HYPERCOMPLEX ANALYSIS

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Abstract. Although the Riemann hypothesis originates as a conjecture concerning the classical zeta function, the conjecture is applied more generally to zeta functions which resemble the classical zeta function in Euler product and functional identity. An asymptotic estimate of the number of primes with a given bound is the application originally intended for the Riemann hypothesis. The Riemann hypothesis for other zeta functions is more distantly related to properties of prime numbers. Other meanings of the Riemann hypothesis are discovered in the context in which zeta functions arise. The zeta functions which most resemble the classical zeta function arise in complex analysis. The present work is concerned with the construction and properties of zeta functions which originate in hypercomplex analysis.

The merit of a large class of zeta functions lies in distracting attention from the properties of particular members of the class. Since the Riemann hypothesis applies to a large class of zeta functions, the significance of the conjecture lies in general properties of the spaces used to construct the functions. Of these spaces the simplest is the complex plane, whose properties are axiomatized in the concept of a field. Since the real line is also a field, it needs to be added that the complex plane admits an automorphism called conjugation. The field of real numbers is recovered as the set of self–conjugate elements of the complex plane. Since conjugated fields other than the complex plane exist, it appears at first sight that the field properties do not penetrate to the essence of the complex plane. Experience however shows that other fields are relevant to the complex plane for a description of its symmetries. The complex plane is not understood apart from other fields.

The attraction of complex analysis is such that those who enter its domain are reluctant to leave it. The reason for doing so is that Cartesian space is more fundamentally important then the complex plane. But Cartesian space lacks the multiplicative structure which accounts for the special properties of the complex plane. Multiplicative structure is recovered by embedding Cartesian space in a skew–field. Hypercomplex analysis is the resulting noncommutative generalization of complex analysis. The properties of field which prepare the analysis of the complex plane have analogues in properties of skew–fields for application to the hypercomplex plane.

The construction of a skew–field is made from the algebra of polynomials with rational coefficients. The Euclidean algorithm determines the structure of ideals in the polynomial algebra. The degree of a nonzero polynomial is the greatest nonnegative integer n such that the coefficient of z^n in the polynomial is nonzero. A product of nonzero polynomials

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is a nonzero polynomial whose degree is the sum of the degrees of the factors. If $f(z)$ is a polynomial and if $g(z)$ is a nonzero polynomial, then a polynomial $h(z)$ exists such that the degree of the polynomial

$$f(z) - g(z)h(z)$$

is less than the degree of $g(z)$ when it is nonzero. An ideal of the polynomial algebra which contains a nonzero element contains a nonzero element $g(z)$ of least degree. The elements of the ideal are generated by $g(z)$ as products $g(z)h(z)$ for a polynomial $h(z)$.

An algebraic plane is a quotient field of the polynomial algebra which is constructed from an isomorphism of the polynomial algebra into itself. If the isomorphism takes the indeterminate z into a polynomial z^- in z, then it takes a polynomial $f(z)$ into the composed polynomial

$$f^-(z) = f(z^-).$$

An ideal of the polynomial algebra is generated by the polynomial

$$1 - z^- z = a_0 + a_1 z + \ldots + a_r z^r.$$

If $f(z)$ and $g(z)$ are polynomials such that

$$g(z) - f(z^-)$$

belongs to the ideal, then

$$f(z) - g(z^-)$$

belongs to the ideal. The generating polynomial is assumed to have even degree and to have integer coefficients satisfying the symmetry condition

$$a_k = a_{r-k}$$

for every $k = 0, \ldots, r$. A linear functional on the polynomial algebra is defined as the trace

$$\mathrm{spur}'(f)$$

of the transformation induced by multiplication by $f(z)$ on the quotient algebra. The trace

$$\mathrm{spur}(f^* f)$$

is assumed to be nonnegative for every polynomial $f(z)$ and to be zero only when $f(z)$ belongs to the ideal. The quotient algebra is said to be an algebraic plane if every nonzero element has an inverse.

The conjugation of the algebraic plane is the automorphism c into c^- induced by the isomorphism $f(z)$ into $f^-(z)$ of the polynomial algebra. The identity

$$a = b^-$$

holds whenever

$$b = a^-.$$

The self–conjugate elements of the algebraic plane form a field. A self–conjugate element of the field is said to be nonnegative if it is a sum of products $c^- c$. A nonnegative element is said to be positive if it is nonzero.

An algebraic skew–plane is constructed from an algebraic plane by adjourning an element k which satisfies the identity

$$ck = kc^-$$

for every element c of the algebraic plane. The elements of the algebraic skew–plane are sums

$$a + kb$$

with a and b in the algebraic plane. The product

$$(a + kb)(c + kd)$$

of elements $a + kb$ and $c + kd$ of the algebraic skew–plane is the element

$$(ac - b^- d) + k(a^- d + bc)$$

of the algebraic skew–plane. The algebraic skew–plane is an associative algebra in which every nonzero element is invertible. Conjugation for the algebraic skew–plane is the anti–automorphism which takes $a + kb$ into

$$(a + kb)^- = a^- - kb.$$

The identity

$$(a + kb)^-(a + kb) = a^- a + b^- b$$

holds for all elements a and b of the algebraic plane.

An example of an algebraic plane is the quotient field of the polynomial algebra modulo the cyclotomic polynomial whose roots are the primitive r–th roots of unity for some positive integer r greater than two. The conjugation of the algebraic plane agrees with the inverse on a primitive r–th root of unity. The trace of a polynomial is the sum of the values of the polynomial at the primitive r–th roots of unity. The trace of the polynomial $f^- f$ is the sum

$$\sum |f(\infty)|^2$$

taken over the primitive r–th roots ω of unity. Every algebraic plane is so obtained.

A topology is defined on an algebraic skew–plane by its convex structure. A convex combination

$$a(1 - h) + bh$$

of elements a and b of the algebraic skew–plane is an element of the algebraic skew–plane defined by a nonnegative rational number h such that $1 - h$ is nonnegative. A subset of the algebraic skew–plane is said to be convex if it contains the convex combinations of any two of its elements.

The closure of a nonempty convex subset B of the algebraic skew–plane is the set B^- of elements c of the algebraic skew–plane such that for some element b of B the convex combination

$$c(1-h)+bh$$

belongs to B for every positive rational number h such that $1-h$ is nonnegative. The empty set is a convex set whose closure is defined to be itself. The closure of a convex set is a convex set