcal{D}(B^*)$ into the space $\mathcal{D}(V^*)$ which takes $(f(z), g(z))$ into $(D^*(z)f(z), g(z))$ acts as a partially isometric transformation of the space $\mathcal{D}(B^*)$ onto the complementary space to the image of the space $\mathcal{D}(D^*)$ in the space $\mathcal{D}(V^*)$. The kernel of the transformation is the set of pairs $(f(z), g(z))$ of elements of $\mathcal{C}(z)$ such that $g(z)$ vanishes and the coefficients of $f(z)$ are annihilated by the adjoint of multiplication by $B^*(z)$. A partially isometric transformation of ext $\mathcal{L}(1-\psi^*)$ onto the image of the space $\mathcal{D}(B^*)$ is defined by taking $f(z) + z^{-1}g(z^{-1})$ into $(V^*(z)f(z), -g(z))$.

The convex decompositions

$$\mathcal{H}(U) = (1-t)\mathcal{H}(C) + t\mathcal{H}(UB)$$

and

$$\mathcal{H}(V^*) = (1-t)\mathcal{H}(D^*) + t\mathcal{H}(V^*A^*)$$

apply with t and $1-t$ equal. The spaces $\mathcal{H}(C)$ and $\mathcal{H}(UB)$ are isometrically equal if, and only if, the coefficients of every element of the space $\mathcal{L}(1-\psi)$ are annihilated by the adjoint of multiplication by $U(z)$ as a transformation of $\mathcal{C}(z)$ into itself, in which case the space $\mathcal{H}(U)$ satisfies the identity for difference quotients. The spaces $\mathcal{H}(D^*)$ and $\mathcal{H}(V^*A^*)$ are isometrically equal if, and only if, the coefficients of every element of the space $\mathcal{L}(1-\psi^*)$ are annihilated by the adjoint of multiplication by $V^*(z)$ as a transformation of $\mathcal{C}(z)$ into itself, in which case the space $\mathcal{H}(V^*)$ satisfies the identity for difference quotients. The space $\mathcal{H}(U)$ satisfies the identity for difference quotients if, and only if, the space $\mathcal{H}(V^*)$ satisfies the identity for difference quotients.

This completes the proof of the theorem.

The convex decomposition is compatible with commuting transformations. If $P(z)$ is a power series with operator coefficients such that multiplication by $P(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that the adjoint of multiplication by $P(z)$ is a contractive transformation of the space $\mathcal{H}(W)$ into itself, and such that the adjoint of multiplication by $P(z)$ is a contractive transformation of the space $\mathcal{H}(U)$ into itself, then the adjoint of multiplication by $P(z)$ is a contractive transformation of the space $\mathcal{H}(C)$ into itself and the adjoint of multiplication by $P(z)$ is a contractive transformation of the space $\mathcal{H}(UB)$ into itself.

A Hilbert space of power series with vector coefficients which satisfies the identity for difference quotients determines an invariant subspace when it is suitably confined.

Theorem 10. *A Hilbert space of power series with vector coefficients which satisfies the identity for difference quotients is contained isometrically in $\mathcal{C}(z)$ if it is contained contractively in a space $\mathcal{H}(W)$ for a power series $W(z)$ with operator coefficients such that multiplication by $W(z)$ and multiplication by $W^*(z)$ are isometric transformations of $\mathcal{C}(z)$ into itself.*

Proof of Theorem 10. A Hilbert space of power series with vector coefficients which satisfies the identity for difference quotients is by Theorem 2 isometrically equal to a space $\mathcal{H}(U)$ for a power series $U(z)$ with operator coefficients such that multiplication by $U(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. If the space $\mathcal{H}(U)$ is contained contractively in the space $\mathcal{H}(W)$, then a power series $V(z)$ with operator coefficients exists by Theorem 3 such that multiplication by $V(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that

$$W(z) = U(z)V(z),$$

and such that the range of multiplication by $V(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself is orthogonal to the kernel of multiplication by $U(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself. Multiplication by $V(z)$ is an isometric transformation of $\mathcal{C}(z)$ into itself since multiplication by $W(z)$ is an isometric transformation of $\mathcal{C}(z)$ into itself. Multiplication by $U^*(z)$ is an isometric transformation of $\mathcal{C}(z)$ into itself since multiplication by $W^*(z)$ is an isometric transformation of $\mathcal{C}(z)$ into itself.

The contractive transformation of the space $\mathcal{D}(U)$ into the space $\mathcal{D}(W)$ which takes $(f(z), g(z))$ into $(f(z), V^*(z)g(z))$ acts as a partially isometric transformation of the space $\mathcal{D}(U)$ onto a Hilbert space which is contained contractively in the space $\mathcal{D}(W)$. The contractive transformation of the space $\mathcal{D}(V)$ into the space $\mathcal{D}(W)$ which takes $(f(z), g(z))$ into $(V(z)f(z), g(z))$ acts as partially isometric transformation of the space $\mathcal{D}(V)$ onto the complementary space to the image of the space $\mathcal{D}(U)$ in the space $\mathcal{D}(W)$.

The adjoint of multiplication by $U(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\phi)$ which is contained contractively in $\mathcal{C}(z)$. The complementary space to the space $\mathcal{L}(\phi)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\phi)$ whose elements are the elements $f(z)$ of $\mathcal{C}(z)$ such that $u(z)f(z)$ belongs to the space $\mathcal{H}(U)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\phi)} = \|f(z)\|^2 + \|U(z)f(z)\|^2_{\mathcal{H}(U)}$$

is satisfied. The adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself takes an element $f(z)$ of the space $\mathcal{L}(1-\phi)$ into an element $g(z)$ of $\mathcal{C}(z)$ such that $W(z)g(z)$

belongs to the space $\mathcal{H}(W)$. Since multiplication by $W(z)$ is an isometric transformation of $\mathcal{C}(z)$ into itself, no nonzero element $g(z)$ of $\mathcal{C}(z)$ exists such that $W(z)g(z)$ belongs to the space $\mathcal{H}(W)$. The adjoint of multiplication by $V(z)$ as a transformation of $\mathcal{C}(z)$ into itself annihilates the elements of the space $\mathcal{L}(1-\phi)$. The space $\mathcal{L}(1-\phi)$ is contained contractively in the space $\mathcal{H}(V)$ which is contained isometrically in $\mathcal{C}(z)$.

The space $\mathcal{D}(V)$ is the graph of a transformation of the space $\mathcal{H}(V)$ onto the space $\mathcal{H}(V^*)$ since the space $\mathcal{D}(W)$ is the graph of a transformation of the space $\mathcal{H}(W)$ onto the space $\mathcal{H}(W^*)$. An element

$$f(z) + z^{-1}g(z^{-1})$$

of ext $\mathcal{C}(z)$ belongs to ext $\mathcal{L}(1-\phi)$ if, and only if, $f(z)$ and $g(z)$ are elements of $\mathcal{C}(z)$ such that $(U(z)f(z), -g(z))$ belongs to the space $\mathcal{D}(U)$. The identity

$$\|f(z) + z^{-1}g(z^{-1})\|^2_{\text{ext } \mathcal{L}(1-\phi)} = \|f(z) + z^{-1}g(z^{-1})\|^2 + \|(U(z)f(z), -g(z))\|^2_{\mathcal{D}(U)}$$

is satisfied. Since $(f(z), -V^*(z)g(z))$ is an element of the space $\mathcal{D}(V)$ which has the same norm as the element $f(z)$ of the space $\mathcal{H}(V)$, the identity

$$\|f(z) + z^{-1}g(z^{-1})\|^2_{\text{ext } \mathcal{L}(1-\phi)} = \|(U(z)f(z), -g(z))\|^2_{\mathcal{D}(U)} + \|(-f(z), V^*(z)g(z))\|^2_{\mathcal{D}(V)}$$

is satisfied.

The adjoint of multiplication by $V^*(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\psi)$ which is contained contractively in $\mathcal{C}(z)$. The complementary space to the space $\mathcal{L}(\psi)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\psi)$ whose elements are the elements $f(z)$ of $\mathcal{C}(z)$ such that $V^*(z)f(z)$ belongs to the space $\mathcal{H}(V^*)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\psi)} = \|f(z)\|^2 + \|V^*(z)f(z)\|^2_{\mathcal{H}(V^*)}$$

is satisfied. The adjoint of multiplication by $U^*(z)$ as a transformation of $\mathcal{C}(z)$ into itself takes an element $f(z)$ of the space $\mathcal{L}(1-\psi)$ into an element $g(z)$ of $\mathcal{C}(z)$ such that $W^*(z)g(z)$ belongs to the space $\mathcal{H}(W^*)$. Since multiplication by $W^*(z)$ is an isometric transformation of $\mathcal{C}(z)$ into itself, no nonzero element $g(z)$ of $\mathcal{C}(z)$ exists such that $W^*(z)g(z)$ belongs to the space $\mathcal{H}(W^*)$ The adjoint of multiplication by $U^*(z)$ as a transformation of $\mathcal{C}(z)$ into itself annihilates the elements of the space $\mathcal{L}(1-\psi)$. The space $\mathcal{L}(1-\psi)$ is contained contractively in the space $\mathcal{H}(U^*)$ which is contained isometrically in $\mathcal{C}(z)$.

The space $\mathcal{D}(U^*)$ is the graph of a transformation of the space $\mathcal{H}(U^*)$ onto the space $\mathcal{H}(U)$ since the space $\mathcal{D}(W^*)$ is the graph of a transformation of the space $\mathcal{H}(W^*)$ onto the space $\mathcal{H}(W)$. An element

$$f(z) + z^{-1}g(z^{-1})$$

of ext $\mathcal{C}(z)$ belongs to ext $\mathcal{L}(1-\psi)$ if, and only if, $f(z)$ and $g(z)$ are elements of $\mathcal{C}(z)$ such that $(V^*(z)f(z), -g(z))$ belongs to the space $\mathcal{D}(V^*)$. The identity

$$\|f(z) + z^{-1}g(z^{-1})\|^2_{\text{ext } \mathcal{L}(1-\psi)} = \|f(z) + z^{-1}g(z^{-1})\|^2 + \|(V^*(z)f(z), -g(z))\|^2_{\mathcal{D}(V^*)}$$

is satisfied. Since $(f(z), -U(z)g(z))$ is an element of the space $\mathcal{D}(U^*)$ which has the same norm as the element of the space $\mathcal{H}(U^*)$, the identity

$$\|f(z) + z^{-1}g(z^{-1})\|^2_{\text{ext } \mathcal{L}(1-\psi)} = \|(-f(z), U(z)g(z))\|^2_{\mathcal{D}(U^*)} + \|(V^*(z)f(z), -g(z)\|^2_{\mathcal{D}(V^*)}$$

is satisfied.

Since an isometric transformation of ext $\mathcal{L}(1-\psi)$ onto ext $\mathcal{L}(1-\phi)$ is defined by taking

$$f(z) + z^{-1}g(z^{-1})$$

into

$$g(z) + z^{-1}f(z^{-1})$$

when $f(z)$ and $g(z)$ are elements of $\mathcal{C}(z)$, the function

$$\phi^*(z) = \psi(z)$$

can be chosen for the function $\psi(z)$ with

$$\phi^*(z) = \Sigma \phi_n^- z^n$$

if

$$\phi(z) = \Sigma \phi_n z^n.$$

A Herglotz space $\mathcal{L}(\theta)$ which is contained contractively in $\mathcal{C}(z)$ and which is the closure of its polynomial elements is defined so that ext $\mathcal{L}(1-\theta)$ is the set of elements $h(z)$ of ext $\mathcal{L}(1-\phi)$ with scalar product determined by the identity

$$\|h(z)\|^2_{\text{ext } \mathcal{L}(1-\phi)} = \|h(z)\|^2 + \|h(z)\|^2_{\text{ext } \mathcal{L}(1-\phi)}.$$

The space $\mathcal{L}(\theta^*)$ is contained contractively in $\mathcal{C}(z)$ and is the closure of its polynomial elements.

A power series $A(z)$ with operator coefficients exists such that multiplication by $A(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that the adjoint of multiplication by $A(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the space $\mathcal{L}(\theta)$, and such that the kernel of the transformation contains $zf(z)$ whenever it contains $f(z)$.

The space $\mathcal{H}(A^*)$ is contained contractively in the space $\mathcal{H}(U^*)$. A power series $C(z)$ with operator coefficients exists such that multiplication by $C(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that

$$U(z) = C(z)A(z),$$

and such that the range of multiplication by $C^*(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself is orthogonal to the kernel of multiplication by $A^*(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself.

The space $\mathcal{H}(C)$ is contained contractively in the space $\mathcal{H}(U)$. Multiplication by $C(z)$ is a partially isometric transformation of the space $\mathcal{H}(A)$ onto the complementary space to the space $\mathcal{H}(C)$ in the space $\mathcal{H}(U)$. A partially isometric transformation of the space $\mathcal{L}(1-\phi)$ into the space $\mathcal{H}(A)$ is defined by taking $f(z)$ into $A(z)f(z)$. The orthogonal complement of the image of the space $\mathcal{L}(1-\phi)$ in the space $\mathcal{H}(A)$ is the set of elements which are annihilated on multiplication by $C(z)$. A partially isometric transformation of the space $\mathcal{L}(1-\phi)$ onto the image of the space $\mathcal{H}(A)$ in the space $\mathcal{H}(U)$ is defined by taking $f(z)$ into $U(z)f(z)$.

A power series $B(z)$ with operator coefficients exists such that multiplication by $B(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that the adjoint of multiplication by $B^*(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the space $\mathcal{L}(\theta^*)$, and such that the kernel of the transformation contains $zf(z)$ whenever it contains $f(z)$.

The space $\mathcal{H}(B)$ is contained contractively in the space $\mathcal{H}(V)$. A power series $D(z)$ with operator coefficients exists such that multiplication by $D(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that

$$V(z) = B(z)D(z),$$

and such that the range of multiplication by $D(z)$ as a transformation of $\mathcal{C}(z)$ into itself is orthogonal to the kernel of multiplication by $B(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself.

The space $\mathcal{H}(D^*)$ is contained contractively in the space $\mathcal{H}(V^*)$. Multiplication by $D^*(z)$ is a partially isometric transformation of the space $\mathcal{H}(B^*)$ onto the complementary space to the space $\mathcal{H}(D^*)$ in the space $\mathcal{H}(V^*)$. A partially isometric transformation of the space $\mathcal{L}(1-\phi^*)$ into the space $\mathcal{H}(B^*)$ is defined by taking $f(z)$ into $B^*(z)f(z)$. The orthogonal complement of the space $\mathcal{L}(1-\phi^*)$ in the space $\mathcal{H}(B^*)$ is the set of elements which are annihilated on multiplication by $D^*(z)$. A partially isometric transformation of the space $\mathcal{L}(1-\phi^*)$ onto the image of the space $\mathcal{H}(B^*)$ in the space $\mathcal{H}(V^*)$ is defined by taking $f(z)$ into $V^*(z)f(z)$.

The contractive transformation of the space $\mathcal{D}(B^*)$ into the space $\mathcal{D}(V^*)$ which takes $(f(z), g(z))$ into $(f(z), B(z)g(z))$ acts as a partially isometric transformation of the space $\mathcal{D}(D^*)$ onto a Hilbert space which is contained contractively in the space $\mathcal{D}(V^*)$. The contractive transformation of the space $\mathcal{D}(B^*)$ into the space $\mathcal{D}(V^*)$ which takes $(f(z), g(z))$ into $(B^*(z)f(z), g(z))$ acts as a partially isometric transformation of the space $\mathcal{D}(B^*)$ onto the complementary space to the image of the space $\mathcal{D}(D^*)$ in the space $\mathcal{D}(V^*)$.

The kernel of the transformation is the set of pairs $(f(z), g(z))$ of elements of $\mathcal{C}(z)$ such that $g(z)$ vanishes and the coefficients of $f(z)$ are annihilated by the adjoint of multiplication by $B^*(z)$. A partially isometric transformation of ext $\mathcal{L}(1-\phi^*)$ onto the image of the space $\mathcal{D}(B^*)$ is defined by taking $f(z) + z^{-1}g(z^{-1})$ into $(V^*(z)f(z), -g(z))$. The space $\mathcal{L}(1-\phi)$ is mapped onto the image of the space $\mathcal{H}(B)$ in the space $\mathcal{H}(U)$ by a partially isometric transformation which takes $f(z)$ into $U(z)f(z)$.

The convex decomposition

$$\mathcal{H}(U) = (1-t)\mathcal{H}(C) + t\mathcal{H}(UB)$$

applies with t and $1-t$ equal. Since the space $\mathcal{H}(U)$ satisfies the identity for difference quotients, the space $\mathcal{H}(C)$ and $\mathcal{H}(UB)$ are isometrically equal to the space $\mathcal{H}(U)$. Since multiplication by $U(z)$ is a partially isometric transformation of the space $\mathcal{L}(1-\phi)$ onto the orthogonal complement of the space $\mathcal{H}(C)$ in the space $\mathcal{H}(U)$, multiplication by $U(z)$ annihilates every element of the space $\mathcal{L}(1-\phi)$. The space $\mathcal{H}(U)$ is contained isometrically in $\mathcal{C}(z)$.

This completes the proof of the theorem.

A topology is defined on the space of continuous transformations of a Hilbert space into itself for the construction of invariant subspaces. The definition is made for the coefficient space because of convenience in notation but is applicable to an arbitrary Hilbert space.

An operator is a continuous transformation of the coefficient space $\mathcal{C}$ into itself. The adjoint of an operator S is an operator S^-. Elements of the space are vectors. The scalar product

$$\langle a, b\rangle = b^- a$$

of vectors a and b is linear in a and conjugate linear in b. The norm $|c|$ of a vector c is the nonnegative square root of the scalar self–product

$$|c|^2 = c^- c.$$

If a and b are vectors, an operator ab^- is defined by the associative law

$$(ab^-)c = a(b^- c)$$

for every vector c. The operator and its adjoint

$$(ab^-)^- = ba^-$$

have range of dimension at most one. Every operator whose range has dimension at most one is equal to ab^- for some vectors a and b.

If $e_1, \ldots, e_r$ is an orthonormal set of vectors whose vector span contains the range of the operator and the range of its adjoint, then

$$a = a_1 e_1 + \ldots + a_r e_r$$

for complex numbers

$$a_k = e_k^- a$$

and

$$b = b_1 e_1 + \ldots + b_r e_r$$

for complex numbers

$$b_k = e_k^- b.$$

The operator

$$ab^- = \sum a_i b_j^- e_i e_j^-$$

is represented by a square matrix with entry

$$a_i b_j^-$$

in the i–th row and j–th column for $i, j = 1, \ldots, r$. The trace

$$b^- a = b_1^- a_1 + \ldots + b_r^- a_r$$

of the matrix is a complex number which is independent of the orthonormal set. The adjoint operator

$$ba^- = \sum b_i a_j^- e_i e_j^-$$

is represented by the conjugate transpose matrix which has entry

$$b_i a_j^-$$

in the i–th row and j–th column for $i, j = 1, \ldots, r$. The trace

$$a^- b = a_1^- b_1 + \ldots + a_r^- b_r$$

of the adjoint operator is the complex conjugate of the trace of the operator.

An operator

$$S = \sum S_{ij} e_i e_j^-$$

whose range is contained in the vector span of the orthonormal set and has adjoint

$$S^- = \sum S_{ji}^- e_i e_j^-$$

whose range is contained in the vector span of the orthonormal set is represented by a square matrix with entry

$$S_{ij}$$

in the i–th row and j–th column for $i, j = 1, \ldots, r$ and has adjoint represented by the conjugate transpose matrix with entry

$$S_{ji}^-$$

in the i–th row and j–th column for $i, j = 1, \ldots, r$. The trace

$$\operatorname{spur} S = S_{11} + \ldots + S_{rr}$$

of the operator S is a sum which is independent of the choice of orthonormal set. The trace

$$\operatorname{spur} S^- = S_{11}^- + \ldots + S_{rr}^-$$

of the adjoint operator is the complex conjugate

$$\operatorname{spur} S^- = (\operatorname{spur} S)^-$$

of the trace of the operator.

If T is an operator and if

$$S = ab^-$$

is an operator whose range has dimension at most one, then the composed operator

$$TS = Tab^-$$

has range of dimension at most one. Since

$$S^- = ba^-$$

has range of dimension at most one, the composed operator

$$S^-T^- = ba^-T^-$$

has range of dimension at most one. The trace

$$\operatorname{spur}(S^-T^-) = a^-T^-b$$

is the complex conjugate of the trace

$$\operatorname{spur}(TS) = b^-Ta.$$

If T is an operator and if S is an operator of finite dimensional range, then the composed operator TS has range of no larger dimension. Since the adjoint operator has range of the same dimension as the dimension of range of S^-, the composed operator S^-T^- has range of no larger dimension. The traces of the composed operators TS and S^-T^- are complex conjugates since the operators are adjoints of each other.

The operator norm

$$|S| = \sup |b^-Sa|$$

of an operator S is defined as a least upper bound taken over all vectors a and b of norm

$$|a| \leq 1$$

and

$$|b| \leq 1$$

at most one.

The norm of a nonzero operator is positive. The identity

$$|wT| = |w||T|$$

holds for every complex number w if T is an operator.

The inequality

$$|S + T| \leq |S| + |T|$$

holds for all operators S and T. A contractive operator is an operator T whose operator norm

$$|T| \leq 1$$

is not greater than one. A contractive operator T does not increase metric distances: The inequality

$$|Tb - Ta| \leq |b - a|$$

holds for all vectors a and b.

The trace norm of an operator S of finite dimensional range is defined as the least upper bound

$$\|S\| = \sup |\operatorname{spur}(TS)|$$

taken over all contractive operators T. Since the adjoint of a contractive operator is contractive, the adjoint of S^- of an operator S of finite dimensional range has the same trace norm

$$\|S^-\| = \|S\|$$

as S. The trace norm of a nonzero operator S of finite dimensional range is positive. The identity

$$\|wS\| = |w|\|S\|$$

holds for every complex number w if S is an operator of finite dimensional range. The inequality

$$\|S + T\| \leq \|S\| + \|T\|$$

holds for all operators S and T of finite dimensional range.

The trace class is defined as the completion of the space of operators of finite dimensional range in the metric topology defined by the trace norm. An operator is said to be completely continuous it is a limit in the metric topology defined by the operator norm of operators of finite dimensional range. Since the inequality

$$|T| \leq \|T\|$$

holds for every operator T of finite dimensional range, the trace class is contained in the space of completely continuous operators. The trace class is a vector space with norm which is the continuous extension of the trace norm.

If T is an operator, the function

$$\operatorname{spur}(TS)$$

of operators S of finite dimensional range admits a unique continuous extension as a function of operators S of trace class. The inequality

$$|\operatorname{spur}(TS)| \leq |T|\|S\|$$

holds for every operator S of trace class with $\|S\|$ defined by continuity in the trace class.

If T is an operator, the function

$$\operatorname{spur}(ST)$$

of operators S of finite dimensional range admits a unique continuous extension as a function of operators S of trace class. The inequality

$$|\operatorname{spur}(TS)| \leq |T|\|S\|$$

holds for every operator S of trace class with $\|S\|$ defined by continuity in the trace class.

If T if an operator, the function

$$\operatorname{spur}(ST)$$

of operators S of finite dimensional range admits a unique continuous extension as a function of operators S of trace class. The inequality

$$|\operatorname{spur}(ST)| \leq \|S\||T|$$

holds for every operator S of trace class. The compositions ST and TS of an operator S of trace class and an operator T are operators of trace class. The identity

$$\operatorname{spur}(S^- T^-) = \operatorname{spur}(ST)^-$$

holds for every operator S of trace class and every operator T.

The class of completely continuous operators is a vector space which is complete in the metric topology defined by the operator norm. Every linear functional on the space of completely continuous operators which is continuous for metric topology is defined by an operator S of trace class and takes a completely continuous operator T into

$$\operatorname{spur}(TS) = \operatorname{spur}(ST).$$

The Dedekind topology of the space of completely continuous operators is derived from a definition of closure for convex sets. The closure B^- of a convex set B of completely continuous operators is defined to be its closure for the metric topology defined by the operator norm. A convex set of completely continuous operators is defined to be open if it is disjoint from the closure to every disjoint convex set of completely continuous operators. A set of completely continuous operators is defined to be open if it is a union of open convex sets. A set of completely continuous operators is defined to be closed if it is the complement of an open set of completely continuous operators. A convex set is closed if, and only if, it is equal to its closure.

The space of completely continuous operators is a Hausdorff space in the Dedekind topology whose open and closed sets are defined from the closure operation on convex sets. The Hahn–Banach theorem applies in the Dedekind topology: If a nonempty open convex set A is disjoint from a nonempty convex set B, then B is contained in a closed convex set whose complement is convex and contains A. The Dedekind topology is identical with the metric topology defined by the operator norm since the space of completely continuous operators is complete in the operator norm.

The Dedekind topology of the space of trace class operators is derived from a closure operation on convex sets. The closure B^- of a convex set B of trace class operators is defined to be its closure for the metric topology defined by the trace norm. A convex set of trace class operators is defined to be open if it is disjoint from the closure of every disjoint convex set of trace class operators. A set of trace class operators is defined to be open if it is a union of open convex sets of trace class operators. A convex set of trace class operators is closed if, and only if, it is equal to its closure.

The trace class is a Hausdorff space in the topology whose open and closed sets are defined by the closure operation on convex sets. The Hahn–Banach theorem applies in the Dedekind topology: If a nonempty open convex set A is disjoint from a nonempty convex set B, then B is contained in a closed convex set whose complement is convex and contains A. The Dedekind topology is identical with the metric topology defined by the trace norm since the trace class is complete in the metric topology.

The Dedekind topology of the space of all operators is derived from a closure operation on convex sets. The closure B^- of a convex set B of operators is defined to be its closure for the metric topology defined by the operator norm. A convex set of operators is defined to be open if it is disjoint from the closure of every disjoint convex set of operators. A set of operators is defined to be open if it is a union of open convex sets of operators. A convex set of operators is closed if, and only if, it is equal to its closure.

The space of all operators is a Hausdorff space in the topology whose open and closed sets are defined by the closure operation on convex sets. The Hahn–Banach theorem applies in the Dedekind topology: If a nonempty open convex set A is disjoint from a nonempty convex set B, then B is contained in a closed convex set whose complement is convex and contains A. The Dedekind topology is identical with the metric topology defined by the operator norm since the space of all operators is complete in the metric topology.

The weak topology of the trace class is defined by the space of completely continuous operators. A basic open set for the weak topology is defined as a finite intersection of inverse images of open convex subsets of the complex plane under transformations of the trace class into the complex plane which take T into

$$\mathrm{spur}(TS)$$

for a completely continuous operator S. An open set is defined as a union of basic open sets. A closed set is defined as the complement of an open set.

The trace class is a Hausdorff space in the topology whose open and closed sets are defined by the space of completely continuous operators. The Hahn–Banach theorem applies in the weak topology: If a nonempty open convex set A is disjoint from a nonempty convex set B, then a closed convex set exists which contains B and whose complement is convex and contains A. A convex is closed for the weak topology if, and only if, its image in the complex plane is a closed convex set under the transformation which takes T into

$$\mathrm{spur}(TS)$$

for every completely continuous operator S.

A subset of the trace class is said to be bounded if its image in the complex plane is bounded under the transformation which takes T into

$$\mathrm{spur}(TS)$$

for every completely continuous operator S. A subset of the trace class is bounded if, and only if, the set of metric distances

$$\|B - A\|$$

between elements A and B of the set is bounded. A closed and bounded subset of the trace class is compact when the trace class is given the weak topology. The inclusion of the trace class given the Dedekind topology in the trace class given the weak topology is continuous.

The weak topology of the space of all operators is defined by the trace class. A basic open set for the weak topology is defined as a finite intersection of inverse images of open convex subsets of the complex plane under transformations of the space of all operators into the complex plane which take T into

$$\mathrm{spur}(TS)$$

for a trace class operator S. An open set is defined by a union of basic open sets. A closed set is defined as the complement of an open set.

The space of all operators is a Hausdorff space in the topology whose open and closed sets are defined by the trace class. The Hahn–Banach theorem applies in the weak topology: If a nonempty open convex set A is disjoint from a nonempty convex set B, then a closed convex set exists which contains B and whose complement is convex and contains A. A convex set is closed for the weak topology if, and only if, its image in the complex plane is a closed convex set under the transformation which takes T into

$$\mathrm{spur}(TS)$$

for every trace class operator S.

A subset of the space of all operators is said to be bounded if its image in the complex plane is bounded under the transformation which takes T into

$$\mathrm{spur}(TS)$$

for every trace class operator S. A subset of the space of all operators is bounded if, and only if, the set of metric distances

$$|B - A|$$

between elements A and B of the set is bounded. A closed and bounded subset of the space of all operators is compact when the space is given the weak topology. The inclusion of the space of all operators given the Dedekind topology in the space of all operators given the weak topology is continuous.

If S is a completely continuous operator, the norm

$$|S| = \sup |b^- Sa|$$

is a least upper bound taken over all vectors a and b of norm at most one which is achieved since the set of all contractive trace class operators whose range has dimension at most one is weakly compact. If a and b are vectors of norm at most one such that

$$|S| = |b^- Sa|$$

and if S does not vanish, then a and b are vectors of norm one which can be chosen so that

$$\lambda b = Sa$$

for a complex number λ such that

$$|\lambda| = |S|.$$

The operator

$$S - \lambda ba^-$$

annihilates a and has adjoint

$$S^- - \lambda^- a^- b$$

which annihilates b.

If S does not have finite dimensional range, a sequence of completely continuous operators S_n is defined inductively for nonnegative integers n starting with

$$S_0 = S.$$

When S_n is defined, vectors a_n and b_n of norm one are chosen so that

$$\lambda_n b_n = S_n a_n$$

for a complex number λ_n such that

$$|\lambda_n| = |S_n|.$$

An orthonormal set of vectors a_n and an orthonormal set of vectors b_n are defined for nonnegative integers n. A property of completely continuous operators implies that the numbers λ_n converge to zero.

Whenever an orthonormal set of vectors e_n is defined for nonnegative integers n, then the action

$$Se_n$$

of a completely continuous operator S on the vectors produces a sequence of vectors which converge to zero. The stated property is immediate when the operator has finite dimensional range. The property follows for every completely continuous operator since the operator is a limit in the metric topology defined by the operator norm of operators of finite dimensional range.

The structure of completely continuous operators determines the structure of trace class operators since every operator which is of trace class is completely continuous. If a trace class operator S does not have finite dimensional range, then an orthonormal set of vectors a_n and an orthonormal set of vectors b_n are defined for nonnegative integers n so that

$$\lambda_n b_n = S a_n$$

for a complex number λ_n for every n and such that

$$\|S\| = \sum |\lambda_n|.$$

A compact convex set is the closed convex span of its extreme points by the Krein–Milman theorem. Examples of compact convex sets are found in the space of all operators given the weak topology.

An extreme point of a convex set is an element c of the set which is not a convex combination

$$c = (1-t)a + tb$$

of distinct elements a and b of the set with t and $1-t$ positive.

The image of a compact convex set B of operators is a compact convex subset of the complex plane under the transformation which takes T into

$$\operatorname{spur}(TS)$$

for every trace class operator S. The image of an extreme point of B is an extreme point of the image of B.

The elements of B which are mapped into an extreme point of the image of B have properties which are taken as the definition of a face of B. A face of B is a nonempty closed convex subset of B such that elements a and b of B belong to the subset whenever some convex combination

$$(1-t)a + tb$$

with t and $1-t$ positive belongs to the subset.

A nonempty compact convex set has a face since the full set meets the definition of a face. An intersection of faces is a face if it is nonempty. If a nonempty class of faces has the property that every finite intersection is nonempty, then the intersection of all members of the class is nonempty. Every face contains a minimal face.

A face which contains more than one element is not minimal since the image of the face contains more than one element under the transformation which takes T into

$$\operatorname{spur}(TS)$$

for some trace class operator S. The unique element of a minimal face of a nonempty compact convex set is an extreme point of the set.

If an operator does not belong to the closed convex span of the extreme points of a nonempty compact convex set, then the image of the operator in the complex plane does not belong to the closed convex span of the images of extreme points under the transformation which takes T into

$$\operatorname{spur}(TS)$$

for some trace class operator S. Such an operator does not belong to the compact convex set since otherwise an extreme point can be constructed which has not previously been observed.

An application of the Krein–Milman theorem is made to the existence of invariant subspaces of a contractive transformation of a Hilbert space into itself which has an isometric adjoint. Since the coefficient space is an unrestricted Hilbert space, it can be assumed without loss of generality that the space is a Herglotz space $\mathcal{L}(\phi)$ and that the transformation takes $f(z)$ into $[f(z) - f(0)]/z$. The kernel of the transformation is an invariant subspace which is of interest when it contains a nonzero element. When the kernel contains no nonzero element, an isometric transformation of ext $\mathcal{L}(\phi)$ onto $\mathcal{L}(\phi)$ is defined by taking a Laurent series into the power series which has the same coefficient of z^n for every nonnegative integer n. The transformation of the space $\mathcal{L}(\phi)$ into itself which takes $f(z)$ into $[f(z) - f(0)]/z$ is unitarily equivalent to the transformation of ext $\mathcal{L}(\phi)$ into itself which takes $f(z)$ into $z^{-1}f(z)$.

The Krein–Milman theorem is applied to the construction of closed subspaces of ext $\mathcal{L}(\phi)$ which are invariant under multiplication by z and under division by z. The Herglotz space $\mathcal{L}(\phi)$ is arbitrary. The Herglotz spaces which are contained contractively in the space $\mathcal{L}(\phi)$ form a convex set. An extreme point of the set is a Herglotz space $\mathcal{L}(\psi)$ such that ext $\mathcal{L}(\psi)$ is contained isometrically in ext $\mathcal{L}(\phi)$. A topology is defined on the convex set to meet the compactness hypothesis of the Krein–Milman theorem.

If a Hilbert space $\mathcal{P}$ is contained contractively in a Hilbert space $\mathcal{H}$, the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$ is a transformation of $\mathcal{H}$ into $\mathcal{P}$ which is treated as a transformation P of $\mathcal{H}$ into itself on composition with the inclusion of $\mathcal{P}$ in $\mathcal{H}$. The self–adjoint transformation P is nonnegative and contractive.

A nonnegative and contractive transformation P of $\mathcal{H}$ into itself coincides with the adjoint of the inclusion in $\mathcal{H}$ of a Hilbert space $\mathcal{P}$ which is contained contractively in $\mathcal{H}$. The range of the transformation P is dense in the space $\mathcal{P}$. The identity

$$\langle Pa, Pb\rangle_{\mathcal{P}} = \langle a, Pb\rangle_{\mathcal{H}}$$

defines the scalar product of those elements of $\mathcal{P}$ which belong to the range of P. The space $\mathcal{P}$ is the completion of the range of P in the metric topology defined by the norm. The scalar product of $\mathcal{P}$ is defined by continuity.

Hilbert spaces which are contained contractively in a Hilbert space $\mathcal{H}$ are in one–to–one correspondence with nonnegative and contractive transformations of $\mathcal{H}$ into itself. The correspondence is a homomorphism of convex structure: If Hilbert spaces $\mathcal{P}$ and $\mathcal{Q}$ are contained contractively n $\mathcal{H}$ and if the Hilbert space

$$(1-t)\mathcal{P} + t\mathcal{Q}$$

is a convex combination of $\mathcal{P}$ and $\mathcal{Q}$, then the adjoint of the inclusion of the space in $\mathcal{H}$ coincides with

$$(1-t)P + tQ$$

with P the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$ and Q the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$.

The adjoint of the inclusion of $\mathcal{H}$ in itself is multiplication by the number one. Multiplication by the number zero coincides with the adjoint of the inclusion in $\mathcal{H}$ of the Hilbert space containing no nonzero element. If a nonnegative and contractive transformation P coincides with the adjoint of the inclusion of a Hilbert space $\mathcal{P}$ in $\mathcal{H}$, then the nonnegative and contractive transformation $1 - P$ coincides with the adjoint of the inclusion in $\mathcal{H}$ of the complementary space to $\mathcal{P}$ in $\mathcal{H}$.

The set of all Hilbert spaces which are contained contractively in a Hilbert space $\mathcal{H}$ is a compact convex set when given the topology obtained from the weak topology of the space of all nonnegative and contractive transformations of $\mathcal{H}$ into itself. The topology of transformations is transferred to a topology of spaces by treating the isomorphism of convex structure as a homeomorphism.

The convex set of all Hilbert spaces which are contained contractively in a Hilbert space $\mathcal{H}$ is the closed convex span of its extreme points by the Krein–Milman theorem. A Hilbert space which is contained contractively in $\mathcal{H}$ is an extreme point of the convex set if, and only if, it is contained isometrically in $\mathcal{H}$. The identity

$$P^2 = P$$

characterizes the adjoint P of the inclusion in $\mathcal{H}$ of a Hilbert space $\mathcal{P}$ which is contained isometrically in $\mathcal{H}$. The transformation is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{P}$.

Another application of the Krein–Milman theorem is given to the convex set of all Herglotz spaces which are contained contractively in a Herglotz space $\mathcal{L}(\phi)$. The convex set is compact since it is closed subset of a compact set. The convex set is the closed convex span of its extreme points. The extreme points are the Herglotz spaces $\mathcal{L}(\psi)$ such that ext $\mathcal{L}(\psi)$ is contained isometrically in ext $\mathcal{L}(\phi)$. Closed subspaces of a Hilbert space which are invariant subspaces for an isometric transformation of the space into itself and for its inverse are obtained by a computation of Herglotz spaces $\mathcal{L}(\psi)$ contained contractively in a Herglotz space $\mathcal{L}(\phi)$ such that ext $\mathcal{L}(\psi)$ is contained isometrically in ext $\mathcal{L}(\phi)$.

A continuous function $f(w)$ of ω in the unit circle is parametrized by the function

$$\omega = \exp(it)$$

of t in the real line. A Herglotz space $\mathcal{L}(\psi)$ which is contained contractively in the space $\mathcal{L}(\phi)$ is constructed when the function $f(\omega)$ of ω takes its values in the interval $[0, 1]$. A construction of ext $\mathcal{L}(\psi)$ is made in ext $\mathcal{L}(\phi)$.

A Laurent series

$$2\pi(1+r)h_r(z) = \sum \int (\tfrac{1}{2}\ e^{-it}z + \tfrac{1}{2}e^{it}z^{-1})^n f(e^{it})dt$$

is defined for every nonnegative integer r as a sum from n equal to zero to n equal r of Stieltjes integrals over the unit circle. The coefficient of z^n in $h_r(z)$ vanishes when n is greater than r or less than $-r$. The Laurent series represents a function whose values on the unit circle belong to the interval $[0, 1]$.

A polynomial $U_r(z)$ of degree at most r exists such that multiplication by $U_r(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself and such that

$$h_r(z) = U_r(z)U_r^*(z^{-1}).$$

Since the polynomial has complex numbers as coefficients, multiplication by $U_r(z)$ is a partially isometric transformation of ext $\mathcal{L}(\phi)$ onto ext $\mathcal{L}(\psi_r)$ for a Herglotz space $\mathcal{L}(\psi_r)$ which is contained in the Herglotz space $\mathcal{L}(\phi)$.

A polynomial $V_r(z)$ of degree at most r exists such that multiplication by $V_r(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself and such that

$$1 - h_r(z) = V_r(z)V_r^*(z^{-1}).$$

Since the polynomial has complex coefficients, multiplication by $V_r(z)$ is a partially isometric transformation of ext $\mathcal{L}(\phi)$ onto ext $\mathcal{L}(\theta_r)$ for a Herglotz space $\mathcal{L}(\theta_r)$ which is contained contractively in the Herglotz space $\mathcal{L}(\phi)$.

The Herglotz spaces $\mathcal{L}(\psi_r)$ and $\mathcal{L}(\theta_r)$ are complementary spaces in the space $\mathcal{L}(\phi)$ since the identity

$$1 = U_r(z)U_r^*(z^{-1}) + V_r(z)V_r^*(z^{-1})$$

implies that ext $\mathcal{L}(\psi_r)$ and ext $\mathcal{L}(\theta_r)$ are contained contractively in ext $\mathcal{L}(\phi)$ as complementary spaces to each other. The adjoint of the inclusion of ext $\mathcal{L}(\psi_r)$ in ext $\mathcal{L}(\phi)$ coincides with multiplication by

$$U_r(z)U_r^*(z^{-1})$$

as a transformation of ext $\mathcal{L}(\phi)$ into itself. The adjoint of the inclusion of ext $\mathcal{L}(\theta_r)$ in ext $\mathcal{L}(\phi)$ coincides with multiplication by

$$V_r(z)V_r^*(z^{-1})$$

as a transformation of ext $\mathcal{L}(\phi)$ into itself.

A Laurent series

$$h(z) = \lim h_r(z)$$

with complex coefficients is defined formally as a limit in which the coefficient of z^n on the left is a limit of coefficients of z^n on the right for every integer n. If

$$h(z) = \sum \lambda_n z^n,$$

then

$$2\pi\lambda_n = \int e^{-int} f(e^{it}) dt$$

is a Stieltjes integral over the unit circle for every integer n. A Herglotz space $\mathcal{L}(\psi)$ which is contained contractively in the Herglotz space $\mathcal{L}(\phi)$ exists such that the adjoint of the inclusion of ext $\mathcal{L}(\psi)$ in ext $\mathcal{L}(\phi)$ coincides with multiplication by $h(z)$ as a transformation of ext $\mathcal{L}(\phi)$ into itself.

The continuous functions $f(\omega)$ of w in the unit circle whose values are taken in the interval $[0, 1]$ form a convex set in which convex combinations of functions are defined by convex combinations of function values. The parametrization of Herglotz spaces which are contained contractively in the Herglotz space $\mathcal{L}(\phi)$ is a homomorphism of convex structure. The space $\mathcal{L}(\phi)$ is parametrized by the function which is identically one. The function which is identically zero parametrizes a Herglotz space which contains no nonzero element.

If $f(\omega)$ and $g(\omega)$ are continuous functions of ω in the unit circle whose values are taken in the interval $[0, 1]$ and which satisfy the inequality

$$f(\omega) \leq g(\omega)$$

for every element ω of the unit circle, then the Herglotz space parametrized by the function $f(\omega)$ of ω is contained contractively in the Herglotz space parametrized by the function $g(\omega)$ of ω.

A Herglotz space $\mathcal{L}(\psi)$ which is contained contractively in the Herglotz space $\mathcal{L}(\phi)$ is parametrized by a closed subset C of the unit circle. The space $\mathcal{L}(\psi)$ is contained contractively in every Herglotz space parametrized by a continuous function with value one at every element of C. When a Herglotz space $\mathcal{L}(\theta)$ is contained contractively in every Herglotz space parametrized by a continuous function with value one at every element of C, the space $\mathcal{L}(\theta)$ is contained contractively in the space $\mathcal{L}(\psi)$. The adjoint of the inclusion of ext $\mathcal{L}(\psi)$ in ext $\mathcal{L}(\phi)$ coincides with multiplication by the Laurent series

$$h(z) = \sum \lambda_n z^n$$

whose n–th coefficient

$$2\pi\lambda_n = \int e^{-int} f(e^{it}) dt$$

is defined as a Stieltjes integral over C for every integer n. The inclusion of ext $\mathcal{L}(\psi)$ in ext $\mathcal{L}(\phi)$ is isometric. The adjoint of the inclusion is an orthogonal projection.

The Baire class of subsets of the unit circle is defined as the smallest class of subsets which contains the closed sets, which contains the complement of every set of the class, and which contains every countable union of sets in the class.

A Herglotz space $\mathcal{L}(\psi)$ which is contained contractively in the Herglotz space $\mathcal{L}(\phi)$ is parametrized by every Baire subset C of the unit circle. The inclusion of ext $\mathcal{L}(\psi)$ in ext $\mathcal{L}(\phi)$ is isometric. The orthogonal projection of ext $\mathcal{L}(\phi)$ onto ext $\mathcal{L}(\psi)$ coincides with multiplication by the Laurent series

$$h(z) = \sum \lambda_n z^n$$

whose n–th coefficient

$$2\pi\lambda_n = \int e^{-int} f(e^{it}) dt$$

is defined as a Lebesgue integral over C for every integer n.

The spectrum of multiplication by z as a transformation of ext $\mathcal{L}(\phi)$ into itself is defined as the smallest closed subset of the unit circle which is assigned the orthogonal projection of ext $\mathcal{L}(\phi)$ onto itself. The orthogonal projection assigned to a Baire subset of the unit circle is identical with the orthogonal projection assigned to the intersection of the Baire set with the spectrum.

A Baire function is a function $f(\omega)$ of ω in the unit circle which is a pointwise limit of elements of a countable set of continuous functions. A function $f(\omega)$ of ω in the unit circle is a Baire function if, and only if, the inverse image of every closed subset of the complex plane is a Baire set.

If $f(\omega)$ is a Baire function of ω in the unit circle which is bounded on the spectrum, a continuous transformation of ext $\mathcal{L}(\phi)$ into itself is defined as a Lebesgue integral

$$\int f(\omega) dP(\omega)$$

of orthogonal projections P assigned to Baire subsets of the spectrum. The transformation commutes with every continuous transformation of ext $\mathcal{L}(\phi)$ into itself which commutes with multiplication by z. When the values of the function are taken in the interval $[0, 1]$, the transformation coincides with the adjoint of the inclusion in ext $\mathcal{L}(\phi)$ of ext $\mathcal{L}(\psi)$ for a Herglotz space $\mathcal{L}(\psi)$ which is contained contractively in the Herglotz space $\mathcal{L}(\phi)$. The transformation is multiplication by z when $f(\omega) = \omega$ and is division by z when $f(\omega) = \omega^{-1}$. If $f(z)$ is a polynomial in z, the transformation defined by the function $f(\omega)$ of ω is multiplication by $f(z)$. If $f(z)$ is a polynomial in z^{-1}, the transformation defined by the function $f(\omega)$ of ω is multiplication by $f(z)$.

Lebesgue integration computes the invariant subspaces whose existence is given by the Krein–Milman theorem. If a Herglotz space $\mathcal{L}(\psi)$ is contained contractively in a Herglotz space $\mathcal{L}(\phi)$ and if ext $\mathcal{L}(\psi)$ is defined as an integral of projections, then ext $\mathcal{L}(\psi)$ is an invariant subspace for every continuous transformation of ext $\mathcal{L}(\phi)$ into itself which commutes with multiplication by z. The restricted transformation of ext $\mathcal{L}(\psi)$ into itself is contractive if the transformation of ext $\mathcal{L}(\phi)$ into itself is contractive.

The construction of invariant subspaces for contractive transformations of a Hilbert space into itself resembles the construction made by Hilbert for transformations with isometric adjoint. Herglotz spaces are replaced by canonical models of contractive transformations.

If $W(z)$ is a power series with operator coefficients such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, then invariant subspaces are constructed for the contractive transformation of the space $\mathcal{H}(W)$ into itself which takes $f(z)$ into $[f(z) - f(0)]/z$ for every element $f(z)$ of the space.

If $P(z)$ is a power series with complex coefficients such that multiplication by $P(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, then the adjoint of multiplication by $P(z)$ acts as a contractive transformation of the space $\mathcal{H}(W)$ into itself which commutes with the transformation taking $f(z)$ into $[f(z) - f(0)]/z$.

The adjoint of multiplication by $P(z)$ acts as a partially isometric transformation of the space $\mathcal{H}(W)$ onto a Hilbert space $\mathcal{H}$ which is contained contractively in the space $\mathcal{H}(W)$. The adjoint of multiplication by $P(z)$ acts as a partially isometric transformation of the augmented space $\mathcal{H}(W')$,

$$W'(z) = zW(z)$$

onto a Hilbert space $\mathcal{H}'$ which is contained contractively in the space $\mathcal{H}(W')$. The space $\mathcal{H}'$ is the set of elements $f(z)$ of $\mathcal{C}(z)$ such that $[f(z) - f(0)]/z$ belongs to $\mathcal{H}$ with scalar product determined by the identity

$$\|[f(z) - f(0)]/z\|_{\mathcal{H}}^2 = \|f(z)\|_{\mathcal{H}'}^2 - |f(0)|^2.$$

The space $\mathcal{H}$ satisfies the inequality for difference quotients since it is contained contractively in the space $\mathcal{H}'$.

If $Q(z)$ is a power series with operator coefficients such that multiplication by $Q(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself and such that

$$Q(z)W(z) = W(z)Q(z),$$

then the adjoint of multiplication by $Q(z)$ acts as a contractive transformation of the space $\mathcal{H}(W)$ into itself. The adjoint of multiplication by $Q(z)$ acts as a contractive transformation of $\mathcal{H}$ into itself since

$$P(z)Q(z) = Q(z)P(z).$$

The existence of invariant subspaces for a contractive transformation of a Hilbert space into itself is an application of the Krein–Milman theorem.

Theorem 11. *If a contractive transformation of a Hilbert space into itself is not a scalar multiple of the identity transformation, then a closed subspace of the Hilbert space other than the least subspace and the greatest subspace exists which is an invariant subspace for every contractive transformation of the Hilbert space into itself which commutes with the given transformation.*

Proof of Theorem 11. Since a contractive transformation T of a Hilbert space $\mathcal{H}$ into itself can be multiplied by a positive number without change of its invariant subspaces, it can be assumed without loss of generality that the limit

$$\lim \|T^n h\|_{\mathcal{H}}$$

taken over the positive integers n vanishes for every element h of the space and that the same condition is satisfied when T is replaced by its adjoint T^*.

A closed subspace is an invariant subspace for every contractive transformation which commutes with T if, and only if, its orthogonal complement is an invariant subspace for every contractive transformation which commutes with T^*. Since T can be replaced by T^*, the dimension of the closure of the range of $1 - T^*T$ can be assumed less than or equal to the dimension of the closure of the range of $1 - TT^*$.

A coefficient space $\mathcal{C}$ is chosen whose dimension is equal to the dimension of the closure of the range of $1-T^*T$. The hypothesis on interates of T implies that the transformation is unitarily equivalent to the transformation which takes $f(z)$ into $[f(z)-f(0)]/z$ in a Hilbert space of power series with vector coefficients which satisfies the identity for difference quotients and is contained isometrically in $\mathcal{C}(z)$.

The space is a space $\mathcal{H}(W)$ for a power series $W(z)$ with operator coefficients such that multiplication by $W(z)$ is a partially isometric transformation of $\mathcal{C}(z)$ into itself. Since the dimension of the coefficient space is less than or equal to the dimension of the closure of the range of $1-TT^*$, the power series can be chosen so that multiplication by $W(z)$ annihilates no nonzero vector. Since the kernel of multiplication by $W(z)$ contains $[f(z) - f(0)]/z$ whenever it contains $f(z)$, multiplication by $W(z)$ is an isometric transformation of $\mathcal{C}(z)$ into itself.

The transformation of the space $\mathcal{H}(W)$ into the space $\mathcal{H}(W^*)$ whose graph is the space $\mathcal{D}(W)$ is injective by the hypothesis on the iterates of T^*. Since the space $\mathcal{D}(W^*)$ is the graph of a transformation of the space $\mathcal{H}(W^*)$ onto the space $\mathcal{H}(W)$, multiplication by $W^*(z)$ annihilates no nonzero vector. Since the space $\mathcal{H}(W^*)$ satisfies the identity for difference quotients, the space is contained isometrically in $\mathcal{C}(z)$. Since multiplication by $W^*(z)$ is a partially isometric transformation of $\mathcal{C}(z)$ into itself whose kernel contains no nonzero element, multiplication by $W^*(z)$ is an isometric transformation of $\mathcal{C}(z)$ into itself.

A compact convex set is constructed whose elements are Hilbert spaces which are contained contractively in the space $\mathcal{H}(W)$ and which satisfy the inequality for difference quotients. The adjoint of multiplication by $P(z)$ is required to act as a contractive transformation of a space into itself whenever $P(z)$ is a power series with operator coefficients such that multiplication by $P(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself such that the adjoint of multiplication by $P(z)$ acts as a contractive transformation of the space $\mathcal{H}(W)$ into itself.

The convex set is compact since it is a closed subset of the compact convex set of Hilbert spaces which are contained contractively in the space $\mathcal{H}(W)$. The convex set is closed since it is defined by contractive inclusions of spaces which are contained contractively in the space $\mathcal{H}(W)$. Contractiveness of an inclusion is tested by trace class operations. The inequality for difference quotients is tested as a contractive inclusion of a space in its augmented space.

Since the transformation of the space $\mathcal{H}(W)$ into itself which takes $f(z)$ into $[f(z) - f(0)]/z$ is by hypothesis not a scalar multiple of the identity transformation, it acts as a partially isometric transformation of the space $\mathcal{H}(W)$ onto a Hilbert space which belongs to the convex set and is not a convex combination of the least subspace and the greatest subspace of the space $\mathcal{H}(W)$. The convex set is the closed convex span of its extreme points by the Krein–Milman theorem since it is compact. An extreme point exists other than the least subspace and the greatest subspace of the space $\mathcal{H}(W)$.

An extreme point is a space which is contained isometrically in the space $\mathcal{H}(W)$ by Theorem 9 since multiplication by $W(z)$ and multiplication by $W^*(z)$ are isometric transformations of $\mathcal{C}(z)$ into itself. The space is an invariant subspace for every commuting transformation by Theorem 7.

This completes the proof of the theorem.

The relationship between factorization and invariant subspaces applies to a larger class of power series with operator coefficients. The power series are transfer functions of canonical linear systems whose state space is given an indefinite scalar product but whose topology is identical with that of a Hilbert space.

A Krein space is a vector space with scalar product which is the orthogonal sum of a Hilbert space and the anti–space of a Hilbert space. A Krein space is characterized as a vector space with scalar product which is self–dual for a norm topology.

Theorem 12. *A vector space with scalar product is a Krein space if it admits a norm which satisfies the convexity identity*

$$\|(1-t)a+tb\|^2+t(1-t)\|b-a\|^2=(1-t)\|a\|^2+t\|b\|^2$$

for all elements a and b of the space when $0<t<1$ and if the linear functionals on the space which are continuous for the metric topology defined by the norm are the linear functionals which are continuous for the weak topology induced by duality of the space with itself.

Proof of Theorem 12. Norms on the space are considered which satisfy the hypotheses of the theorem. The hypotheses imply that the space is complete in the metric topology defined by any such norm. If a norm $\|c\|_+$ is given for elements c of the space, a dual norm $\|c\|_-$ for elements c of the space is defined by the least upper bound

$$\|a\|_-=\sup|\langle a,b\rangle|$$

taken over the elements b of the space such that

$$\|b\|_+<1.$$

The least upper bound is finite since every linear functional which is continuous for the weak topology induced by self–duality is assumed continuous for the metric topology. Since every linear functional which is continuous for the metric topology is continuous for the weak topology induced by self–duality, the set of elements a of the space such that

$$\|a\|_-\le 1$$

is compact in the weak topology induced by self–duality. The set of elements a of the space such that
$$\|a\|_- < 1$$
is open for the metric topology induced by the plus norm. The set of elements b of the space such that
$$\|b\|_+ \leq 1$$
is compact in the weak topology induced by self–duality.

The convexity identity

$$\|(1-t)a + tb\|_+^2 + t(1-t)\|b-a\|_+^2 = (1-t)\|a\|_+^2 + t\|b\|_+^2$$

holds by hypothesis for all elements a and b of the space when $0 < t < 1$. It will be shown that the convexity identity

$$\|(1-t)u + tv\|_-^2 + t(1-t)\|v-u\|_-^2 = (1-t)\|u\|_-^2 + t\|v\|_-^2$$

holds for all elements u and v of the space when $0 < t < 1$. Use is made of the convexity identity
$$\begin{gathered}\langle (1-t)a + tb, (1-t)u + tv\rangle + t(1-t)\langle b-a, v-u\rangle \\ = (1-t)\langle a, u\rangle + t\langle b, v\rangle\end{gathered}$$
for elements $a, b, u,$ and v of the space when $0 < t < 1$. Since the inequality

$$\begin{gathered}|(1-t)\langle a, u\rangle + t\langle b, v\rangle| \\ \leq \|(1-t)a + tb\|_+\|(1-t)u + tv\|_- + t(1-t)\|b-a\|_+\|v-u\|_-\end{gathered}$$

holds by the definition of the minus norm, the inequality

$$\begin{gathered}|(1-t)\langle a, u\rangle + t\langle b, v\rangle|^2 \leq [\|(1-t)a + tb\|_+^2 + t(1-t)\|b-a\|_+^2] \\ \times[\|(1-t)u + tv\|_-^2 + t(1-t)\|v-u\|_-^2]\end{gathered}$$

is satisfied. The inequality

$$\begin{gathered}|(1-t)\langle a, u\rangle + t\langle b, v\rangle|^2 \leq [(1-t)\|a\|_+^2 + t\|b\|_+^2] \\ \times[\|(1-t)u + tv\|_-^2 + t(1-t)\|v-u\|_-^2]\end{gathered}$$

holds by the convexity identity for the plus norm. The inequality is applied for all elements a and b of the space such that the inequalities

$$\|a\|_+ \leq \|u\|_-$$

and

$$\|b\|_+ \leq \|v\|_-$$

are satisfied. The inequality

$$(1-t)\|u\|_-^2 + t\|v\|_-^2 \leq \|(1-t)u + tv\|_-^2 + t(1-t)\|v-u\|_-^2$$

follows by the definition of the minus norm. Equality holds since the reverse inequality is a consequence of the identities

$$(1-t)[(1-t)u + tv] + t[(1-t)u - (1-t)v] = (1-t)u$$

and

$$[(1-t)u + tv] - [(1-t)u - (1-t)v] = v.$$

It has been verified that the minus norm satisfies the hypotheses of the theorem. The dual norm to the minus norm is the plus norm. Another norm which satisfies the hypotheses of the theorem is defined by

$$\|c\|_t^2 = (1-t)\|c\|_+^2 + t\|c\|_-^2$$

when $0 < t < 1$. Since the inequalities

$$|\langle a, b\rangle| \leq \|a\|_+\|b\|_-$$

and

$$|\langle a, b\rangle| \leq \|a\|_-\|b\|_+$$

hold for all elements a and b of the space, the inequality

$$|\langle a, b\rangle| \leq (1-t)\|a\|_+\|b\|_- + t\|a\|_-\|b\|_+$$

holds when $0 < t < 1$. The inequality

$$|\langle a, b\rangle| \leq \|a\|_t\|b\|_{1-t}$$

follows for all elements a and b of the space when $0 < t < 1$. The inequality implies that the dual norm of the t norm is dominated by the $1-t$ norm. A norm which dominates its dual norm is obtained when $t = \frac{1}{2}$.

Consider the norms which satisfy the hypotheses of the theorem and which dominate their dual norms. Since a nonempty totally ordered set of such norms has a greatest lower bound, which is again such a norm, a minimal such norm exists by the Zorn lemma. If a minimal norm is chosen as the plus norm, it is equal to the t–norm obtained when $t = \frac{1}{2}$. It follows that a minimal norm is equal to its dual norm.

If a norm satisfies the hypotheses of the theorem and is equal to its dual norm, a related scalar product is introduced on the space which may be different from the given scalar product. Since the given scalar product assumes a subsidiary role in the subsequent argument, it is distinguished by a prime. A new scalar product is defined by the identity

$$4\langle a, b\rangle = \|a+b\|^2 - \|a-b\|^2 + i\|a+ib\|^2 - i\|a-ib\|^2.$$

The symmetry of a scalar product is immediate. Linearity will be verified.

The identity

$$\langle wa, wb\rangle = w^- w\langle a, b\rangle$$

holds for all elements a and b of the space if w is a complex number. The identity

$$\langle ia, b\rangle = i\langle a, b\rangle$$

holds for all elements a and b of the space. The identity

$$\langle ta, b\rangle = t\langle a, b\rangle$$

will be verified for all elements a and b of the space when t is a positive number. It is sufficient to verify the identity

$$\|ta + b\|^2 - \|ta - b\|^2 = t\|a + b\|^2 - t\|a - b\|^2$$

since a similar identity follows with b replaced by ib. The identity holds since

$$\|ta + b\|^2 + t\|a - b\|^2 = t(1 + t)\|a\|^2 + (1 + t)\|b\|^2$$

and

$$\|ta - b\|^2 + t\|a + b\|^2 = t(1 + t)\|a\|^2 + (1 + t)\|b\|^2$$

by the convexity identity.

If a, b, and c are elements of the space and if $0 < t < 1$, the identity

$$4\langle(1 - t)a + tb, c\rangle = \|(1 + t)(a + c) + t(b + c)\|^2$$
$$-\|(1 - t)(a - c) + t(b - c)\|^2 + i\|(1 - t)(a + ic) + t(b + ic)\|^2$$
$$-i\|(1 - t)(a - ic) + t(b - ic)\|^2$$

is satisfied with the right side equal to

$$(1 - t)\|a + c\|^2 + t\|b + c\|^2 - (1 - t)\|a - c\|^2 - t\|b - c\|^2$$
$$+i(1 - t)\|a + ic\|^2 + it\|b + ic\|^2 - i(1 - t)\|a - ic\|^2 - it\|b - ic\|^2$$
$$= 4(1 - t)\langle a, c\rangle + 4t\langle b, c\rangle.$$

The identity

$$\langle(1 - t)a + tb, c\rangle = (1 - t)\langle a, c\rangle + t\langle b, c\rangle$$

follows.

Linearity of a scalar product is now easily verified. Scalar self–products are nonnegative since the identity

$$\langle c, c\rangle = \|c\|^2$$

holds a for every element c of the space. A Hilbert space is obtained whose norm is the minimal norm. Since the inequality

$$|\langle a, b\rangle'| \leq \|a\|\|b\|$$

holds for all elements a and b of the space, a contractive transformation J of the Hilbert space into itself exists such that the identity

$$\langle a, b\rangle' = \langle Ja, b\rangle$$

holds for all elements a and b of the space. The symmetry of the given scalar product implies that the transformation J is self–adjoint. Since the Hilbert space norm is self–dual with respect to the given scalar product, the transformation J is also isometric with respect to the Hilbert space scalar product. The space is the orthogonal sum of the space of eigenvectors of J for the eigenvalue one and the space of eigenvectors of J for the eigenvalue minus one. These spaces are also orthogonal with respect to the given scalar product. They are the required Hilbert space and anti–space of a Hilbert space for the orthogonal decomposition of the vector space with scalar product to form a Krein space.

This completes the proof of the theorem.

The orthogonal decomposition of a Krein space is not unique since equivalent norms can be used. The dimension of the anti–space of a Hilbert space in the decomposition is however an invariant called the Pontryagin index of the Krein space. Krein spaces are a natural context for a complementation theory which was discovered in Hilbert spaces [5].

A generalization of the concept of orthogonal complement applies when a Krein space $\mathcal{P}$ is contained continuously and contractively in a Krein space $\mathcal{H}$. The contractive property of the inclusion means that the inequality

$$\langle a, a\rangle_{\mathcal{H}} \leq \langle a, a\rangle_{\mathcal{P}}$$

holds for every element a of $\mathcal{P}$. Continuity of the inclusion means that an adjoint transformation of $\mathcal{H}$ into $\mathcal{P}$ exists. A self–adjoint transformation P of $\mathcal{H}$ into $\mathcal{H}$ is obtained on composing the inclusion with the adjoint. The inequality

$$\langle Pc, Pc\rangle_{\mathcal{H}} \leq \langle Pc, Pc\rangle_{\mathcal{P}}$$

for elements c of $\mathcal{H}$ implies the inequality

$$\langle P^2c, c\rangle_{\mathcal{H}} \leq \langle Pc, c\rangle_{\mathcal{H}}$$

for elements c of $\mathcal{H}$, which is restated as an inequality

$$P^2 \leq P$$

for self–adjoint transformations in $\mathcal{H}$.

The properties of adjoint transformations are used in the construction of a complementary space $\mathcal{Q}$ to $\mathcal{P}$ in $\mathcal{H}$.

Theorem 13. *If a Krein space $\mathcal{P}$ is contained continuously and contractively in a Krein space $\mathcal{H}$, then a unique Krein space $\mathcal{Q}$ exists, which is contained continuously and contractively in $\mathcal{H}$, such that the inequality*

$$\langle c, c\rangle_{\mathcal{H}} \leq \langle a, a\rangle_{\mathcal{P}} + \langle b, b\rangle_{\mathcal{Q}}$$

holds whenever $c = a+b$ with a in $\mathcal{P}$ and b in $\mathcal{Q}$ and such that every element c of $\mathcal{H}$ admits some such decomposition for which equality holds.

Proof of Theorem 13. Define $\mathcal{Q}$ to be the set of elements b of $\mathcal{H}$ such that the least upper bound

$$\langle b, b\rangle_{\mathcal{Q}} = \sup[\langle a+b, a+b\rangle_{\mathcal{H}} - \langle a, a\rangle_{\mathcal{P}}]$$

taken over all elements a of $\mathcal{P}$ is finite. It will be shown that $\mathcal{Q}$ is a vector space with scalar product having the desired properties. Since the origin belongs to $\mathcal{P}$, the inequality

$$\langle b, b\rangle_{\mathcal{H}} \leq \langle b, b\rangle_{\mathcal{Q}}$$

holds for every element b of $\mathcal{Q}$. Since the inclusion of $\mathcal{P}$ in $\mathcal{H}$ is contractive, the origin belongs to $\mathcal{Q}$ and has self–product zero. If b belongs to $\mathcal{Q}$ and if w is a complex number, then wb is an element of $\mathcal{Q}$ which satisfies the identity

$$\langle wb, wb\rangle_{\mathcal{Q}} = w^{-}w\langle b, b\rangle_{\mathcal{Q}}.$$

The set $\mathcal{Q}$ is invariant under multiplication by complex numbers. The set $\mathcal{Q}$ is shown to be a vector space by showing that it is closed under convex combinations.

It will be shown that $(1-t)a+tb$ belongs to $\mathcal{Q}$ whenever a and b are elements of $\mathcal{Q}$ and t is a number, $0 < t < 1$. Since an arbitrary pair of elements of $\mathcal{P}$ can be written in the form $(1-t)a+tv$ and $v-u$ for elements u and v of $\mathcal{P}$, the identity

$$\begin{aligned}
&\langle (1-t)a+tb, (1-t)a+tb\rangle_{\mathcal{Q}} + t(1-t)\langle b-a, b-a\rangle_{\mathcal{Q}} \\
&= \sup[\langle (1-t)(a+u)+t(b+v), (1-t)(a+u)+t(b+v)\rangle_{\mathcal{H}} \\
&\quad +t(1-t)\langle (b+v)-(a+u), (b+v)-(a+u)\rangle_{\mathcal{H}} \\
&\quad -\langle (1-t)u+tv, (1-t)u+tv\rangle_{\mathcal{P}} - t(1-t)\langle v-u, v-u\rangle_{\mathcal{P}}]
\end{aligned}$$

holds with the least upper bound taken over all elements u and v of $\mathcal{P}$. By the convexity identity the least upper bound

$$\begin{aligned}
&\langle (1-t)a+tb, (1-t)a+tb\rangle_{\mathcal{Q}} + t(1-t)\langle b-a, b-a\rangle_{\mathcal{Q}} \\
&= \sup[\langle a+u, a+u\rangle_{\mathcal{H}} - \langle u, u\rangle_{\mathcal{P}}] + \sup[\langle b+v, b+v\rangle_{\mathcal{H}} - \langle v, v\rangle_{\mathcal{P}}]
\end{aligned}$$

holds over all elements u and v of $\mathcal{P}$. It follows that the identity

$$\begin{aligned}
&\langle (1-t)a+tb, (1-t)a+tb\rangle_{\mathcal{Q}} + t(1-t)\langle b-a, b-a\rangle_{\mathcal{Q}} \\
&= (1-t)\langle a, a\rangle_{\mathcal{Q}} + t\langle b, b\rangle_{\mathcal{Q}}
\end{aligned}$$

is satisfied.

This completes the verification that $\mathcal{Q}$ is a vector space. It will be shown that a scalar product is defined on the space by the identity

$$4\langle a, b\rangle_{\mathcal{Q}} = \langle a+b, a+b\rangle_{\mathcal{Q}} - \langle a-b, a-b\rangle_{\mathcal{Q}} + i\langle a+ib, a+ib\rangle_{\mathcal{Q}} - i\langle a-ib, a-ib\rangle_{\mathcal{Q}}.$$

Linearity and symmetry of a scalar product are verified as in the characterization of Krein spaces. The nondegeneracy of a scalar product remains to be verified.

Since the inclusion of $\mathcal{P}$ in $\mathcal{H}$ is continuous, a self–adjoint transformation P of $\mathcal{H}$ into itself exists which coincides with the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$. If c is an element of $\mathcal{H}$ and if a is an element of $\mathcal{P}$, the inequality

$$\langle a - Pc, a - Pc\rangle_{\mathcal{H}} \leq \langle a - Pc, a - Pc\rangle_{\mathcal{P}}$$

implies the inequality

$$\langle (1-P)c, (1-P)c\rangle_{\mathcal{Q}} \leq \langle c, c\rangle_{\mathcal{H}} - \langle Pc, Pc\rangle_{\mathcal{P}}.$$

Equality holds since the reverse inequality follows from the definition of the self–product in $\mathcal{Q}$. If b is an element of $\mathcal{Q}$ and if c is an element of $\mathcal{H}$, the inequality

$$\langle b - c, b - c\rangle_{\mathcal{H}} \leq \langle Pc, Pc\rangle_{\mathcal{P}} + \langle b - (1-P)c, b - (1-P)c\rangle_{\mathcal{Q}}$$

can be written

$$\langle b, b\rangle_{\mathcal{H}} - \langle b, c\rangle_{\mathcal{H}} - \langle c, b\rangle_{\mathcal{H}} \leq \langle b, b\rangle_{\mathcal{Q}} - \langle b, (1-P)c\rangle_{\mathcal{Q}} - \langle (1-P)c, b\rangle_{\mathcal{Q}}.$$

Since b can be replaced by wb for every complex number w, the identity

$$\langle b, c\rangle_{\mathcal{H}} = \langle b, (1-P)c\rangle_{\mathcal{Q}}$$

is satisfied. The nondegeneracy of a scalar product follows in the space $\mathcal{Q}$. The space $\mathcal{Q}$ is contained continuously in the space $\mathcal{H}$ since $1 - P$ coincides with the adjoint of the inclusion of $\mathcal{Q}$ in the space $\mathcal{H}$.

The intersection of $\mathcal{P}$ and $\mathcal{Q}$ is considered as a vector space $\mathcal{P} \wedge \mathcal{Q}$ with scalar product

$$\langle a, b\rangle_{\mathcal{P}\wedge\mathcal{Q}} = \langle a, b\rangle_{\mathcal{P}} + \langle a, b\rangle_{\mathcal{Q}}.$$

Linearity and symmetry of a scalar product are immediate, but nondegeneracy requires verification. If c is an element of $\mathcal{H}$,

$$P(1-P)c = (1-P)Pc$$

is an element of $\mathcal{P} \wedge \mathcal{Q}$ which satisfies the identity

$$\langle a, P(1-P)c\rangle_{\mathcal{P}\wedge\mathcal{Q}} = \langle a, c\rangle_{\mathcal{H}}$$

for every element a of $\mathcal{P} \wedge \mathcal{Q}$. Nondegeneracy of a scalar product in $\mathcal{P} \wedge \mathcal{Q}$ follows from nondegeneracy of the scalar product in $\mathcal{H}$. The space $\mathcal{P} \wedge \mathcal{Q}$ is contained continuously in the space $\mathcal{H}$. The self–adjoint transformation $P(1-P)$ in $\mathcal{H}$ coincides with the adjoint of the inclusion of $\mathcal{P} \wedge \mathcal{Q}$ in $\mathcal{H}$. The inequality

$$0 \le \langle c, c\rangle_{\mathcal{P} \wedge \mathcal{Q}}$$

holds for every element c of $\mathcal{P} \wedge \mathcal{Q}$ since the identity

$$0 = c - c$$

with c in $\mathcal{P}$ and $-c$ in $\mathcal{Q}$ implies the inequality

$$0 \le \langle c, c\rangle_{\mathcal{P}} + \langle c, c\rangle_{\mathcal{Q}}.$$

The intersection space is shown to be a Hilbert space by showing that it is complete in the metric topology defined by its norm. A Cauchy sequence of elements c_n of the intersection space converges in the weak topology of $\mathcal{P}$. The limit is an element c of $\mathcal{P}$ such that the identity

$$\langle c, a\rangle_{\mathcal{P}} = \lim \langle c_n, a\rangle_{\mathcal{P}}$$

holds for every element a of $\mathcal{P}$. The identity

$$\langle c, c\rangle_{\mathcal{P}} = \lim \langle c_n, c_n\rangle_{\mathcal{P}}$$

is satisfied. The identity

$$\langle c, a\rangle_{\mathcal{H}} = \lim \langle c_n, a\rangle_{\mathcal{H}}$$

holds for every element a of $\mathcal{H}$. The identity

$$\langle c, c\rangle_{\mathcal{H}} = \lim \langle c_n, c_n\rangle_{\mathcal{H}}$$

is satisfied.

Since the limit

$$\lim \langle c_n, b\rangle_{\mathcal{Q}}$$

exists for every element b of $\mathcal{Q}$ and since the limit

$$\lim \langle c_n, c_n\rangle_{\mathcal{Q}}$$

exists, c is an element of $\mathcal{Q}$ which is the limit of the elements c_n in the metric topology of $\mathcal{Q}$. The element c of the intersection space is the limit of the elements c_n of the intersection space in the metric topology of the space.

The Cartesian product of $\mathcal{P}$ and $\mathcal{Q}$ is isomorphic to the Cartesian product of $\mathcal{H}$ and $\mathcal{P} \wedge \mathcal{Q}$. If a is an element of $\mathcal{P}$ and if b is an element of $\mathcal{Q}$, a unique element c of $\mathcal{P} \wedge \mathcal{Q}$ exists such that the identity

$$\langle a - c, a - c\rangle_{\mathcal{P}} + \langle b + c, b + c\rangle_{\mathcal{Q}} = \langle a + b, a + b\rangle_{\mathcal{H}} + \langle c, c\rangle_{\mathcal{P} \wedge \mathcal{Q}}$$

is satisfied. Every element of the Cartesian product of $\mathcal{H}$ and $\mathcal{P} \wedge \mathcal{Q}$ is a pair $(a+b, c)$ for elements a of $\mathcal{P}$ and b of $\mathcal{Q}$ for such an element c of $\mathcal{P} \wedge \mathcal{Q}$. Since $\mathcal{H}$ is a Krein space and since $\mathcal{P} \wedge \mathcal{Q}$ is a Hilbert space, the Cartesian product of $\mathcal{P}$ and $\mathcal{Q}$ is a Krein space. Since $\mathcal{P}$ is a Krein space, it follows that $\mathcal{Q}$ is a Krein space.

The existence of a Krein space $\mathcal{Q}$ with the desired properties has now been verified. Uniqueness is proved by showing that a Krein space $\mathcal{Q}'$ with these properties is isometrically equal to the space $\mathcal{Q}$ constructed. Such a space $\mathcal{Q}'$ is contained contractively in the space $\mathcal{Q}$. The self–adjoint transformation $1 - P$ in $\mathcal{H}$ coincides with the adjoint of the inclusion of $\mathcal{Q}'$ in $\mathcal{H}$. The space $\mathcal{P} \wedge \mathcal{Q}'$ is a Hilbert space which is contained contractively in the Hilbert space $\mathcal{P} \wedge \mathcal{Q}$. Since the inclusion is isometric on the range of $P(1-P)$, which is dense in both spaces, the space $\mathcal{P} \wedge \mathcal{Q}'$ is isometrically equal to the space $\mathcal{P} \wedge \mathcal{Q}$. Since the Cartesian product of $\mathcal{P}$ and $\mathcal{Q}'$ is isomorphic to the Cartesian product of $\mathcal{P}$ and $\mathcal{Q}$, the spaces $\mathcal{Q}$ and $\mathcal{Q}'$ are isometrically equal.

This completes the proof of the theorem.

The space $\mathcal{Q}$ is called the complementary space to $\mathcal{P}$ in $\mathcal{H}$. The space $\mathcal{P}$ is recovered as the complementary space to the space $\mathcal{Q}$ in $\mathcal{H}$. The decomposition of an element c of $\mathcal{H}$ as $c = a + b$ with a an element of $\mathcal{P}$ and b an element of $\mathcal{Q}$ such that equality hold in the inequality

$$\langle c, c\rangle_{\mathcal{H}} \leq \langle a, a\rangle_{\mathcal{P}} + \langle b, b\rangle_{\mathcal{Q}}$$

is unique. The minimal decomposition results when a is obtained from c under the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$ and b is obtained from c under the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$.

A construction is made of complementary subspaces whose inclusion in the full space have adjoints coinciding with given self–adjoint transformations.

Theorem 14. *If a self–adjoint transformation P of a Krein space into itself satisfies the inequality*

$$P^2 \leq P,$$

then unique Krein spaces $\mathcal{P}$ and $\mathcal{Q}$ exist, which are contained continuously and contractively in $\mathcal{H}$ and which are complementary spaces in $\mathcal{H}$, such that P coincides with the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$ and $1 - P$ coincides with the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$.

Proof of Theorem 14. The proof repeats the construction of a complementary space under a weaker hypothesis. The range of P is considered as a vector space $\mathcal{P}'$ with scalar product determined by the identity

$$\langle Pc, Pc\rangle_{\mathcal{P}'} = \langle Pc, c\rangle_{\mathcal{H}},$$

for every element c of $\mathcal{H}$. The space $\mathcal{P}'$ is contained continuously and contractively in the space $\mathcal{H}$. The transformation P coincides with the adjoint of the inclusion of $\mathcal{P}'$ in $\mathcal{H}$. A Krein space $\mathcal{Q}$, which is contained continuously and contractively in $\mathcal{H}$, is defined as the set of elements b of $\mathcal{H}$ such that the least upper bound

$$\langle b, b\rangle_{\mathcal{Q}} = \sup[\langle a+b, a+b\rangle_{\mathcal{H}} - \langle a, a\rangle_{\mathcal{P}'}]$$

taken over all elements a of $\mathcal{P}'$ is finite. The adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$ coincides with $1 - P$. The complementary space to $\mathcal{Q}$ in $\mathcal{H}$ is a Krein space $\mathcal{P}$ which contains the space $\mathcal{P}'$ isometrically and which is contained continuously and contractively in $\mathcal{H}$. The adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$ coincides with $1 - P$.

This completes the proof of the theorem.

A factorization of continuous and contractive transformations in Krein spaces is an application of complementation theory.

Theorem 15. *The kernel of a continuous and contractive transformation T of a Krein space $\mathcal{P}$ into a Krein space $\mathcal{Q}$ is a Hilbert space which is contained continuously and isometrically in $\mathcal{P}$ and whose orthogonal complement in $\mathcal{P}$ is mapped isometrically onto a Krein space which is contained continuously and contractively in $\mathcal{Q}$.*

Proof of Theorem 15. Since the transformation T of $\mathcal{P}$ into $\mathcal{Q}$ is continuous and contractive, the self–adjoint transformation $P = TT^*$ in $\mathcal{Q}$ satisfies the inequality $P^2 \leq P$. A unique Krein space $\mathcal{M}$, which is contained continuously and contractively in $\mathcal{Q}$, exists such that P coincides with the adjoint of the inclusion of $\mathcal{M}$ in $\mathcal{Q}$. It will be shown that T maps $\mathcal{P}$ contractively into $\mathcal{M}$.

If a is an element of $\mathcal{P}$ and if b is an element of $\mathcal{Q}$, then

$$\langle Ta + (1-P)b, Ta + (1-P)b \rangle_{\mathcal{Q}}$$

$$= \langle T(a - T^*b), T(a - T^*b) \langle_{\mathcal{Q}} + \rangle b, b \rangle_{\mathcal{Q}} + \langle b, T(a - T^*b) \rangle_{\mathcal{Q}} + \langle T(a - T^*b), b \rangle_{\mathcal{Q}}$$

is less than or equal to

$$\langle a - T^*b, a - T^*b \rangle_{\mathcal{P}} + \langle b, b \rangle_{\mathcal{Q}} + \langle T^*b, a - T^*b \rangle_{\mathcal{P}} + \langle a - T^*b, T^*b \rangle_{\mathcal{P}}$$

$$= \langle a, a \rangle_{\mathcal{P}} + \langle (1 - TT^*)b, b \rangle_{\mathcal{Q}}.$$

Since b is an arbitrary element of $\mathcal{Q}$, Ta is an element of $\mathcal{M}$ which satisfies the inequality

$$\langle Ta, Ta \rangle_{\mathcal{M}} \leq \langle a, a \rangle_{\mathcal{P}}.$$

Equality holds when $a = T^*b$ for an element b of $\mathcal{Q}$ since

$$\langle TT^*b, TT^*b \rangle_{\mathcal{M}} = \langle TT^*b, b \rangle_{\mathcal{Q}} = \langle T^*b, T^*b \rangle_{\mathcal{P}}.$$

Since the transformation of $\mathcal{P}$ into $\mathcal{M}$ is continuous, the adjoint transformation is an isometry. The range of the adjoint transformation is a Krein space which is contained continuously and isometrically in $\mathcal{P}$ and whose orthogonal complement is the kernel of T. Since T is contractive, the kernel of T is a Hilbert space.

This completes the proof of the theorem.

A continuous transformation of a Krein space $\mathcal{P}$ into a Krein space $\mathcal{Q}$ is said to be a partial isometry if its kernel is a Krein space which is contained continuously and isometrically in $\mathcal{P}$ and whose orthogonal complement is mapped isometrically into $\mathcal{Q}$. A partially isometric transformation of a Krein space into a Krein space is contractive if, and only if, its kernel is a Hilbert space. Complementation is preserved under contractive partially isometric transformations of a Krein space onto a Krein space.

Theorem 16. *If a contractive partially isometric transformation T maps a Krein space $\mathcal{H}$ onto a Krein space $\mathcal{H}'$ and if Krein spaces $\mathcal{P}$ and $\mathcal{Q}$ are contained continuously and contractively as complementary subspaces of $\mathcal{H}$, then Krein spaces $\mathcal{P}'$ and $\mathcal{Q}'$, which are contained continuously and contractively as complementary subspaces of $\mathcal{H}'$, exist such that T acts as a contractive partially isometric transformation of $\mathcal{P}$ onto $\mathcal{P}'$ and of $\mathcal{Q}$ onto $\mathcal{Q}'$.*

Proof of Theorem 16. Since the Krein spaces $\mathcal{P}$ and $\mathcal{Q}$ are contained continuously and contractively in $\mathcal{H}$ and since T is a continuous and contractive transformation of $\mathcal{H}$ into $\mathcal{H}'$, T acts as a continuous and contractive transformation of $\mathcal{P}$ into $\mathcal{H}'$ and of $\mathcal{Q}$ into $\mathcal{H}'$. Krein spaces $\mathcal{P}'$ and $\mathcal{Q}'$, which are contained continuously and contractively in $\mathcal{H}'$, exist such that T acts as a contractive partially isometric transformation of $\mathcal{P}$ onto $\mathcal{P}'$ and of $\mathcal{Q}$ onto $\mathcal{Q}'$. It will be shown that $\mathcal{P}'$ and $\mathcal{Q}'$ are complementary subspaces of $\mathcal{H}'$.

An element a of $\mathcal{P}'$ is of the form Ta for an element a of $\mathcal{P}$ such that

$$\langle Ta, Ta \rangle_{\mathcal{P}'} = \langle a, a \rangle_{\mathcal{P}}.$$

An element b of $\mathcal{Q}'$ is of the form Tb for an element b of $\mathcal{Q}$ such that

$$\langle Tb, Tb \rangle_{\mathcal{Q}'} = \langle b, b \rangle_{\mathcal{Q}}.$$

The element $c = a + b$ of $\mathcal{H}$ satisfies the inequalities

$$\langle c, c \rangle_{\mathcal{H}} \leq \langle a, a \rangle_{\mathcal{P}} + \langle b, b \rangle_{\mathcal{Q}}$$

and

$$\langle Tc, Tc \rangle_{\mathcal{H}'} \leq \langle c, c \rangle_{\mathcal{H}}.$$

The element $Tc = Ta + Tb$ of $\mathcal{H}'$ satisfies the inequality

$$\langle Tc, Tc \rangle_{\mathcal{H}'} \leq \langle Ta, Ta \rangle_{\mathcal{P}'} + \langle Tb, Tb \rangle_{\mathcal{Q}'}.$$

An element of $\mathcal{H}'$ is of the form Tc for an element c of $\mathcal{H}$ such that

$$\langle Tc, Tc \rangle_{\mathcal{H}'} = \langle c, c \rangle_{\mathcal{H}}.$$

An element a of $\mathcal{P}$ and an element b of $\mathcal{Q}$ exist such that $c = a + b$ and

$$\langle c, c \rangle_{\mathcal{H}} = \langle a, a \rangle_{\mathcal{P}} + \langle b, b \rangle_{\mathcal{Q}}.$$

Since the element Ta of $\mathcal{P}'$ satisfies the inequality

$$\langle Ta, Tb \rangle_{\mathcal{P}'} \leq \langle a, a \rangle_{\mathcal{P}}$$

and since the element Tb of $\mathcal{Q}'$ satisfies the inequality

$$\langle Tb, Tb \rangle_{\mathcal{Q}'} \leq \langle b, b \rangle_{\mathcal{Q}},$$

the element Tc of $\mathcal{H}$ satisfies the inequality

$$\langle Tc, Tc\rangle_{\mathcal{H}'} \geq \langle Ta, Ta\rangle_{\mathcal{P}'} + \langle Tb, Tb\rangle_{\mathcal{Q}'}.$$

Equality holds since the reverse inequality is satisfied.

This completes the proof of the theorem.

A canonical linear system which is conjugate isometric and whose state space is a Krein space is constructed whose transfer function is a given power series $W(z)$ with operator coefficients when multiplication by $W(z)$ is densely defined as a transformation with domain and range in $\mathcal{C}(z)$.

A self–adjoint transformation H with domain and range in the Cartesian product $\mathcal{C}(z) \times \mathcal{C}(z)$ is defined by taking $(f(z), g(z))$ into $(u(z), v(z))$ when multiplication by $W(z)$ takes $f(z)$ into $v(z)$ and the adjoint of multiplication by $W(z)$ takes $g(z)$ into $u(z)$. The spectral subspace of contractivity for the self–adjoint transformation H is an invariant subspace in which the restriction of H is contractive and whose orthogonal complement is an invariant subspace in which the inverse of H is contractive. Eigenfunctions for the eigenvalues one and minus one are included in the spectral subspace of contractivity for the transformation H.

The existence and uniqueness of the spectral subspace of contractivity for the self–adjoint transformation H are given by the Hilbert construction of invariant subspaces for the isometric transformation

$$(H - w^-)^{-1}(H - w)$$

of the Cartesian product space $\mathcal{C}(z) \times \mathcal{C}(z)$ into itself when w is a nonreal number. The construction is independent of the choice of w.

An element of the spectral subspace of contractivity for H is the sum of an element $(f(z), g(z))$ for which $g(z)$ vanishes and an element $(f(z), g(z))$ for which $f(z)$ vanishes.

An element of the orthogonal complement of the spectral subspace of contractivity for H is the sum of an element $(f(z), g(z))$ for which $g(z)$ vanishes and an element $(f(z), g(z))$ for which $f(z)$ vanishes.

The spectral subspace of contractivity for multiplication by $W(z)$ is defined as the set of elements $f(z)$ of $\mathcal{C}(z)$ such that $(f(z), 0)$ belongs to the spectral subspace of contractivity for H.

The spectral subspace of contractivity for the adjoint of multiplication by $W(z)$ is the set of elements $g(z)$ of $\mathcal{C}(z)$ such that $(0, g(z))$ belongs to the spectral subspace of contractivity for H.

Since multiplication by z is an isometric transformation of $\mathcal{C}(z)$ into itself which commutes with multiplication by $W(z)$, the adjoint transformation which takes $f(z)$ into $[f(z) - f(0)]/z$ acts as a partially isometric transformation of the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$ into itself. The transformation which takes $f(z)$ into $[f(z) - f(0)]/z$ acts as a partially isometric transformation of the orthogonal complement of the spectral subspace of contractivity for multiplication by $W(z)$

onto itself. The spectral subspace of contractivity for multiplication by $W(z)$ contains $zf(z)$ for an element $f(z)$ of $\mathcal{C}(z)$ if, and only if, it contains $f(z)$. The orthogonal complement of the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$ contains $zf(z)$ for an element $f(z)$ of $\mathcal{C}(z)$ if, and only if, it contains $f(z)$.

Since multiplication by $W(z)$ is a contractive transformation of the spectral subspace of contractivity for multiplication by $W(z)$ into the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$, it acts as a partially isometric transformation of the spectral subspace of contractivity for multiplication by $W(z)$ onto a Hilbert space which is contained contractively in the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$. The complementary space in the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$ of the image of the spectral subspace of contractivity for multiplication by $W(z)$ is a Hilbert space $\mathcal{P}$ which is contained contractively in the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$.

Since the inverse of multiplication by $W(z)$ is an injective and contractive transformation of the orthogonal complement of the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$ into the orthogonal complement of the spectral subspace of contractivity for multiplication by $W(z)$, it acts as an isometric transformation of the orthogonal complement of the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$ onto a Hilbert space which is contained contractively in the orthogonal complement of the spectral subspace of contractivity for multiplication by $W(z)$. The complementary space in the orthogonal complement of the spectral subspace of contractivity for multiplication by $W(z)$ in the image of the orthogonal complement of the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$ is a Hilbert space $\mathcal{Q}$ which is contained contractively in the orthogonal complement of the spectral subspace of contractivity for multiplication by $W(z)$.

A transformation is defined of the set of elements of the spectral subspace of contractivity for multiplication by $W(z)$ which are orthogonal to the elements with constant coefficient zero into the space $\mathcal{Q}$ is defined by taking $f(z)$ into $[f(z) - f(0)]/z$. The elements of $\mathcal{Q}$ obtained are included isometrically in $\mathcal{C}(z)$.

A transformation is defined of the set of elements of the orthogonal complement of the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$ which are orthogonal to the elements with constant coefficient zero into the space $\mathcal{P}$ is defined by taking $f(z)$ into $[f(z) - f(0)]/z$. The elements of $\mathcal{P}$ obtained are included isometrically in $\mathcal{C}(z)$.

The augmented space $\mathcal{P}'$ is the set of elements $f(z)$ of $\mathcal{C}(z)$ such that $[f(z) - f(0)]/z$ belongs to $\mathcal{P}$ with scalar product determined by the identity for difference quotients

$$\|[f(z) - f(0)]/z\|_{\mathcal{P}}^2 = \|f(z)\|_{\mathcal{P}'}^2 - |f(0)|^2.$$

The space $\mathcal{P}$ is contained contractively in the space $\mathcal{P}'$. The elements of the orthogonal complement of the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$ which are orthogonal to elements with constant coefficient zero belongs to $\mathcal{P}'$ and are orthogonal to $\mathcal{P}$. They are elements of the complementary space to $\mathcal{P}$ in $\mathcal{P}'$ which are included isometrically in $\mathcal{C}(z)$ and whose orthogonal complement is the set of elements of the complementary space to $\mathcal{P}$ in $\mathcal{P}'$ which belong to the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$. Multiplication by $W(z)$ acts as a partially isometric transformation of the set of elements of the spectral subspace of contractivity for multiplication by $W(z)$ which are orthogonal to elements with constant coefficient zero onto the set of elements of the complementary space to $\mathcal{P}$ in $\mathcal{P}'$ which belong to the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$.

The augmented space $\mathcal{Q}'$ is the set of elements $f(z)$ of $\mathcal{C}(z)$ such that $[f(z) - f(0)]/z$ belongs to $\mathcal{Q}$ with scalar product determined by the identity for difference quotients

$$\|[f(z) - f(0)]/z\|_{\mathcal{Q}}^2 = \|f(z)\|_{\mathcal{Q}'}^2 - |f(0)|^2.$$

The space $\mathcal{Q}$ is contained contractively in the space $\mathcal{Q}'$. The elements of the spectral subspace of contractivity for multiplication by $W(z)$ which are orthogonal to elements with constant coefficient zero belong to $\mathcal{Q}'$ and are orthogonal to $\mathcal{Q}$. They are elements of the complementary space to $\mathcal{Q}$ in $\mathcal{Q}'$ which are included isometrically in $\mathcal{C}(z)$ and whose orthogonal complement is the set of elements of the complementary space to $\mathcal{Q}$ in $\mathcal{Q}'$ which belong to the orthogonal complement of the spectral subspace of contractivity for multiplication by $W(z)$. The inverse of multiplication by $W(z)$ acts as an isometric transformation of the set of elements of the orthogonal complement of the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$ which are orthogonal to elements with constant coefficient zero onto the set of elements of the complementary space to $\mathcal{Q}$ in $\mathcal{Q}'$ which belong to the orthogonal complement of the spectral subspace of contractivity for multiplication by $W(z)$.

Multiplication by $W(z)$ acts as a partially isometric transformation of the complementary space to $\mathcal{Q}$ in $\mathcal{Q}'$ onto the complementary space to $\mathcal{P}$ in $\mathcal{P}'$. An injective and contractive transformation of the complementary space to $\mathcal{Q}$ in $\mathcal{Q}'$ into the coefficient space is defined by taking a power series into its constant coefficient. Since the dimension of the complementary space to $\mathcal{Q}$ in $\mathcal{Q}'$ does not exceed the dimension of the coefficient space, a power series $V(z)$ with operator coefficients exists such that multiplication by $V(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself and such that multiplication by $V(z)$ acts as a partially isometric transformation of the coefficient space onto the complementary space to $\mathcal{Q}$ in $\mathcal{Q}'$. The space $\mathcal{Q}$ is isometrically equal to the state space $\mathcal{H}(V)$ of a canonical linear system which is conjugate isometric and has transfer function $V(z)$.

Multiplication by the power series

$$U(z) = W(z)V(z)$$

with operator coefficients is a contractive transformation of $\mathcal{C}(z)$ onto itself which acts as a partially isometric transformation of the coefficient space onto the complementary space to $\mathcal{P}$ in $\mathcal{P}'$. The space $\mathcal{P}$ is isometrically equal to the state space $\mathcal{H}(U)$ of a canonical linear system which is conjugate isometric and has transfer function $U(z)$.

The spaces $\mathcal{H}(U)$ and $\mathcal{H}(V)$ are applied in the construction of a canonical linear system which is conjugate isometric and whose transfer function is $W(z)$. The state space $\mathcal{H}(W)$ of the linear system is a Krein space.

Theorem 17. *If $W(z)$ is a power series with operator coefficients such that multiplication by $W(z)$ is densely defined as a transformation with domain and range in $\mathcal{C}(z)$, then $W(z)$ is the transfer function of a canonical linear system which is conjugate isometric and whose state space $\mathcal{H}(W)$ is a Krein space. Power series $V(z)$ and*

$$U(z) = W(z)V(z)$$

with operator coefficients exist such that multiplication by $U(z)$ and multiplication by $V(z)$ are contractive transformations of $\mathcal{C}(z)$ into itself, such that the space $\mathcal{H}(U)$ is contained contractively in the space $\mathcal{H}(W)$, and such that multiplication by $W(z)$ is an anti–isometric transformation of the space $\mathcal{H}(V)$ onto the orthogonal complement of the space $\mathcal{H}(U)$ in the space $\mathcal{H}(W)$.

Proof of Theorem 17. It will be shown that the Krein space $\mathcal{H}$ which is the orthogonal sum of the Hilbert space $\mathcal{P}$ and the anti–isometric image of the Hilbert space $\mathcal{Q}$ under multiplication by $W(z)$ is the state space of a canonical linear system which is conjugate isometric and has transfer function $W(z)$.

The augmented space $\mathcal{H}'$ is defined as the set of power series $f(z)$ with vector coefficients such that $[f(z) - f(0)]/z$ belongs to $\mathcal{H}$ with scalar product determined by the identity for difference quotients

$$\langle [f(z) - f(0)]/z, [f(z) - f(0)]/z \rangle_{\mathcal{H}} = \langle f(z), f(z) \rangle_{\mathcal{H}'} - |f(0)|^2.$$

The augmented space $\mathcal{H}'$ is a Krein space which is the orthogonal sum of the augmented Hilbert space $\mathcal{P}'$ and the anti–isometric image of the Hilbert space $\mathcal{Q}$ under multiplication by $zW(z)$.

The Hilbert space $\mathcal{P}$ is contained contractively in the Krein space $\mathcal{H}'$. The complementary space to $\mathcal{P}$ in $\mathcal{H}'$ is the orthogonal sum of a Hilbert space, which is the partially isometric image of the coefficient space under multiplication by

$$U(z) = W(z)V(z),$$

and the anti–isometric image of the Hilbert space $\mathcal{Q}$ under multiplication by $zW(z)$.

The partially isometric image of the coefficient space under multiplication by $U(z)$ is the partially isometric image under multiplication by $W(z)$ of the complementary space to $\mathcal{Q}$ in $\mathcal{Q}'$.

Multiplication by $W(z)$ is a partially anti–isometric transformation of the augmented Hilbert space $\mathcal{Q}'$ onto the complementary space to $\mathcal{P}$ in $\mathcal{H}'$ whose kernel is the complementary space to $\mathcal{Q}$ in $\mathcal{Q}'$. Multiplication by $W(z)$ is an anti–isometric transformation of $\mathcal{Q}$ into the complementary space to $\mathcal{P}$ in $\mathcal{H}$.

Multiplication by $W(z)$ acts as a partially isometric transformation of the coefficient space onto the complementary space to the image of $\mathcal{Q}$ in the complementary space to $\mathcal{P}$ in $\mathcal{H}'$. This verifies that the Krein space $\mathcal{H}$ is contained contractively in the augmented Krein space $\mathcal{H}'$ and that multiplication by $W(z)$ is a partially isometric transformation of the coefficient space onto the complementary space to $\mathcal{H}$ in $\mathcal{H}'$.

This completes the proof of the theorem.

Chapter 2. The Proof of the Bieberbach Conjecture

A complex valued function $f(z)$ of $z = x + iy$ in a region of the complex plane is said to be differentiable at an element w of the region if the function

$$[f(z) - f(w)]/(z - w)$$

is continuous at w when suitably defined at w. The value at w is taken as the definition of the derivative $f'(w)$ at w. A function is continuous at w if it is differentiable at w.

A square summable power series $f(z)$ with complex coefficients converges in the unit disk and defines a function in the unit disk. The value

$$f(w) = \langle f(z), (1 - w^- z)^{-1} \rangle$$

at w of the function represented by a square summable power series $f(z)$ is a scalar product in the space of square summable power series with the square summable power series

$$(1 - w^- z)^{-1} = 1 + (w^-)z + (w^-)^2 z^2 + \dots$$

The function represented by a square summable power series is continuous since the identity

$$f(\beta) - f(\alpha) = \langle f(z), (1 - \beta^- z)^{-1} - (1 - \alpha^- z)^{-1} \rangle$$

holds when α and β are in the unit disk and since the square summable power series

$$(1 - \beta^- z)^{-1} - (1 - \alpha^- z)^{-1} = (\beta - \alpha)^- z + (\beta^2 - \alpha^2)^- z^2 + \dots$$

satisfies the inequality

$$\|(1 - \beta^- z)^{-1} - (1 - \alpha^- z)^{-1}\|^2 \le |\beta - \alpha|^2 (1 + |\alpha + \beta|^2 + |\alpha^2 + \alpha\beta + \beta^2|^2 + \dots)$$

If $f(z)$ is a square summable power series, a sequence of square summable power series $f_n(z)$ is defined inductively by

$$f_0(z) = f(z)$$

and

$$f_{n+1}(z) = [f_n(z) - f_n(0)]/z$$

for every nonnegative integer n. Since the inequality

$$\|f_n(z)\| \le \|f(z)\|$$

holds for every nonnegative integer n, the square summable power series

$$[f(z) - f(\alpha)]/(z - \alpha) = f_1(z) + \alpha f_2(z) + \alpha^2 f_3(z) + \dots$$

is a sum in the metric topology of the space of square summable power series when α is in the unit disk. Since the power series represents a continuous function in the disk, the power series $f(z)$ represents a differentiable function in the disk. The function

$$[f(w) - f(\alpha)]/(w - \alpha)$$

of w in the disk is continuous at α when given a definition $f'(\alpha)$ at α.

Square summable power series which represent the same function are identical since the coefficients of a square summable power series are all zero if the function represented vanishes identically. A square summable power series is identified with the function it represents. The reproducing kernel function

$$(1 - w^- z)^{-1}$$

for function values at w in the space of square summable power series is the element of the space which in a scalar product determines the value of the represented function at w when w is in the unit disk.

If $W(z)$ is a nontrivial power series such that multiplication by $W(z)$ is a contractive transformation of the space of square summable power series into itself, then

$$W(z)W(w)^- /(1 - w^- z)$$

is the reproducing kernel function for function values at w in the range space $\mathcal{M}(W)$ when w is in the unit disk. For if

$$g(z) = W(z)f(z)$$

is an element of the space $\mathcal{M}(W)$, the identity

$$g(w) = \langle g(z), W(z)W(w)^- /(1 - w^- z)\rangle_{\mathcal{M}(W)}$$

is a consequence of the identity

$$f(w) = \langle f(z), (1 - w^- z)^{-1}\rangle$$

since multiplication by $W(z)$ is an isometric transformation of the space $\mathcal{C}(z)$ onto the space $\mathcal{M}(W)$ and since the identity

$$g(w) = W(w)f(w)$$

is satisfied. The reproducing kernel function

$$W(z)W(w)^- /(1 - w^- z)$$

for function values at w in the space $\mathcal{M}(W)$ is obtained from the reproducing kernel function

$$(1 - w^- z)^{-1}$$

for function values at w in the space of square summable power series under the adjoint of the inclusion of $\mathcal{M}(W)$ in $\mathcal{C}(z)$.

The reproducing kernel function

$$[1 - W(z)W(w)^-]/(1 - w^- z)$$

for function values at w in the space $\mathcal{H}(W)$ is obtained from the reproducing kernel function

$$(1 - w^- z)^{-1}$$

for function values at w in the space of square summable power series under the adjoint of the inclusion of the space $\mathcal{H}(W)$ in $\mathcal{C}(z)$. The identity

$$f(w) = \langle f(z), [1 - W(z)W(w)^-]/(1 - w^- z)\rangle_{\mathcal{H}(W)}$$

holds for every element $f(z)$ of the space $\mathcal{H}(W)$. Since the identity applies when

$$f(z) = [1 - W(z)W(w)^-]/(1 - w^- z),$$

the function represented by the power series $W(z)$ is bounded by one in the unit disk.

Reproducing kernel functions are applied to determine the structure of a Hilbert space $\mathcal{H}$ whose elements are functions in the unit disk. A continuous linear functional on the space is assumed to be defined for every element w of the unit disk by taking function values at w. The reproducing kernel function for function values at w is th unique element $K(w, z)$ of the space which represents the value

$$f(w) = \langle f(z), K(w, z)\rangle_{\mathcal{H}}$$

for every element $f(z)$ of the space. The indeterminate z is treated as a dummy variable in the notation for a function. The function

$$K(\alpha, \beta) = \langle K(\alpha, z), K(\beta, z)\rangle_{\mathcal{H}}$$

of α and β in the unit disk is treated as an infinite matrix. The symmetry of a scalar product implies the Hermitian symmetry

$$K(\beta, \alpha) = K(\alpha, \beta)^-$$

of the matrix. The infinite matrix is nonnegative in a sense which is determined by its finite square submatrices. If $\gamma_1, \dots, \gamma_r$ are in the unit disk, then the $r \times r$ matrix with entry

$$K(\gamma_i, \gamma_j)$$

in the i–th row and j–th column is nonnegative. A nonnegative number results when the matrix is multiplied on the right by a column vector with r entries and on the left by the conjugate transpose row vector. The nonnegative number is a sum of products

$$c_i^- K(\gamma_i, \gamma_j) c_j$$

taken over i and j equal to $1, \ldots, r$ for complex numbers $c_1, \ldots, c_r$.

Reproducing kernel functions are applied in interpolation. If $\gamma_1, \ldots, \gamma_r$ are distinct elements of disk, the set of elements of the Hilbert space which vanish at these elements is a closed vector subspace whose orthogonal complement consists of functions which are determined by their values at these elements. A function on the finite set is extended to the unit disk so as to be orthogonal to functions which vanish on the finite set. The space of functions on the finite set is a Hilbert space in the scalar product inherited from the full space. Every function on the finite set is a linear combination of reproducing kernel functions which represent values taken on the set. A reproducing kernel function for values on a set is its own extrapolation to the full space. The nonnegativity of a reproducing kernel function is the condition for the existence of a scalar product for the functions on the finite set which creates a Hilbert space compatible with the reproducing property. The finite linear combinations of reproducing kernel functions form a dense vector subspace of the Hilbert space of functions defined on the unit disk. The Hilbert space is the metric completion of the dense subspace. The reproducing property permits the elements of the completion to be realized as functions defined on the unit disk.

The Jordan curve theorem states that the complex complement of a simple closed curve in the complex plane is the union of a bounded region and an unbounded region. The Cauchy formula states that the Stieltjes integral

$$\int f(z)dz = 0$$

of a continuous function over the closed curve is equal to zero if the curve has finite length, if the function has a continuous extension to the closure of the bounded region, and if the function is differentiable at all but a finite number of elements of the bounded region. An example of a simple closed curve is the unit circle, which bounds the unit disk. The Cauchy formula for the unit circle is proved by decomposing the unit disk into regions which are bounded by circles centered at the origin and straight lines through the origin.

Points of nondifferentiability are constructed for a function $f(z)$ of z in the unit disk, which has a continuous extension to the closed disk, when the Cauchy integral

$$S(1) = \int_0^{2\pi} f(e^{i\theta}) ie^{i\theta} d\theta$$

for the unit circle is nonzero. A point of nondifferentiability is constructed in the annulus

$$a < |z| < b$$

when the inequality

$$(b-a)|S(1)| \leq |\int_0^{2\pi} f(be^{i\theta}) ibe^{i\theta} d\theta - \int_0^{2\pi} f(ae^{i\theta}) iae^{i\theta} d\theta|$$

is satisfied. If the length of an interval (α, β) is less than 2π, a simple closed curve is constructed from $ae^{i\alpha}$ to $be^{i\alpha}$ along a radial line away from the origin, from $be^{i\alpha}$ to $be^{i\beta}$

counterclockwise along a circle of radius b centered at the origin, from $be^{i\beta}$ to $ae^{i\beta}$ along a radial line towards the origin, and from $ae^{i\beta}$ to $ae^{i\alpha}$ clockwise along a circle of radius a about the origin. The Cauchy integral for the curve is

$$S(a,b;\alpha,\beta)$$
$$= \int_a^b f(re^{i\alpha})e^{i\alpha}dr - \int_a^b f(re^{i\beta})e^{i\beta}dr + \int_\alpha^\beta f(be^{i\theta})ibe^{i\theta}d\theta - \int_\alpha^\beta f(ae^{i\theta})iae^{i\theta}d\theta.$$

The Cauchy integral is zero for a linear function since it is zero for a constant and for z. The nonzero nature of the integral measures the difficulty in approximating the given function by a linear function.

A point of nondifferentiability is found in the region bounded by the curve when the inequality
$$(\beta-\alpha)(b-a)|S(1)| \le 2\pi|S(a,b;\alpha,\beta)|$$
is satisfied. A point w of nondifferentiability is obtained when the regions containing w and satisfying the inequality form a basis for the neighborhoods of w. If the inequality

$$|f(z)-g(z)| \le \epsilon|z-w|$$

holds in the region for some linear function $g(z)$ for a positive number ϵ, then

$$|S(1)| \le \epsilon$$

since the inequality
$$2\pi|S(a,b;\alpha,\beta)| \le (\beta-\alpha)(b-a)\epsilon$$
is satisfied.

The maximum principle states that the real part of a function $f(z)$ of z in the unit disk, which is differentiable at all but a finite number of points in the disk and which has a continuous extension to the closed disk, vanishes in the unit disk if it is nonpositive on the unit circle and nonnegative at the origin. The function $f(z)/z$ is differentiable at all but a finite number of points in the annulus

$$a < |z| < 1$$

when a is in the interval $(0,1)$. Since the identity

$$\int_0^{2\pi} f(ae^{i\theta})d\theta = \int_0^{2\pi} f(e^{i\theta})d\theta$$

holds by the proof of the Cauchy formula, the value of the function at the origin is an average
$$2\pi f(0) = \int_0^{2\pi} f(e^{i\theta})d\theta$$

of values on the boundary. If the real part of the integrand is nonpositive and real part of the integral is nonnegative, then the real part of the integral and the real part of the integrand are zero. The function is a constant since its real part vanishes in the unit disk.

An example of a function which is differentiable and bounded by one in the unit disk is

$$W(z) = (\alpha - z)/(1 - \alpha^- z)$$

when α is in the unit disk. A Hilbert space $\mathcal{H}$ of functions in the unit disk exists whose reproducing kernel function for function values at w is

$$[1 - W(z)W(w)^-]/(1 - w^- z) = (1 - \alpha^- \alpha)(1 - \alpha^- z)^{-1}(1 - \alpha w^-)^{-1}$$

when w is in the unit disk. The space is contained isometrically in the space of square summable power series since

$$(1 - \alpha^- z)^{-1}$$

is the reproducing kernel function for function values at α in $\mathcal{C}(z)$. The orthogonal complement of $\mathcal{H}$ in $\mathcal{C}(z)$ is a Hilbert space $\mathcal{M}$ which is contained isometrically in $\mathcal{C}(z)$ and which contains the functions which vanish at α. Since the reproducing kernel function for function values at w in $\mathcal{M}$ is

$$W(z)W(w)^-/(1 - w^- z),$$

multiplication by $W(z)$ is an isometric transformation of $\mathcal{C}(z)$ onto $\mathcal{M}$. Since $\mathcal{M}$ is contained isometrically in $\mathcal{C}(z)$, multiplication by $W(z)$ is an isometric transformation of $\mathcal{C}(z)$ into itself.

Applications of the maximum principle are made when a continuous function $W(z)$ of z in the unit disk is bounded by one and differentiable at all but a finite number of points in the disk. If the inequality

$$|W(\alpha)| < 1$$

holds for some α in the disk, then it holds for all α in the disk. If the inequality holds for a point α of differentiability, then a continuous function $W'(z)$ of z in the unit disk, which is bounded by one and differentiable at all but a finite number of points in the disk, is defined by the identity

$$W'(z)(\alpha - z)/(1 - \alpha^- z) = [W(\alpha) - W(z)]/[1 - W(\alpha)^- W(z)].$$

The identity is applied as a parametrization of the continuous functions $V(z)$, which are bounded by one in the unit disk and differentiable at all but a finite number of points in the disk, such that

$$V(\alpha) = W(\alpha).$$

Such a function is obtained on replacing $W(z)$ by $V(z)$ in the identity and replacing $W'(z)$ by a continuous function $V'(z)$ which is bounded by one in the unit disk and differentiable at all but a finite number of points in the disk.

If a continuous function $W(z)$ of z in the unit disk is bounded by one in the disk and is differentiable at all but a finite number of points in the disk and if a Hilbert space $\mathcal{H}$ exists whose elements are functions of z in the disk and which has the function

$$[1 - W(z)W(w)^-]/(1 - w^- z)$$

of z as reproducing kernel function for function values at w when w is in the unit disk, then multiplication by $W(z)$ is an isometric transformation of $\mathcal{C}(z)$ onto a Hilbert space $\mathcal{M}$ whose elements are functions of z in the unit disk and which has the function

$$W(z)W(w)^-/(1 - w^- z)$$

of z as reproducing kernel function for function values at w when w is in the unit disk. A Hilbert space $\mathcal{H} \vee \mathcal{M}$ exists in which the spaces $\mathcal{H}$ and $\mathcal{M}$ are contained contractively as complementary spaces. The elements of the space $\mathcal{H} \vee \mathcal{M}$ are functions defined in the unit disk. Since the reproducing kernel function for function values at w in the space $\mathcal{H} \vee \mathcal{M}$ is the sum of the reproducing kernel functions for function values at w in the spaces $\mathcal{H}$ and $\mathcal{M}$, the function

$$(1 - w^- z)^{-1}$$

of z is the reproducing kernel function for function values at w in the space $\mathcal{H} \vee \mathcal{M}$ when w is in the unit disk. The space $\mathcal{H} \vee \mathcal{M}$ is isometrically equal to $\mathcal{C}(z)$ since the space of square summable power series has the same reproducing kernel functions. Since the space $\mathcal{M}$ is contained contractively in $\mathcal{C}(z)$, multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. The function $W(z)$ is represented by a square summable power series. The space $\mathcal{H}$ is isometrically equal to the space $\mathcal{H}(W)$. The space $\mathcal{H}(W)$ is interpreted as $\mathcal{C}(z)$ when $W(z)$ is identically zero.

If a continuous function $U(z)$ of z in the unit disk is bounded by one and is differentiable at all but a finite number of points in the disk and if the inequality

$$|U(\alpha)| < 1$$

holds at a point α of the disk, then the continuous function

$$V(z) = [U(\alpha) - U(z)]/[1 - U(z)U(\alpha)^-]$$

of z is bounded by one in the disk and is differentiable at all but a finite number of points in the disk. Multiplication by $U(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself if, and only if, multiplication by $V(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. For a Hilbert space $\mathcal{H}(U)$ exists whose elements are functions of z in the disk and which contains the function

$$[1 - U(z)U(w)^-]/(1 - w^- z)$$

of z as reproducing kernel function for function values at w when w is in the disk if, and only if, a Hilbert space $\mathcal{H}(V)$ exists whose elements are functions of z in the disk and which contains the function

$$[1 - V(z)V(w)^-]/(1 - w^- z)$$

of z as reproducing kernel function for function values at w when w is in the disk. Since the identity

$$[1 - U(z)U(\alpha)^-][1 - V(z)V(w)^-][1 - U(\alpha)U(w)^-]$$
$$= [1 - U(\alpha)U(\alpha)^-][1 - U(z)U(w)^-]$$

is satisfied, multiplication by

$$[1 - U(\alpha)U(\alpha)^-]^{-\frac{1}{2}}[1 - U(z)U(\alpha)^-]$$

is an isometric transformation of the space $\mathcal{H}(V)$ onto the space $\mathcal{H}(U)$.

If a continuous function $U(z)$ of z in the disk is bounded by one and differentiable at all but a finite number of points in the disk and if

$$U(\alpha) = 0$$

at a point α of differentiability, then the identity

$$U(z) = V(z)(\alpha - z)/(1 - \alpha^- z)$$

holds for a continuous function $V(z)$ of z in the disk which is bounded by one and which is differentiable at all but a finite number of points in the disk. Multiplication by $U(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself if, and only if, multiplication by $V(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. A space $\mathcal{H}(U)$, whose elements are functions of z in the unit disk and which contains the function

$$[1 - U(z)U(w)^-]/(1 - w^- z)$$

of z as reproducing kernel function for function values at w when w is in the disk, exists if, and only if, a Hilbert space $\mathcal{H}(V)$ exists whose elements are functions of z in the disk and which contains the function

$$[1 - V(z)V(w)^-]/(1 - w^- z)$$

of z as reproducing kernel function for function values at w when w is in the disk. The space $\mathcal{H}(V)$ is contained isometrically in the space $\mathcal{H}(U)$ and contains the elements of the space $\mathcal{H}(U)$ which vanish at α.

If a continuous function $W(z)$ of z in the unit disk is bounded by one in the disk and is differentiable at all but a finite number of points in the disk and if $\alpha_1, \dots, \alpha_r$ are distinct points of differentiability in the disk, then continuous functions $W_n(z)$ of z in the disk, which are bounded by one in the disk and which are differentiable at all but a finite number of points in the disk, are defined inductively by

$$W_0(z) = W(z)$$

and

$$W_n(z)(\alpha_n - z)/(1 - \alpha_n^- z) = [W_{n-1}(\alpha_n) - W_{n-1}(z)]/[1 - W_{n-1}(z)W_{n-1}(\alpha_n)^-]$$

when n is positive and $W_{n-1}(z)$ is not a constant of absolute value one. A parametrization results of the continuous functions of z in the unit disk, which are bounded by one in the disk and which are differentiable at all but a finite number of points in the disk, having the same values as $W(z)$ at the points $\alpha_1, \ldots, \alpha_r$. Such functions are obtained on replacing $W_r(z)$ by an arbitrary continuous function of z which is bounded by one in the unit disk and which is differentiable at all but a finite number of points in the disk. A Hilbert space $\mathcal{H}(W)$, whose elements are functions of z in the disk and which contains the function

$$[1 - W(z)W(w)^-]/(1 - w^- z)$$

of z as reproducing kernel function for function values at w when w is in the disk, exists if, and only if, a Hilbert space $\mathcal{H}(W_r)$ exists whose elements are functions of z in the disk and which contains the function

$$[1 - W_r(z)W_r(w)^-]/(1 - w^- z)$$

of z as reproducing kernel function for function values at w when w is in the disk. If $W_r(z)$ is a constant of absolute value one, the space $\mathcal{H}(W_r)$ contains no nonzero element and the space $\mathcal{H}(W)$ has dimension r. The condition that the space $\mathcal{H}(W)$ has dimension at least r is necessary and sufficient for the construction of the function $W_r(z)$.

A theorem of Cauchy states that a continuous function of z in the unit disk is represented by a power series if it is differentiable at all but a finite number of points in the disk. If a continuous function $W(z)$ of z is bounded by one in the disk and is differentiable at all but a finite number of points in the disk, then multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. A proof is given by showing that for every finite set of distinct points $\alpha_1, \ldots, \alpha_r$ in the disk the matrix whose entry in the i–th row and j–th column is

$$[1 - W(\alpha_i)W(\alpha_j)^-]/(1 - \alpha_j^- \alpha_i)$$

is nonnegative. The conclusion is immediate when $\alpha_1, \ldots, \alpha_r$ are points of differentiability since multiplication by $V(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself for a power series $V(z)$ representing a function which agrees with $W(z)$ at the given points. The same conclusion holds by continuity when the points are not points of differentiability.

A function $f(z)$ of z is said to be analytic in the unit disk if it is represented by a power series. The Cauchy theorem states that a function $f(z)$ of z is analytic in the unit disk if it is continuous in the disk and is differentiable at all but a finite number of points in the disk.

A function $\phi(z)$ of z, which is analytic and has nonnegative real part in the unit disk, admits a Poisson representation. When the function is continuous in the closed disk, the integral representation

$$2\pi \, \frac{\phi(z) + \phi(w)^-}{1 - w^- z} = \int_0^{2\pi} \frac{\phi(e^{i\theta}) + \varphi(e^{i\theta})^-}{(1 - e^{-i\theta} z)(1 - w^- e^{i\theta})} \, d\theta$$

holds when z and w are in the unit disk. The Poisson representation is an application of the Cauchy integrals

$$2\pi\phi(z) = \int_0^{2\pi} \frac{\phi(e^{i\theta})d\theta}{1 - e^{-i\theta}z}$$

and

$$0 = \int_0^{2\pi} \frac{\phi(e^{i\theta})e^{i\theta}d\theta}{1 - w^- e^{i\theta}}.$$

When the function $\phi(z)$ of z is not continuous in the closed disk, a nonnegative measure μ on the Baire subsets of the real line is constructed whose value

$$\mu(E) = \lim \int_E \tfrac{1}{2}[\varphi(e^{ix-y}) + \varphi(e^{ix-y})^-]dx$$

is a limit as y decreases to zero of integrals of the real part of

$$\varphi(e^{ix-y}).$$

The Poisson representation reads

$$\pi \frac{\varphi(z) + \varphi(w)^-}{1 - w^- z} = \int_0^{2\pi} \frac{d\mu(e^{i\theta})}{(1 - e^{-i\theta}z)(1 - w^- e^{i\theta})}$$

when z and w are in the unit disk.

A Hilbert space is constructed whose elements are equivalence classes of Baire measurable functions $f(e^{i\theta})$ of $e^{i\theta}$ on the unit circle for which the integral

$$2\pi\|f\|^2 = \int_0^{2\pi} |f(e^{i\theta})|^2 d\mu(e^{i\theta})$$

is finite. A partially isometric transformation of the space onto the Herglotz space $\mathcal{L}(\phi)$ is defined by taking a function $f(e^{i\theta})$ of $e^{i\theta}$ on the unit circle into the function

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})d\mu(e^{i\theta})}{1 - e^{-i\theta}z}$$

of z in the unit disk. Multiplication by $e^{-i\theta}$ in the Hilbert space of functions on the boundary corresponds to the difference–quotient transformation in the Herglotz space. A related isometric transformation exists of the Hilbert space of functions on the unit circle onto the extension space of the Herglotz space. Multiplication by $e^{i\theta}$ in the Hilbert space of functions on the unit circle corresponds to multiplication by z in the extension space $\mathcal{E}(\phi)$ to the Herglotz space $\mathcal{L}(\phi)$.

A Riemann mapping function is a power series

$$f(z) = \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 + \ldots$$

with vanishing constant coefficient which represents an injective mapping of the unit disk into the complex plane.

The area theorem is the source of estimates of coefficients of Riemann mapping functions. Analyticity and injectivity imply a contractive property of composition in a Hilbert space whose elements are functions analytic in the unit disk.

An isomorphic Hilbert space $\mathcal{G}$ is the set of equivalence classes of power series

$$h(z) = c_0 + c_1 z + c_2 z^2 + \dots$$

such that the sum

$$\|h(z)\|_{\mathcal{G}}^2 = |c_1|^2 + 2|c_2|^2 + 3|c_3|^2 + \dots$$

converges. Power series are defined a equivalent if they have equal coefficients of z^n for every positive integer n. Representatives are chosen in equivalence classes with vanishing constant coefficient for the definition of analytic functions. An element of the space represents an analytic function $h(z)$ of z in the unit disk such that the integral

$$\pi\|h(z)\|_{\mathcal{G}}^2 = \iint |h'(z)|^2 dxdy$$

with respect to area measure in the unit disk computes the scalar self–product.

Contractive composition is obtained for a Riemann mapping function $f(z)$ which maps the unit disk onto a region which is contained in the unit disk. If

$$h(z) = c_0 + c_1 z + c_2 z^2 + \dots$$

is an element of the space $\mathcal{G}$,

$$g(z) = c_0 + c_1 f(z) + c_2 f(z)^2 + \dots$$

is an element of the space whose scalar self–product is computed by the integral

$$\pi\|g(z)\|_{\mathcal{G}}^2 = \iint |g'(z)|^2 dxdy$$

with respect to area measure for the unit disk. Since the chain rule

$$g'(z) = h'(f(z))f'(z)$$

applies to complex differentiation and since the mapping defined by $f(z)$ is injective, the change of variable theorem produces the integral

$$\pi\|g(z)\|_{\mathcal{G}}^2 = \iint |h'(z)|^2 dxdy$$

with respect to area measure over the region onto which $f(z)$ maps the unit disk. Since the region is contained in the unit disk, the integral

$$\pi\|h(z)\|_{\mathcal{G}}^2 - \|g(z)\|_{\mathcal{G}}^2 = \iint |h'(z)|^2 dxdy$$

with respect to area measure over the complement of the region in the unit disk verifies the contractive property of composition.

The Hilbert space $\mathcal{G}$ is contained isometrically in a Krein space ext $\mathcal{G}$ whose elements are equivalence classes of Laurent series. Laurent series are defined as equivalent if the coefficients of z^n are equal for every nonzero integer n. The orthogonal complement of the Hilbert space $\mathcal{G}$ in the Krein space ext $\mathcal{G}$ is the anti–space of a Hilbert space which is the anti–isometric image of $\mathcal{G}$ under the transformation which takes $f(z)$ into $f(z^{-1})$.

If $h(z)$ is an element of ext $\mathcal{G}$ whose coefficient of z^n vanishes for all but a finite number of negative integers n, then $h(z)$ represents a function which is analytic in the region obtained from the unit disk on deleting the origin. The composition

$$g(z) = h(f(z))$$

is an element of ext $\mathcal{G}$ whose coefficient of z^n vanishes for all but a finite number of negative integers n. The integral

$$\pi\langle h(z), h(z)\rangle_{\text{ext } \mathcal{G}} - \pi\langle g(z), g(z)\rangle_{\text{ext } \mathcal{G}} = \iint |g'(z)|^2 dxdy$$

with respect to area over the complement in the unit disk of the region onto which $f(z)$ maps the unit disk verifies the contractive property of composition on a dense set of elements of ext $\mathcal{G}$. The contractive property follows by continuity for all elements of ext $\mathcal{G}$.

A proof of the contractive property of composition in the Krein space is not essential at the outset since this property is taken as a hypothesis.

The Grunsky transformation is defined under hypotheses of contractivity. If $W(z)$ is a power series with vanishing constant coefficient such that a contractive transformation of the space $\mathcal{G}$ into itself is defined by taking $f(z)$ into $f(W(z))$, then the composition acts a a partially isometric transformation of the Hilbert space $\mathcal{G}$ onto a Hilbert space which is contained contractively in $\mathcal{G}$. The Grunsky space $\mathcal{G}(W)$ is defined as the Hilbert space which is the complementary space in $\mathcal{G}$ to the range of the transformation.

Elements of $\mathcal{G}$ define functions analytic in the unit disk when representatives with vanishing constant coefficient are chosen in equivalence classes. The reproducing kernel function for function values at w is the function

$$\log \frac{1}{1 - zw^-} = \frac{(zw^-)}{1} + \frac{(zw^-)^2}{2} + \frac{(zw^-)^3}{3} + \dots$$

of z when w is in the unit disk. The reproducing kernel function for function values at w in the range of the transformation is the function

$$\log \frac{1}{1 - W(z)W(w)^-} = \frac{[W(z)W(w)^-]}{1} + \frac{[W(z)W(w)^-]^2}{2} + \frac{[W(z)W(w)^-]^3}{3} + \dots$$

of z when w is in the unit disk. The reproducing kernel function for function values at w in the space $\mathcal{G}(W)$ is the function

$$K(w, z) = \log \frac{1 - W(z)W(w)^-}{1 - zw^-}$$

of z when w is in the unit disk.

Multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself since $W(z)$ represents a function which is analytic in the unit disk and which is bounded by one by the positivity properties of reproducing kernel functions. A relationship between the Grunsky space $\mathcal{G}(W)$ and the space $\mathcal{H}(W)$ of the invariant subspace construction is implied by the resemblance between reproducing kernel functions.

For every positive integer r a Hilbert space is constructed whose elements are functions of the complex variables $z_1, \dots, z_r$ in the unit disk for each variable. The reproducing kernel function at $w_1, \dots, w_r$ is the function

$$K(w_1, z_1) \dots K(w_r, z_r)$$

of $z_1, \dots, z_r$ for $w_1, \dots, w_r$ in the unit disk. A partially isometric transformation of the product space onto a Hilbert space $\mathcal{G}^r(W)$ whose elements are functions analytic in the unit disk is defined by taking a function $f(z_1, \dots, z_r)$ of $z_1, \dots, z_r$ into the function

$$f(z, \dots, z)$$

of z. The reproducing kernel function for function values at w in the space $\mathcal{G}^r(W)$ is the function

$$K(w, z)^r$$

of z when w is in the unit disk.

The complex numbers are a Hilbert space $\mathcal{G}^0(W)$ of functions analytic in the unit disk whose reproducing kernel function for function values at w is the function

$$1 = K(w, z)^0$$

of z in the unit disk when the scalar product is determined by the choice of absolute value as norm.

If an element $f_r(z)$ of the space $\mathcal{G}^r(W)$ is chosen for every nonnegative integer r, the sum

$$f(z) = f_0(z) + \frac{1}{1!}\ f_1(z) + \frac{1}{2!}\ f_2(z) + \dots$$

is an element of the space $\mathcal{H}(W)$ which satisfies the inequality

$$\|f(z)\|^2_{\mathcal{H}(W)} \leq \|f_0(z)\|^2_{\mathcal{G}^0(W)} + \frac{1}{1!}\ \|f_1(z)\|^2_{\mathcal{G}'(W)} + \frac{1}{2!}\|f_2(z)\|^2_{\mathcal{G}^2(W)} + \dots$$

whenever the sum converges. Every element $f(z)$ of the space $\mathcal{H}(W)$ admits a representation for which equality holds. If $f(z)$ is an element of the space $\mathcal{G}(W)$, then

$$\exp f(z)$$

is an element of the space $\mathcal{H}(W)$ which satisfies the inequality

$$\| \exp f(z)\|^2_{\mathcal{H}(W)} \leq \exp \|f(z)\|^2_{\mathcal{G}(W)}.$$

If $W(z)$ is a power series with vanishing constant coefficient such that a contractive transformation of the space $\mathcal{G}$ into itself is defined by taking $f(z)$ into $f(W(z))$, then

$$W^*(z) = W(z^-)^-$$

is a power series with vanishing constant coefficient such that a contractive transformation of the space $\mathcal{G}$ into itself is defined by taking $f(z)$ into $f(W^*(z))$. If a contractive transformation of the space ext $\mathcal{G}$ into itself is defined by taking $f(z)$ into $f(W(z))$, then a contractive transformation of the space ext $\mathcal{G}$ into itself is defined by taking $f(z)$ into $f(W^*(z))$.

The Grunsky transformation of the space $\mathcal{G}(W)$ into the space $\mathcal{G}(W^*)$ is defined when the composition $f(z)$ into $f(W(z))$ is contractive in ext $\mathcal{G}$.

Theorem 18. *If for a power series $W(z)$ with vanishing constant coefficient a contractive transformation of ext $\mathcal{G}$ into itself is defined by taking $f(z)$ into $f(W(z))$, then the function*

$$\log \frac{1 - W(w^-)/W(z)}{1 - w^-/z}$$

of z is represented by an element of the space $\mathcal{G}(W)$ and the function

$$\log \frac{1 - W^*(z)/W(w)^-}{1 - z/w^-}$$

of z is represented by an element of the space $\mathcal{G}(W^*)$ when w is in the unit disk. The Grunsky transformation is a contractive transformation of the space $\mathcal{G}(W)$ into the space $\mathcal{G}(W^*)$ which takes $f(z)$ into $g(z)$ when the identity

$$g(w) = \langle f(z), \log \frac{1 - W(w^-)/W(z)}{1 - w^-/z} \rangle_{\mathcal{G}(W)}$$

holds for w in the unit disk and whose adjoint is a contractive transformation of the space $\mathcal{G}(W^*)$ into the space $\mathcal{G}(W)$ which takes $f(z)$ into $g(z)$ when the identity

$$g(w) = \langle f(z), \log \frac{1 - W^*(z)/W(w)^-}{1 - z/w^-} \rangle_{\mathcal{G}(W^*)}$$

holds for w in the unit disk.

Proof of Theorem 18. Since a contractive transformation of ext $\mathcal{G}$ into itself is defined by taking $f(z)$ into $f(W(z))$, the transformation acts as a partially isometric transformation of ext $\mathcal{G}$ onto a Krein space which is contained contractively in ext $\mathcal{G}$. Since the transformation takes $\mathcal{G}$ contractively into itself, it acts as a partially isometric transformation of $\mathcal{G}$ onto a Hilbert space which is contained contractively in $\mathcal{G}$ and whose complementary space in ext $\mathcal{G}$ is the orthogonal sum of $\mathcal{G}(W)$ and the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$. The transformation acts as a partially isometric transformation of the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$ onto a Krein space $\mathcal{M}$ which is contained contractively in the orthogonal sum of the space $\mathcal{G}(W)$ and the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$.

An element

$$f(z) + g(z)$$

of $\mathcal{M}$ is the sum of an element $f(z)$ of the space $\mathcal{G}(W)$ and an element $g(z)$ of the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$ which satisfies the inequality

$$\|f(z)\|^2_{\mathcal{G}(W)} + \langle g(z), g(z)\rangle_{\text{ext } \mathcal{G}} \leq \langle f(z) + g(z), f(z) + g(z)\rangle_{\mathcal{M}}.$$

An anti–isometric transformation of $\mathcal{G}$ onto the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$ is defined by taking $f(z)$ into $f(z^{-1})$. The transformation takes

$$\log(1 - zw^-)^{-1}$$

into

$$\log(1 - w^-/z)^{-1}$$

when w is in the unit disk. Since the identity

$$f(w) = \langle f(z), \log(1 - zw^-)^{-1}\rangle_{\mathcal{G}}$$

holds for every element $f(z)$ of $\mathcal{G}$, the identity

$$f(1/w) = \langle f(z), \log(1 - w^-/z)\rangle_{\text{ext } \mathcal{G}}$$

holds for every element $f(z)$ of the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$. Since the function represented by $W(z)$ maps the unit disk into itself,

$$\log(1 - W(w^-)/z)$$

is an element of the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$ which satisfies the identity

$$f(1/W^*(w)) = \langle f(z), \log(1 - W(w^-)/z)\rangle_{\text{ext } \mathcal{G}}$$

for every element $f(z)$ of the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$.

Since a partially isometric transformation of the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$ onto $\mathcal{M}$ is defined by taking $f(z)$ into

$$g(z) = f(W(z)),$$

the element

$$\log(1 - W(w^-)/W(z))$$

of $\mathcal{M}$ satisfies the identity

$$f(1/W^*(w)) = \langle g(z), \log(1 - W(w^-)/W(z))\rangle_{\mathcal{M}}$$

for every element $g(z)$ of $\mathcal{M}$. The element $f(z)$ of the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$ is uniquely determined by its image $g(z)$ in $\mathcal{M}$.

Since the element

$$\log(1 - W(w^-)/W(z)) = \log \frac{1 - W(w^-)/W(z)}{1 - w^-/z} + \log\,(1 - w^-/z)$$

of $\mathcal{M}$ is the sum of an element of $\mathcal{G}$ and an element of the orthogonal complement of $\mathcal{G}$ in ext $\mathcal{G}$ and since the identities

$$\langle \log(1 - w^-/z), \log(1 - w^-/z)\rangle_{\text{ext } \mathcal{G}} = \log(1 - ww^-)$$

and

$$\langle \log(1 - W(w^-)/W(z)), \log(1 - W(w^-)/W(z))\rangle_{\mathcal{M}} = \log(1 - W(w^-)W^*(w))$$

are satisfied, the element

$$\log \frac{1 - W(w^-)/W(z)}{1 - w^-/z}$$

of $\mathcal{G}$ is an element of the space $\mathcal{G}(W)$ which satisfies the inequality

$$\| \log \frac{1 - W(w^-)/W(z)}{1 - w^-/z} \|^2_{\mathcal{G}(W)} \leq \log \frac{1 - W(w^-)W^*(w)}{1 - ww^-}$$

A contractive transformation of the space $\mathcal{G}^*(W)$ into the space $\mathcal{G}(W)$ exists which takes a finite linear combination

$$\sum c_k \log \frac{1 - W^*(z)W(w_k^-)}{1 - zw_k^-}$$

of reproducing kernel functions for the space $\mathcal{G}(W^*)$ into the finite linear combination

$$\sum c_k \; \log \frac{1 - W(w_k^-)/W(z)}{1 - w_k^-/z}$$

of elements of the space $\mathcal{G}(W)$ since the identity

$$\|\sum c_k \log \frac{1 - W^*(z)W(w_k^-)}{1 - zw_k^-}\|^2_{\mathcal{G}(W^*)} = \sum c_k c_i^- \log \frac{1 - W^*(w_i)W(w_k^-)}{1 - w_i w_k^-}$$

and the inequality

$$\|\sum c_k \log \frac{1 - W(w_k^-)/W(z)}{1 - w_k^-/z}\|^2_{\mathcal{G}(W)} \leq \sum c_k c_i^- \log \frac{1 - W^*(w_i)W(w_k^-)}{1 - w_i w_k^-}$$

are satisfied.

The adjoint transformation of the space $\mathcal{G}(W)$ into the space $\mathcal{G}(W^*)$ takes $f(z)$ into $g(z)$ when the identity

$$g(w) = \langle f(z), \log \frac{1 - W(w^-)/W(z)}{1 - w^-/z} \rangle_{\mathcal{G}(W)}$$

holds for w in the unit disk. This completes the construction of the Grunsky transformation of the space $\mathcal{G}(W)$ into the space $\mathcal{G}(W^*)$.

Since the transformation takes

$$\log \frac{1 - W(z)W(w)^-}{1 - zw^-}$$

into

$$\log \frac{1 - W^*(z)/W(w)^-}{1 - z/w^-}$$

when w is in the unit disk, the adjoint transformation of the space $\mathcal{G}(W^*)$ into the space $\mathcal{G}(W)$ takes $f(z)$ into $g(z)$ when the identity

$$g(w) = \langle f(z), \log \frac{1 - W^*(z)/W(w)^-}{1 - z/w^-} \rangle_{\mathcal{G}(W^*)}$$

holds for w in the unit disk.

The Grunsky transformation originates as a characterization of power series $W(z)$ with vanishing constant coefficient which represent injective mappings of the unit disk. Since the function

$$\frac{1 - W(w^-)/W(z)}{1 - w^-/z}$$

of z admits an analytic logarithm in the unit disk when w is in the unit disk, the numerator is nonzero whenever the denominator is nonzero. In the present formulation the contractive property of the composition $f(z)$ into $f(w(z))$ in ext $\mathcal{G}$ implies that the function represented by $W(z)$ is not only injective but bounded by one in the unit disk. The converse implication has not yet been verified. The original Grunsky transformation is a limiting case of the present transformation which gives a weaker conclusion under a weaker hypothesis.

The Koebe function as a power series

$$f(z) = z + 2z^2 + 3z^3 + \dots$$

represents a function

$$f(z) = z/(1-z)^2$$

which maps the unit disk injectively onto a region obtained from the complex plane on deleting the real numbers not greater than minus one–quarter. The analytic function

$$zf'(z)/f(z) = (1+z)/(1-z)$$

of z in the unit disk has positive real part and has value one at the origin.

A related power series

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \dots$$

with vanishing constant coefficient is defined by

$$zf'(z)/f(z) = 1/\phi(z)$$

for every analytic function of z in the unit disk which has positive real part and which has value one at the origin. The series represents an injective mapping of the unit disk onto a region which contains the origin and which contains every convex combination of one of its elements with the origin. When t is positive and not greater than one, the function

$$tf(z)$$

of z maps the unit disk injectively onto a region which is contained in the given region. A power series $W(t,z)$ with vanishing constant coefficient which represents an injective mapping of the unit disk into itself is defined by the composition

$$tf(z) = f(W(t,z)).$$

The composing functions form a semi–group under composition: The identity

$$W(ab,z) = W(a, W(b,z))$$

holds when a and b are positive and not greater than one. The evolution equation

$$t \frac{\partial}{\partial t} W(t,z) = \phi(z)\, z \frac{\partial}{\partial z} W(t,z)$$

generates the functions belonging to the semi–group. The function $W(t,z)$ has derivative at the origin equal to t.

The Grunsky spaces of analytic functions are Hilbert spaces of analytic functions derived from the spaces applied in the construction of invariant subspaces on a hypothesis of injectivity for the transfer function. Exponentiation is contractive from the initial Grunsky space $\mathcal{G}$ into the initial space $\mathcal{C}(z)$ of the invariant subspace construction. If

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

is an element of $\mathcal{G}$, then

$$\exp f(z) = b_0 + b_1 z + b_2 z^2 + \ldots$$

is an element of $\mathcal{C}(z)$ which satisfies the inequality

$$\sum |b_n|^2 \leq \exp(\sum n(a_n|^2).$$

A generalization is due to Lebedev and Milin.

Theorem 19. *Assume that a nonincreasing sequence of nonnegative numbers ρ_n has a convergent positive sum, that*

$$\sigma_r = \sum_{n=r}^{\infty} \rho_n / \sum_{n=0}^{\infty} \rho_n$$

is defined for every positive integer r, and that the sum

$$\sum \sigma_n / n$$

over the positive integers n converges. If

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

and

$$\exp f(z) = b_0 + b_1 z + b_2 z^2 + \ldots ,$$

then the inequality

$$(\sum \rho_n |b_n|^2) \exp(\sum \sigma_n / n) \leq (\sum \rho_n) \exp(\sum n \sigma_n |a_n|^2)$$

is satisfied.

Proof of Theorem 19. The inequality is verified by maximizing

$$\exp(-\sum n \sigma_n |a_n|^2) \sum \rho_n |b_n|^2$$

under the constraint of convergent sums. If a differentiable function $\alpha_n(t)$ of positive t is given for every positive integer n, a differentiable function $\beta_n(t)$ of positive t is defined for every nonnegative integer n by the equation

$$\sum \beta_n(t) z^n = \exp(\sum \alpha_n(t) z^n).$$

The differential equation

$$\beta_n'(t) = \sum \beta_{n-k}(t)\alpha_k'(t)$$

is satisfied for every nonnegative integer n with summation over the positive integers k which are not greater than n.

The derivative with respect to t of the sum

$$\log(\sum \rho_n \beta_n(t)^- \beta_n(t)) - \sum n\sigma_n \alpha_n(t)^- \alpha_n(t)$$

is the sum

$$\sum [s_k(t) - k\sigma_k \alpha_k(t)]^- \alpha_k'(t) + \sum [s_k(t) - k\sigma_k \alpha_k(t)] \alpha_k'(t)^-$$

over the positive integers k with

$$s_k(t) = \frac{\sum \rho_{n+k} \beta_{n+k}(t) \beta_n(t)^-}{\sum \rho_n \beta_n(t) \beta_n(t)^-}$$

defined by sums over the nonnegative integers n.

The derivative is nonnegative when $\alpha_k(t)$ is defined as the solution of the differential equation

$$\alpha_k'(t) = s_k(t) - k\sigma_k \alpha_k(t)$$

with initial condition

$$\alpha_k(0) = a_k$$

for every positive integer k. The inequality

$$|\alpha_k(t) - a_k \exp(-k\sigma_k t)| \leq \frac{1 - \exp(-k\sigma_k t)}{k\sigma_k}$$

applies when t is positive since

$$|s_k(t)| \leq 1.$$

Since the inequality

$$\exp(-\sum n\sigma_n |a_n|^2) \sum \rho_n |b_n|^2 \leq \exp(-\sum n\sigma_n |\alpha_n(t)|^2) \sum \rho_n |\beta_n(t)|^2$$

is satisfied, it is sufficient to obtain an estimate of

$$\exp(-\sum n\sigma_n |a_n|^2) \sum \rho_n |b_n|^2$$

when the inequality

$$k\sigma_k |a_k| \leq 1$$

is satisfied. A maximum of the continuous function is obtained by compactness. The maximum is attained on the set of coefficients defined by the equations

$$k\sigma_k a_k = \frac{\sum \rho_{n+k} b_{n+k} b_n^-}{\sum \rho_n b_n^- b_n}.$$

The set of noncritical points is mapped continuously into the set of critical points by the solutions of the differential equations. Since the set of noncritical points is connected, the set of noncritical points is mapped onto a compact connected set of critical points. The function

$$\exp(-\sum k\sigma_k|a_k|^2)\sum \rho_n|b_n|^2$$

of coefficients is a constant on the set of critical points. This computes the maximum value since an element of the set of critical points is defined by

$$ka_k = \omega^k$$

for every positive integer k with

$$b_n = \omega^n$$

for every nonnegative integer n.

This completes the proof of the theorem.

Lebedev and Milin state the inequality only when the coefficients ρ_n are a sequence of zeros and ones. The inequality is the motivation for contractive properties of composition which are found in the Koebe function and related mappings defining composition semigroups.

If a power series $f(z)$ has vanishing constant coefficient, the power series

$$f(tz/(1+z)^2) = \sum \alpha_n(t)z^n$$

has vanishing constant coefficient for every positive number t. Since the differential equation

$$t\,\frac{\partial}{\partial t}\,f(tz/(1+z)^2) = \frac{1+z}{1-z}\,z\,\frac{\partial}{\partial z}\,f(tz/(1+z)^2)$$

is satisfied, the coefficients $\alpha_n(t)$ satisfy the differential equations

$$t\alpha_n'(t) = s_n(t) + s_{n-1}(t)$$

in terms of the coefficients $s_n(t)$ of the power series

$$(1+z)z\,\frac{\partial}{\partial z}\,f(tz/(1+z)^2) = \sum\ s_n(t)z^n$$

which satisfy the equations

$$n\ \alpha_n(t) = s_n(t) - s_{n-1}(t).$$

Nonnegative differentiable functions $\sigma_n(t)$ of $t \geq 1$, defined for positive integers n, are said to be admissible as a family if the differential equations

$$\sigma_n(t) + \frac{t\sigma_n'(t)}{n} = \sigma_{n+1}(t) - \frac{t\sigma_{n+1}'(t)}{n+1}$$

are satisfied and if the solutions are nonincreasing functions of t. These conditions imply that the sum

$$\sum \sigma_n(t) \ \frac{[s_n(t) - s_{n-1}(t)]^-[s_n(t) - s_{n-1}(t)]}{n}$$

is a nondecreasing function of t since the inequality

$$[s_n(t) - s_{n-1}(t)]^-[s_n(t) - s_{n-1}(t)] \le 2s_n(t)^- s_n(t) + 2s_{n-1}(t)^- s_{n-1}(t)$$

is satisfied. The sum

$$\sum n\sigma_n(t)\alpha_n(t)^-\alpha_n(t)$$

is a nondecreasing function of t.

The formal sum

$$\sum \frac{\sigma_n(t)}{n} \ (z^n + z^{-n})$$

over the positive integers n satisfies the differential equation

$$\begin{aligned} t \frac{\partial}{\partial t} \sum \frac{\sigma_n(t)}{n} \ (z^n + z^{-n}) &= \frac{1-z}{1+z} \ z \ \frac{\partial}{\partial z} \sum \frac{\sigma_n(t)}{n} \ (z^n + z^{-n}) \\ &= \frac{1-z}{1+z} \sum \sigma_n(t)(z^n - z^{-n}). \end{aligned}$$

The equation admits a unique solution defining an admissible family for initial conditions $\sigma_n(1)$ an arbitrary nonincreasing sequence of nonnegative numbers such that the increments

$$\sigma_n(1) - \sigma_{n+1}(1)$$

are nonincreasing and have finite sum. It is sufficient to make the verification when a positive integer r exists such that

$$\sigma_n(1) = r + 1 - n$$

when n is not greater than r and such that $\sigma_n(1)$ vanishes otherwise.

Since the identity

$$\sum (r+1-n)z^n = \sum \ \frac{z^{n+1} - z}{z-1} = \frac{z^{r+2} - z^2}{(z-1)^2} - \frac{rz}{z-1}$$

holds with summation over the positive integers n which are not greater than r_1 the identity

$$\sum \ (r+1-n)(z^n - z^{-n}) = \frac{z^{r+1} - z^{-r-1} - (r+1)(z - z^{-1})}{(z^{\frac{1}{2}} - z^{-\frac{1}{2}})^2}$$

holds with summation over the positive integers n which are not greater than r.

Since the identity

$$(2r+2)\sum \frac{(2r+1-k)!}{k!(2r+1-2k)!} \, (z^{\frac{1}{2}} - z^{-\frac{1}{2}})^{2r+2-2k} = z^{r+1} + z^{-r-1}$$

holds with summation over the nonnegative integers k which are not greater than $r+1$, the identity

$$-\frac{1-z}{1+z} \, (z^{r+1} - z^{-r-1}) = \sum \frac{(2r+1-k)!}{k!(2r+1-2k)!} \, (z^{\frac{1}{2}} - z^{-\frac{1}{2}})^{2r+2-2k}$$

holds with summation over the nonnegative integers k which are not greater than r.

The solution of the differential equation is

$$-t \, \frac{\partial}{\partial t} \sum \frac{\sigma_n(t)}{n} \, (z^n + z^{-n}) = \sum \frac{(2r+1-k)!}{k!(2r+1-2k)!} \, t^{k-r} (z^{\frac{1}{2}} - z^{-\frac{1}{2}})^{2r-2k}$$

with summation over the nonnegative integers k which are not greater than r. Since the binomial expansion

$$(z^{\frac{1}{2}} - z^{-\frac{1}{2}})^{2r-2k} = \sum (-1)^m \, \frac{(2r-2k)!}{m!(2r-2k-m)!} \, z^{r-k-m}$$

applies with summation of the integers m such that

$$k - r \le m \le r - k,$$

the identity

$$-t \, \frac{\partial}{\partial t} \, \frac{\sigma_n(t)}{n} = \sum (-1)^k \, \frac{(r+n+1+k)!}{k!(r-n-k)!(2n+k)!} \, \frac{t^{-n-k}}{2n+1+2k}$$

holds for every positive integer n which is not greater than r with summation over the nonnegative integers k which are not greater than $r-n$. The equation reads

$$-t \, \frac{\partial}{\partial t} \, \frac{\sigma_n(t)}{n} = \frac{(r+n+1)! \, t^{-n}}{(r-n)!(2n+1)!} \, F(n-r, n+2+r, n+\tfrac{1}{2}; n+\tfrac{3}{2}, 2n+1; t^{-1})$$

in the hypergeometric notation

$$F(a,b,c;d,e;z) = 1 + \frac{abc}{1!de} \, z + \frac{a(a+1)b(b+1)c(c+1)}{2!d(d+1)e(e+1)} \, z^2 + \dots .$$

Another derivation of the equation appears in *A proof of the Bieberbach conjecture*, Acta Mathematica 154 (1985), 137–152.

Since

$$\frac{(r+n+1+k)!}{(r-n-k)!} - \frac{(r+n+k)!}{(r-1-n-k)!} = (2n+1+2k)\ \frac{(r+n+k)!}{(r-n-k)!}$$

when $n+k$ is less than r, the identity reads

$$-(2n)!\ t^{n+1}\ \frac{\partial}{\partial t}\ \frac{\sigma_n(t)}{n} = \sum \frac{(m+n)!}{(m-n)!}\ F(n-m, n+1+m; 2n+1; t^{-1})$$

with summation over the positive integers m which are not greater than r.

The hypergeometric series

$$F(a,b;c;z) = 1 + \frac{ab}{1!c}\ z + \frac{a(a+1)b(b+1)}{2!c(c+1)}\ z^2 + \ldots$$

satisfies the differential equations

$$F'(a,b;c;z) = a[F(a+1,b;c;z) - F(a,b;c;z)]/z$$

and

$$F'(a,b;c;z) = b[F(a,b+1,c;z) - F(a,b;c;z)]/z$$

and

$$F'(a,b;c;z) = (c-1)[F(a,b;c-1;z) - F(a,b;c;z)]/z$$

as well as the differential equations

$$(1-z)F'(a,b;c;z) - bF(a,b;c;z) = (a-c)[F(a,b;c;z) - F(a-1,b;c;z)]/z$$

and

$$(1-z)F'(a,b;c;z) - aF(a,b;c;z) = (b-c)[F(a,b;c;z) - F(a,b-1;c)z)]/z$$

and

$$(1-z)F'(a,b;c;z) - (a+b-c)F(a,b;c;z) = \frac{(c-a)(c-b)}{c}\ F(a,b;c+1;z)$$

which imply the differential equation

$$z(1-z)F''(a,b;c;z) + [c-(a+b+1)z]F'(a,b;c;z) - abF(a,b;c;z) = 0$$

and the recurrence relation

$$F(a,b;c;z) = \frac{b}{a-b-1}\ \frac{a-c}{a-b}\ [F(a-1,b+1;c;z) - F(a,b;c;z)]/z$$
$$+\frac{a}{b-a-1}\ \frac{b-c}{b-a}\ [F(a+1,b-1;c;z) - F(a,b;c;z)]/z.$$

Another consequence is the identity

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

when $c-a-b$ has positive real part.

For every integer r which is not less than a given positive integer n the polynomial

$$F(n-r, n+1+r; 2n+1; z)$$

of degree $r-n$ is an eigenfunction of the differential operator taking $F(z)$ into

$$z(1-z)F''(z) + [2n+1-(2n+2)z]F'(z)$$

for the eigenvalue

$$(n-r)(n+1+r).$$

The operator on polynomials admits a unique self–adjoint extension in the Hilbert space of functions defined in the interval $(0,1)$ which are square integrable with respect to the measure whose value on a Baire subset of the interval is the integral

$$\frac{2n+1}{(2n)!(2n)!} \int t^{2n} dt$$

taken over the set. An orthonormal basis for the Hilbert space is the set of polynomials

$$\frac{(r+n)!}{(r-n)!} F(n-r, n+1+r; 2n+1; z)$$

for integers r which are not less than n. A computation of scalar products is made from the identity

$$\begin{aligned}\left[\frac{(n+r+1)^2}{(2r+1)(2r+2)} + \frac{(r-n)^2}{(2r)(2r+1)} - z\right] F(n-r, n+1+r; 2n+1; z) \\ = \frac{(n+r+1)^2}{(2r+1)(2r+2)} F(n-r-1, n+2+r; 2n+1; z) \\ + \frac{(r-n)^2}{(2r)(2r+1)} F(n-r+1, n+r; 2n+1; z)\end{aligned}$$

from which the recurrence relation

$$\begin{aligned}(n+r+1)^2 \int_0^1 t^{2n}|F(n-r-1, n+2+r; 2n+1; t)|^2 dt \\ = (r+1-n)^2 \int_0^1 t^{2n}|F(n-r, n+1+r; 2n+1; t)|^2 dt\end{aligned}$$

follows.

A theorem of Richard Askey and George Gasper, *Positive Jacobi sums II*, American Journal of Mathematics 98 (1976), 709–737, states that, for every positive integer n and every integer r which is not less than n, the sum

$$\sum \frac{(m+n)!}{(m-n)!} F(n-m, n+1+m; 2n+1; z)$$

over the integers m such that $n \leq m \leq r$ is a polynomial whose values in the interval $(0,1)$ are positive.

Chapter 3. Conformal Mapping

The Lagrange skew–plane is a generalization of the Gauss plane. A Lagrange number

$$\xi = d + ia + jb + kc$$

has rational numbers a, b, c, and d as coordinates. The addition and multiplication of Lagrange numbers are defined from the addition and multiplication of rational numbers by the multiplication table

$$\begin{aligned} ij &= k, & jk &= i, & ki &= j, \\ ji &= -k, & kj &= -i, & ik &= -j, \\ ii &= -1, & jj &= -1, & kk &= -1. \end{aligned}$$

The properties of the Lagrange skew–plane resemble those of the Gauss plane except for the noncommutativity of multiplication.

The associative law

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

holds for all Lagrange numbers α, β, and γ. The commutative law

$$\alpha + \beta = \beta + \alpha$$

holds for all Lagrange numbers α and β. The origin 0 of the Lagrange skew–plane, which has vanishing coordinates, satisfies the identity

$$0 + \gamma = \gamma = \gamma + 0$$

for every element γ of the Lagrange skew–plane. For every element α of the Lagrange skew–plane a unique element

$$\beta = -\alpha$$

of the Lagrange skew–plane exists such that

$$\alpha + \beta = 0 = \beta + \alpha.$$

The identity

$$(\alpha + \beta)^- = \alpha^- + \beta^-$$

holds for all Lagrange numbers α and β.

Multiplication by a Lagrange number γ is a homomorphism of additive structure. The identity

$$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$$

holds for all Lagrange numbers α and β. The parametrization of homomorphisms is consistent with additive structure: The identity

$$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$$

holds for all Lagrange numbers α, β, and γ. Multiplication by γ is the homomorphism which annihilates every element of the Lagrange skew–plane when γ is the origin. Multiplication by γ is the identity homomorphism when γ is the unit.

The composition of homomorphisms is consistent with multiplicative structure: The associative law

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

holds for all Lagrange numbers α, β, and γ. Conjugation is an anti–homomorphism of multiplicative structure: The identity

$$(\alpha\beta)^- = \beta^-\alpha^-$$

holds for all Lagrange numbers α and β.

A rational number is a Lagrange number

$$\gamma = \gamma^-$$

which is self–conjugate. If

$$\gamma = d + ia + jb + kc$$

is a nonzero Lagrange number, then

$$\gamma^-\gamma = a^2 + b^2 + c^2 + d^2$$

is a positive rational number. A nonzero Lagrange number α has an inverse

$$\beta = \alpha^-/(\alpha^-\alpha)$$

such that

$$\beta\alpha = 1 = \alpha\beta.$$

A Lagrange number is said to be integral if its coordinates are either all integers or all halves of odd integers. Sums and products of integral Lagrange numbers are integral. The conjugate of an integral Lagrange number is integral. If ξ is a nonzero integral Lagrange number, $\xi^-\xi$ is a positive integer. The Euclidean algorithm is adapted to the search for integral Lagrange numbers ξ which represent a given positive integer

$$r = \xi^-\xi.$$

If α is an integral Lagrange number and if β is a nonzero integral Lagrange number, then an integral Lagrange number γ exists which satisfies the inequality

$$(\alpha - \beta\gamma)^-(\alpha - \beta\gamma) < \beta^-\beta.$$

The choice of the coordinates of γ is made so that the coordinates of

$$\beta^-\alpha - \beta^-\beta\gamma = d + ia + jb + kc$$

satisfy the inequalities

$$-\beta^-\beta \le 2a \le \beta^-\beta,$$

and

$$-\beta^-\beta \le 2b \le \beta^-\beta,$$

and

$$-\beta^-\beta \le 2c \le \beta^-\beta,$$

and

$$-\beta^-\beta \le 2d \le \beta^-\beta$$

and so that a strict inequality

$$(\beta^-\alpha - \beta^-\beta\gamma)(\beta^-\alpha - \beta^-\beta\gamma) < (\beta^-\beta)^2$$

is obtained.

A nonempty set of integral Lagrange numbers is said to be a left ideal if it contains the sum

$$\alpha + \beta$$

of any elements α and β and if it contains the product

$$\alpha\beta$$

of any element β with an integral Lagrange number α.

A nonempty set of integral Lagrange numbers is said to be a right ideal if it contains the sum

$$\alpha + \beta$$

of any elements α and β and if it contains the product

$$\alpha\beta$$

of any element α with an integral Lagrange number β.

Conjugation transforms a left ideal into a right ideal and a right ideal into a left ideal. A determination of structure is made for right ideals.

A nonzero integral Lagrange number β belongs to a right ideal whose elements are the products $\beta\gamma$ with integral Lagrange numbers γ. A right ideal which contains a nonzero element contains a nonzero element β which minimizes the positive integer $\beta^-\beta$. If α is an element of the ideal, an integral Lagrange number γ exists which satisfies the inequality

$$(\alpha - \beta\gamma)^-(\alpha - \beta\gamma) < \beta^-\beta.$$

The identity

$$\alpha = \beta\gamma$$

follows since $\alpha - \beta\gamma$ is an element of the ideal which is not nonzero.

The Euclidean algorithm solves the equation

$$r = \xi^- \xi$$

for an integral Lagrange number ξ when r is a given positive integer. The solution is obtained from an approximate solution in a quotient ring of the ring of integral Lagrange numbers.

A ring of Lagrange numbers is a nonempty set of Lagrange numbers which contains the difference

$$\alpha - \beta$$

and the product

$$\alpha\beta$$

of any elements α and β of the set. The set of integral Lagrange numbers is a conjugated ring: The ring contains ξ^- whenever it contains ξ.

A quotient ring of the ring of integral Lagrange numbers is defined for every positive integer r. Integral Lagrange numbers α and β are said to be congruent modulo r if

$$\beta - \alpha = r\gamma$$

is divisible by r: The equation admits an integral Lagrange number γ as solution. Congruence modulo r is an equivalence relation on integral Lagrange numbers. The ring is a union of disjoint equivalence classes.

Equivalence classes inherit addition and multiplication since $\alpha_1 + \beta_1$ and $\alpha_2 + \beta_2$ are congruent modulo r and since $\alpha_1\beta_1$ and $\alpha_2\beta_2$ are congruent modulo r whenever α_1 and α_2 are congruent modulo r and β_1 and β_2 are congruent modulo r. Equivalence classes inherit conjugation since γ_1^- and γ_2^- are congruent modulo r whenever γ_1 and γ_2 are congruent modulo r. Addition and multiplication of equivalence classes have the properties required of a ring:

The associative law

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

holds for all integral Lagrange numbers α, β, and γ modulo r. The commutative law

$$\alpha + \beta = \beta + \alpha$$

holds for all integral Lagrange numbers α and β modulo r. The image of the origin of the Lagrange numbers is an origin 0 for the Lagrange numbers modulo r: The identity

$$0 + \gamma = \gamma = \gamma + 0$$

holds for every integral Lagrange number γ modulo r. For every integral Lagrange number α modulo r an integral Lagrange number

$$\beta = -\alpha$$

modulo r exists such that

$$\alpha + \beta = 0 = \beta + \alpha.$$

Multiplication by an integral Lagrange number γ modulo r is a homomorphism of additive structure: The identity

$$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$$

holds for all integral Lagrange numbers α and β modulo r. The parametrization of homomorphisms is consistent with additive structure: The identity

$$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$$

holds for all integral Lagrange numbers α, β, and γ modulo r. Multiplication by γ is the homomorphism which annihilates every integral Lagrange number modulo r when γ is the origin. Multiplication by γ is the identity homomorphism when γ is the image 1 of the unit of the Lagrange numbers.

The composition of homomorphisms is consistent with multiplicative structure: The associative law

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

holds for all integral Lagrange numbers α, β, and γ modulo r.

The ring of integral Lagrange numbers modulo r is conjugated: The identity

$$(\alpha\beta)^- = \beta^-\alpha^-$$

holds for all integral Lagrange numbers α and β modulo r.

There are twenty–four integral Lagrange numbers ξ which represent

$$1 = \xi^-\xi.$$

These Lagrange units form a group under multiplication. The eight elements of the group which are fourth roots of unity form a normal subgroup whose quotient is a cyclic group of three elements.

If r is an odd positive integer, every integral Lagrange number is congruent modulo r to a unique Lagrange number whose coordinates are nonnegative integers less than r. The number of integral Lagrange numbers modulo r is equal to r^4.

If r and s are relatively prime positive integers, the equation

$$1 = ra + sb$$

admits a solution in integers a and b. A canonical homomorphism of the ring of integral Lagrange numbers modulo rs onto the ring of integral Lagrange numbers modulo r exists whose kernel is the conjugated ideal of elements divisible by s. A canonical homomorphism of the ring of integral Lagrange numbers modulo rs onto the ring of integral Lagrange numbers modulo s exists whose kernel is the conjugated ideal of elements divisible by r. The conjugated ring of integral Lagrange numbers modulo rs is canonically isomorphic to the Cartesian product of the conjugated ring of integral Lagrange numbers modulo r and the conjugated ring of integral Lagrange numbers modulo s.

The ring of integral Lagrange numbers modulo two contains sixteen elements. The invertible elements of the ring are represented by Lagrange units. There are twelve integral Lagrange numbers modulo two since a Lagrange unit ω and its negative $-\omega$ are congruent modulo two. A canonical homomorphism exists of the ring of integral Lagrange numbers modulo $2r$ onto the ring of integral Lagrange numbers modulo r whose kernel is the conjugated ideal of elements divisible by r. Since the ideal contains sixteen elements, every integral Lagrange number modulo r is represented by sixteen integral Lagrange numbers modulo $2r$. The number of integral Lagrange numbers modulo r is equal to r^4 for every positive integer r.

The multiplicative group of nonzero integers modulo p is cyclic for every odd prime p. The number of nonzero integers modulo p which are square of integers modulo p is $\frac{1}{2}(p-1)$ as is the number of integers modulo p which are nonsquares. The product of two squares and the product of two nonsquares are squares. The product of a square and a nonsquare is a nonsquare. Since a nonsquare exists, some sum of two squares exists which is a nonsquare.

A skew–conjugate integral Lagrange number

$$\iota = ia + jb$$

modulo p is defined by the choice of integers a and b modulo p such that the equation

$$a^2 + b^2 = c^2$$

admits no solution c in the integers modulo p. If u and v are integers modulo p such that

$$(u + iv)^-(u + iv) = u^2 - \iota^2 v^2$$

vanishes, then u and v both vanish. A conjugated field of p^2 elements is obtained whose elements are integral Lagrange numbers

$$u + \iota v$$

modulo p with integers u and v modulo p as coordinates.

An integer a modulo p exists such that

$$-1 - a^2$$

is a square since $\frac{1}{2}(p+1)$ integers modulo p are represented whereas there are only $\frac{1}{2}(p-1)$ nonsquares. A skew–conjugate integral Lagrange number

$$\kappa = ia + jb + k$$

modulo p exists for some integer b modulo p such that

$$\kappa^- \kappa = 0.$$

Every integral Lagrange number is represented as

$$\alpha + \kappa\beta$$

for unique elements α and β of the field. The identity

$$(\alpha + \kappa\beta)^- (\alpha + \kappa\beta) = \alpha^- \alpha$$

is satisfied.

If p is a prime, a canonical homomorphism of the ring of integral Lagrange numbers modulo rp onto the ring of integral Lagrange numbers modulo r exists whose kernel is the conjugated ideal of elements divisible by r.

If I is a right ideal of the ring of integral Lagrange numbers modulo r, then the set of integral Lagrange numbers which represent elements of the ideal is a right ideal which contains r. An integral Lagrange number ξ exists such that the elements of the ideal are the products $\xi\eta$ with η an integral Lagrange number. The representation

$$r = \xi^- \xi$$

holds if I contains no nonzero element which is self–conjugate.

The number of right ideals of the ring of integral Lagrange numbers modulo r which contain no nonzero self–conjugate element is equal to the sum of the odd divisors of r. The number of integral Lagrange numbers ξ which represent

$$r = \xi^- \xi$$

is equal to twenty–four times the sum of the odd divisors of r.

The Lagrange skew–plane admits topologies which are compatible with addition and multiplication. The Dedekind topology is derived from convex structure.

A convex combination

$$(1 - t)\xi + t\eta$$

of elements ξ and η of the Lagrange skew–plane is an element of the Lagrange skew–plane when t is a rational number in the interval $[0, 1]$. A subset of the Lagrange skew–plane is said to be preconvex if it contains all elements of the Lagrange skew–plane which are convex combinations of elements of the set. The preconvex span of a subset of the Lagrange skew–plane is defined as the smallest preconvex subset of the Lagrange skew–plane which contains the given set.

The closure in the Lagrange skew–plane of a preconvex subset B is the set B^- of elements ξ of the Lagrange skew–plane such that the set whose elements are ξ and the elements of B is preconvex. The closure of a preconvex set is a preconvex set which is its own closure.

A nonempty preconvex set is defined as open if it is disjoint from the closure of every disjoint nonempty preconvex set. The intersection of two nonempty open preconvex sets is an open preconvex set if it is nonempty.

A subset of the Lagrange skew–plane is said to be open if it is a union of nonempty open preconvex sets. The empty set is open since it is an empty union of such sets. Unions of open subsets are open. Finite intersections of open sets are open.

An example of an open set is the complement in the Lagrange skew–plane of the closure of a nonempty preconvex set. A subset of the Lagrange skew–plane is said to be closed if it is the complement in the Lagrange skew–plane of an open set. Intersections of closed sets are closed. Finite unions of closed sets are closed. The Lagrange skew–plane is a Hausdorff space in the topology whose open and closed sets are defined by convexity. These open and closed sets define the Dedekind topology of the Lagrange skew–plane.

If a nonempty open preconvex set A is disjoint from a nonempty preconvex set B, then a maximal preconvex set exists which contains B and is disjoint from A. The maximal preconvex set is closed and has preconvex complement. The existence of the maximal preconvex set is an application of the Kuratowski–Zorn lemma.

Addition and multiplication are continuous as transformations of the Cartesian product of the Lagrange skew–plane with itself into the Lagrange skew–plane. Conjugation is continuous as a transformation of the Lagrange skew–plane into the Lagrange skew–plane. The Dedekind skew–plane is the completion of the Lagrange skew–plane in the uniform Dedekind topology. Neighborhoods of a Lagrange number are determined by neighborhoods of the origin. If an open set A contains the origin and if ξ is a Lagrange number, then the set of sums of ξ and elements of A is an open set which contains ξ. Every open set which contains ξ is obtained from an open set which contains the origin.

A Cauchy class of closed subsets of the Lagrange skew–plane is a nonempty class of closed subsets such that the intersection of the members of any finite subclass is nonempty and such that for every open set A containing the origin some member B of the class exists such that all differences of elements of B belong to A.

A Cauchy class of closed subsets is contained in a maximal Cauchy class of closed subsets. A Cauchy sequence is a sequence of elements $\xi_1, \xi_2, \xi_3, \dots$ of the Lagrange skew–plane such that a Cauchy class of closed subsets is defined whose members are the closed preconvex spans of $\xi_r, \xi_{r+1}, \xi_{r+2}, \dots$ for every positive integer r. A Cauchy sequence determines a maximal Cauchy class. Every maximal Cauchy class is determined by a Cauchy sequence.

An element of the Dedekind skew–plane is defined by a maximal Cauchy class of elements of the Lagrange skew–plane. An element of the Lagrange skew–plane determines the maximal Cauchy class of closed sets which contain the element. The Lagrange skew–plane is contained in the Dedekind skew–plane.

If B is a closed subset of the Lagrange skew–plane, the closure B^- of B in the Dedekind skew–plane is defined as the set of elements of the Dedekind skew–plane whose maximal Cauchy class has B as a member. A subset of the Dedekind skew–plane is defined as open if it is disjoint from the closure in the Dedekind skew–plane of every disjoint closed subset of the Lagrange skew–plane. Unions of open subsets of the Dedekind skew–plane are open. Finite intersections of open subsets of the Dedekind skew–plane are open. A subset of the Lagrange skew–plane is open if, and only if, it is the intersection with the Lagrange skew–plane of an open subset of the Dedekind skew–plane.

A subset of the Dedekind skew–plane is defined as closed if its complement in the Dedekind skew–plane is open. Intersections of closed subsets of the Dedekind skew–plane are closed. Finite unions of closed subset of the Dedekind skew–plane are closed. The closure of a subset of the Dedekind skew–plane is defined as the smallest closed set containing the given set. The closure in the Lagrange skew–plane of a subset of the Lagrange skew–plane is the intersection with the Lagrange skew–plane of the closure of the set in the Dedekind skew–plane.

The Dedekind skew–plane is a Hausdorff space in the topology whose open sets and closed sets are determined by convexity. These open sets and closed sets define the Dedekind topology of the Dedekind skew–plane.

The Lagrange skew–plane is dense in the Dedekind skew–plane. Addition and multiplication admit unique continuous extensions as transformations of the Cartesian product of the Dedekind skew–plane with itself into the Dedekind skew–plane. Conjugation admits a unique continuous extension as a transformation of the Dedekind skew–plant into itself.

Properties of addition in the Lagrange skew–plane are preserved in the Dedekind skew–plane. The associative law

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

holds for all elements α, β, and γ of the Dedekind skew–plane. The commutative law

$$\alpha + \beta = \beta + \alpha$$

holds for all elements α and β of the Dedekind skew–plane. The origin 0 of the Lagrange skew–plane satisfies the identities

$$0 + \gamma = \gamma = \gamma + 0$$

for every element γ of the Dedekind skew–plane. For every element α of the Dedekind skew–plane a unique element

$$\beta = -\alpha$$

of the Dedekind skew–plane exists such that

$$\alpha + \beta = 0 = \beta + \alpha.$$

Conjugation is a homomorphism of additive structure: The identity

$$(\alpha + \beta)^- = \alpha^- + \beta^-$$

holds for all elements α and β of the Dedekind skew–plane.

Multiplication by an element γ of the Dedekind skew–plane is a homomorphism of additive structure: The identity

$$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$$

holds for all elements α and β of the Dedekind skew–plane. The parametrization of homomorphisms is consistent with additive structure: The identity

$$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$$

holds for all elements α, β, and γ of the Dedekind skew–plane. Multiplication by γ is the homomorphism which annihilates every element of the Dedekind skew–plane when γ is the origin. Multiplication by γ is the identity homomorphism when γ is the unit 1 of the Lagrange skew–plane.

The composition of homomorphisms is consistent with multiplicative structure: The associative law

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

holds for all elements α, β, and γ of the Dedekind skew–plane. Conjugation is an anti–homomorphism of multiplicative structure: The identity

$$(\alpha\beta)^- = \beta^-\alpha^-$$

holds for all elements α and β of the Dedekind skew–plane.

The inclusion of the complex plane in the Dedekind skew–plane is a homomorphism of additive and multiplicative structure which commutes with conjugation. The complex plane is a closed subset of the Dedekind skew–plane. The Dedekind topology of the Dedekind plane is the subspace topology of the Dedekind topology of the Dedekind skew–plane.

If γ is a nonzero element of the Dedekind skew–plane, the real number

$$\gamma^-\gamma$$

is positive. If α is a nonzero element of the Dedekind skew–plane, the nonzero element

$$\beta = \alpha^-/(\alpha^-\alpha)$$

satisfies the identities

$$\beta\alpha = 1 = \alpha\beta.$$

The Dedekind skew–plane is complete in the uniform Dedekind topology: Every Cauchy class of closed subsets of the Dedekind skew–plane has a nonempty intersection. Closed and bounded subsets of the Dedekind skew–plane compact: A subset of the Dedekind skew–plane is said to be bounded if a positive number c exists such that the inequality

$$\gamma^{-}\gamma \leq c$$

holds for every element γ of the set. A nonempty class of closed subsets has a nonempty intersection if every finite subclass has a nonempty intersection and if some member of the class is bounded.

The axiomatization of topology has consequences which are unfamiliar to those whose experience is limited to Dedekind topologies. A topology is defined for a set by a class of subsets which are said to be open or equivalently by a class of subsets which are said to be closed. The two formulations of topology are equivalent since a set is assumed to be open if, and only if, its complement is closed. The union of every class of open sets is assumed to be open. Equivalently the intersection of every class of closed sets is assumed to be closed. The intersection of every finite class of open sets is assumed to be open. Equivalently the union of every finite class of closed sets is assumed to be closed. This definition of topology is supplemented by a condition which defines a Hausdorff space: Distinct elements a and b of the space are contained in disjoint open sets A and B, a contained in A and b contained in B.

A trivial example of such a topology is defined for a finite set. A finite set is a Hausdorff space in a unique topology: All subsets are both open and closed. This discrete topology of a finite set is applied in the construction of nontrivial topologies.

If a nonempty class $\mathcal{C}$ of nonempty sets is given, the Cartesian product of the sets is defined as the set of all functions defined on the members of the class such that the value of the function on a member set is always an element of the set. The usual function notation is however replaced by the notation applied to sequences: if N is a member of the class, the value of the function at N is written C_N. When the members of the class are parametrized by positive integers, the notation C_n means C_N with n the positive integer which parametrizes the member set N. The concept of a Cartesian product is applied to classes $\mathcal{C}$ which are unlimited in cardinality. The class $\mathcal{C}$ need not be finite. If it is infinite, it need not be countable. The concept of a Cartesian product can be applied more generally when the class $\mathcal{C}$ is empty or when some member of the class is empty. The Cartesian product is then defined to be empty. (The graph of the function contains no element.)

When the member sets are Hausdorff spaces, the Cartesian product is a Hausdorff space in the Cartesian product topology. The product topology is defined by two conditions: The projection of the product onto each factor space is continuous. A transformation of a topological space into the product space is continuous whenever every composition with a projection into a factor space is continuous.

When the factor spaces are compact Hausdorff spaces, the Cartesian product is a compact Hausdorff space. The proof of compactness is an application of the axiom of choice. The axiom of choice is equivalent to the assertion that a Cartesian product of nonempty sets is nonempty. The Kuratowski–Zorn lemma is a consequence of the axiom of choice: A partially ordered set contains a maximal element if every well–ordered subset admits an upper bound in the set.

Compactness of a Hausdorff space is formulated as the assertion that a nonempty class of closed subsets has a nonempty intersection whenever every finite subclass has the property. Every such class is contained in a maximal such class by the Kuratowski–Zorn lemma. When the class is maximal, the intersection of the members of the class contains a unique element.

If $\mathcal{C}$ is a maximal such class of closed subsets of the Cartesian product, then a maximal such class is seen in every factor space. Seen in a factor space are those closed sets whose inverse image in the Cartesian product are members of the class $\mathcal{C}$. The element determined in every factor space defines the desired element of the Cartesian product.

The adic topology of the Lagrange skew–plane resembles the Dedekind topology in its good relationship to addition and multiplication. The open sets are defined as unions of sets which are both open and closed. The closed sets are defined as intersections of sets which are both open and closed. A basic example of a set which is both open and closed and which contains a given Lagrange number ξ is defined by a positive rational number λ and consists of the Lagrange numbers η such that

$$\lambda(\xi - \eta)^{-}(\xi - \eta)$$

is integral. Every open set is a union of finite intersections of basic open and closed sets. Every closed set is an intersection of basic open and closed sets.

The Lagrange skew–plane is a Hausdorff space in the adic topology. Addition and multiplication are continuous as transformations of the Cartesian product of the Lagrange skew–plane with itself into the Lagrange skew–plane. Conjugation is continuous as a transformation of the Lagrange skew–plane into itself.

The adic skew–plane is defined as the Cauchy completion of the Lagrange skew–plane in the uniform adic topology. Addition and multiplication admit unique continuous extensions as transformations of the Cartesian product of the adic skew–plane with itself into the adic skew–plane. Conjugation admits a unique continuous extension as a transformation of the adic skew–plane into itself. An element of the adic skew–plane is said to be integral if it belongs to the closure of the integral elements of the Lagrange skew–plane. The adic skew–plane is a conjugated ring which contains the set of integral elements as a conjugated subring. Compactness of the subring is proved by a construction as a closed subset of a Cartesian product of compact Hausdorff spaces.

The Cartesian product of the conjugated ring of integral Lagrange numbers modulo r is taken over the positive integers r. The Cartesian product is a conjugated ring whose addition, multiplication, and conjugation are defined by addition, multiplication, and conjugation of projections in factor rings. Since the factor rings are compact Hausdorff spaces in the discrete topology, the Cartesian product is a compact Hausdorff space in the Cartesian product topology. When r_1 is a divisor of r_2, a canonical homomorphism exists of the factor ring modulo r_2 onto the factor ring modulo r_1 whose kernel is the conjugated ideal of elements divisible by r_1.

A closed subring of the Cartesian product is defined as the set of elements of the Cartesian product such that the projection of the factor ring modulo r_2 is mapped into the projection in the factor ring modulo r_1 whenever r_1 is a divisor of r_2. The subring is conjugated and is a compact Hausdorff space in the subspace topology. A continuous conjugated homomorphism of the subring onto the ring of integral elements of the adic skew–plane is defined by taking an element of the subring into the limit of a Cauchy sequence whose r–term is an integral element of the Lagrange skew–plane which represents the projection in the factor ring modulo r.

The adic skew–plane is a ring of quotients of the subring of its integral elements. A conjugated isomorphism of additive structure of the adic skew–plane onto itself is defined on multiplication by r for every positive integer r. The transformation is continuous and has a continuous inverse. Every element of the adic skew–plane is mapped into an integral element on multiplication by some positive integer.

An integral element of the adic skew–plane is said to be p–adic for some prime p if its quotient by r is integral for every positive integer r which is not divisible by p. The set of p–adic elements of the ring of integral elements of the adic skew–plane is a conjugated ideal which is closed in the adic topology. The conjugated ring of integral elements of the adic skew–plane is isomorphic to the Cartesian product of its p–adic ideals taken over all primes p. The topology of the ring of integral elements is the Cartesian product topology of its p–adic ideals.

A decomposition of the adic skew–plane results from the decomposition of its ring of integral elements. An element of the adic skew–plane is said to be p–adic if for some prime p its product with a positive integer is a p–adic integral element of the adic skew–plane. The set of p–adic elements of the adic skew–plane is a conjugated ideal of the adic skew–plane which is closed in the adic topology. The p–adic component of an element of the adic skew–plane is integral for all but a finite number of primes p. If a p–adic element of the adic skew–plane is chosen for every prime p and if all but a finite number of elements are integral, an element of the adic skew–plane exists whose p–adic component is the given p–adic element for every prime p.

The p–adic skew–plane is defined for a prime p as the conjugated ring of p–adic elements of the adic skew–plane. The p–adic topology of the ring is defined as the subspace topology of the adic topology of the adic skew–plane. The set of self–conjugate elements of the ring is the field of p–adic numbers. An element

$$\xi = d + ia + jb + kc$$

of the p–adic skew–plane has coordinates a, b, c, and d in the p–adic field which do not all vanish when ξ does not vanish. The product

$$\xi^- \xi = a^2 + b^2 + c^2 + d^2$$

is a p–adic number which does not vanish when the coordinates of ξ do not all vanish. An inverse

$$\xi^{-1} = \xi^- / (\xi^- \xi)$$

exists in the p–adic skew–plane which satisfies the identities

$$\xi^{-1}\xi = 1 = \xi\xi^{-1}$$

with 1 the unit of the p–adic field and also of the p–adic skew–plane.

The value of the adic skew–plane lies in its relationship to the Dedekind skew–plane which is found in their Cartesian product. The product skew–plane is the set of pairs $\xi = (\xi_+, \xi_-)$ of elements ξ_+ of the Dedekind skew–plane and elements ξ_- of the adic skew–plane. The sum

$$\gamma = \alpha + \beta$$

of elements α and β is defined by

$$\gamma_+ = \alpha_+ + \beta_+$$

and

$$\gamma_- = \alpha_- + \beta_-.$$

The product

$$\gamma = \alpha\beta$$

of elements α and β is defined by

$$\gamma_+ = \alpha_+\beta_+$$

and

$$\gamma_- = \alpha_-\beta_-.$$

The conjugate

$$\beta = \alpha^-$$

of an element α is defined by

$$\beta_+ = \alpha_+^-$$

and

$$\beta_- = \alpha_-^-.$$

The product skew–plane is a Hausdorff space in the Cartesian product topology of the Dedekind skew–plane and the adic skew–plane.

The Dedekind skew–plane and the adic skew–plane are spliced by the construction of a quotient space. A closed subset of the product skew–plane consists of the elements whose components in the Dedekind skew–plane and the adic skew–plane are elements of the Lagrange skew–plane with vanishing sum. If α and β are elements of the subset, then so is $\alpha + \beta$. If α is an element of the subset and if λ is an element of the Lagrange skew–plane, then an element

$$\beta = \lambda\alpha$$

of the subset is defined by

$$\beta_+ = \lambda\alpha_+$$

and

$$\beta_- = \lambda \alpha_- .$$

If α is an element of the subset, then an element

$$\beta = \alpha^-$$

of the subset is defined by

$$\beta_+ = \alpha_+^-$$

and

$$\beta_- = \alpha_-^- .$$

An equivalence relation is defined of the product skew–plane by defining elements α and β to be equivalent when $\beta - \alpha$ belongs to the subset. A fundamental domain for the equivalence relation is the set of elements ξ of the product skew–plane whose adic component is integral and whose Dedekind component satisfies the inequality

$$\xi_+^- \xi_+ < (\xi_+ - \omega)^- (\xi_+ - \omega)$$

for every integral element ω of the Lagrange skew–plane with integral inverse. Every element of the product skew–plane is equivalent to an element of the closure of the fundamental domain. Equivalent elements of the fundamental domain are equal.

Appendix. Cardinality

The cardinality of set A is said to be less than or equal to the cardinality of set B if an injective transformation of set A into set B exists. If the cardinality of set A is less than or equal to the cardinality of set B and if the cardinality of set B is less than or equal to the cardinality of set A, then an injective transformation exists of set A onto set B. Sets A and B are said to have the same cardinality. The cardinality of set A is said to be less than the cardinality of set B if A and B are sets of unequal cardinality such that the cardinality of set A is less than or equal to the cardinality of set B.

Experience with finite sets creates the expectation that any two sets are comparable in cardinality. If A and B are sets of unequal cardinality, then either the cardinality of set A is less than the cardinality of set B or the cardinality of set B is less than the cardinality of set A. The desired conclusion, or its equivalent, is accepted as a hypothesis in the axiomatic definition of sets.

The axiom of choice is the most plausible of the hypotheses which are equivalent to the desired comparability of cardinalities of sets. If a transformation T takes set A onto set B, then a transformation S of set B into set A exists such that the composed transformation TS is the inclusion transformation of set B in itself.

The axiom of choice displaces the previous hypothesis which is equivalent to the comparability of cardinalities of sets. A partial ordering of a set S is determined by distinguished pairs (a, b) of elements a and b of S. The inequality $a \leq b$ is written when (a, b) is a distinguished pair. It is assumed that the inequality $a \leq c$ holds whenever a and c are elements of the set for which the inequalities $a \leq b$ and $b \leq c$ hold for some element b of the set. The inequality $c \leq c$ is assumed for every element c of the set. Elements a and b of the set are assumed to be equal if the inequalities $a \leq b$ and $b \leq a$ are satisfied. A set is said to be well–ordered if every nonempty subset contains a least element. An equivalent of the axiom of choice is the hypothesis that every set admits a well–ordering.

The Kuratowski–Zorn lemma is a flexible formulation of the principle of induction implicit in well–ordering. A partially ordered set admits a maximal element if every well–ordered subset has an upper bound in the set.

The proof of the Kuratowski–Zorn lemma from the axiom of choice is an application of induction. Assume that S is a partially ordered set in which every well–ordered subset has an upper bound. An augmentation of a well–ordered subset A is a well–ordered subset B whose elements are the elements of A and some upper bound of A which does not belong to A. The axiom of choice is applied to a set whose elements are the pairs (A, B) consisting of an augmentable well–ordered subset A and an augmentation B of A. The set is mapped onto the set of augmentable well–ordered subsets by taking (A, B) into A. The axiom of choice asserts the existence of a transformation which takes every augmentable well–ordered subset A into an augmentation (A, A') of A.

The proof of the Kuratowski–Zorn lemma is facilitated by the introduction of notation. A ladder is well–ordered subset A which is constructed by the chosen augmentation procedure. For every element b of A the augmentation of the set of elements of A which are less than b is the set of elements of A which are less than or equal to b. The intersection of ladders A and B is a ladder which is either equal to A or equal to B. If A and B are ladders, then either A is contained in B or B is contained in A. The union of all ladders is a ladder which contains every ladder. Since the greatest ladder is assumed to have an upper bound, it has a greatest element. The greatest element of the greatest ladder is a maximal element of the given partially ordered set S.

Cardinal numbers are constructed by a theorem of Cantor which states that no transformation maps a set onto the class of all its subsets. If a transformation T maps a set $\mathcal{S}$ into the subsets of $\mathcal{S}$, then a subset $\mathcal{S}_\infty$ of $\mathcal{S}$ is constructed which does not belong to the range of T. The set $\mathcal{S}_\infty$ is the set of elements s of $\mathcal{S}$ for which no elements s_n of $\mathcal{S}$ can be chosen for every nonnegative integer n so that s_0 is equal to s and so that s_n belongs to Ts_{n-1} when n is positive. An element s of $\mathcal{S}$ belongs to $\mathcal{S}_\infty$ if, and only if, Ts is contained in $\mathcal{S}_\infty$. This property implies that S_∞ is not equal to Ts for an element s of $\mathcal{S}$.

If γ is a cardinal number, a continuum of order γ is defined as a set of least cardinality which has the same cardinality as the class of its subsets which are continua of order less than γ. The empty set is a continuum of order equal to its cardinality. A set with one element is a continuum of order equal to its cardinality. No other finite set is a continuum of order γ for a cardinal number γ. A countably infinite set is a continuum of order equal to its cardinality.

A parametrization of a continuum $\mathcal{S}$ of order γ is an injective transformation J of $\mathcal{S}$ onto the class of its subsets which are continua of order less than γ such that no elements s_n of $\mathcal{S}$ can be chosen for every nonnegative integer n so that s_n belongs to Js_{n-1} when n is positive. A continuum of order γ admits a parametrization since an injective transformation T exists of $\mathcal{S}$ onto the class of its subsets which are continua of order less than γ. Since S_∞ is then a continuum of order γ, it has the same cardinality as $\mathcal{S}$. The restriction of T to $\mathcal{S}_\infty$ is a parametrization of $\mathcal{S}_\infty$. If W is an injective transformation of $\mathcal{S}$ onto $\mathcal{S}_\infty$, then a parametrization J of $\mathcal{S}$ is defined so that Ja is the set of elements b of $\mathcal{S}$ such that Wb belongs to TWa.

A parametrization J of a continuum $\mathcal{S}$ of order γ is essentially unique. If an injective transformation T maps $\mathcal{S}$ onto the class of its subsets which are continua of order less than γ, then an injective transformation W of $\mathcal{S}$ onto $\mathcal{S}_\infty$ exists such that Ja is always the set of elements b such that Wb belongs to TWa. The construction of T is an application of the Kuratowski–Zorn lemma. Consider the class $\mathcal{C}$ of injective transformations W with domain contained in $\mathcal{S}$ and with range contained in $\mathcal{S}_\infty$ such that every element of Ja belongs to the domain of W whenever a belongs to the domain of W and such that Ja is always the set of elements b of $\mathcal{S}$ such that Wb belongs to JWa. The class $\mathcal{C}$ is partially ordered by the inclusion ordering of the graph. A well–ordered subclass of $\mathcal{C}$ has an upper bound in $\mathcal{C}$ whose graph is a union of graphs. A maximal member of the class $\mathcal{C}$ has $\mathcal{S}$ as its domain.

A nonempty set of cardinal numbers contains a least element since a ladder of well–ordered sets can be constructed with these cardinalities.

A continuum of order γ exists when γ is the cardinality of an uncountable set. It is sufficient to construct a set which has the same cardinality as the class of its subsets which are continua of cardinality less than γ. If a cardinal number α is greater than the cardinality of every continuum of order less than γ, it is sufficient to construct a set which has the same cardinality as the class of its subsets of cardinality less than α. Such a set is constructed when α is the least cardinality greater than the cardinality of an infinite set $\mathcal{S}$. The class $\mathcal{C}$ of all subsets of $\mathcal{S}$ is a set which has the same cardinality as the class of its subsets of cardinality less than α. The cardinality of the class of all subsets of $\mathcal{C}$ of cardinality less than α is less than or equal to the cardinality of all transformations of $\mathcal{S}$ into the set of functions defined on $\mathcal{S}$ with values zero or one. The cardinality of the class of all subsets of $\mathcal{C}$ with values zero or one is less than or equal to the cardinality of the set of all functions defined on the Cartesian product $\mathcal{S} \times \mathcal{S}$ with values zero or one. Since $\mathcal{S}$ is an infinite set, the cardinality of $\mathcal{S} \times \mathcal{S}$ is equal to the cardinality of $\mathcal{S}$. The cardinality of the class of all subsets of $\mathcal{C}$ of cardinality less than α is less than or equal to the cardinality of $\mathcal{C}$.

A hypothesis is required for the determination of cardinalities of continua. The choice of hypothesis depends on the desired applications. When the largest logical structure is wanted in which the accepted methods of analysis apply, then the cardinalities of continua are dependent on hypotheses whose consistency is necessarily untested (as are the accepted hypotheses of analysis). When the smallest logical structure is wanted in which the accepted methods of analysis apply (which is the conventional view in mathematics), then the cardinalities of continua are determined. This is the best choice for a student since it establishes a logical structure with minimal hypotheses which can serve as a guide to generalizations should he want this direction of research. A minimal structure is therefore chosen here.

When a minimal structure is chosen, there are essentially only two ways in which a new cardinality can be constructed from given cardinalities. The cardinality of the class of all subsets of a set is greater than the cardinality of the set. A set of cardinality γ can be obtained as a union of a class of cardinality less than γ of sets whose cardinalities are less than γ. Both constructions produce continua from continua. It follows that every infinite set is a continuum whose order is equal to its cardinality. An uncountable continuum is either the class of all subsets of an infinite set in cardinality or it is a union of a class of smaller cardinality of sets of smaller cardinality.

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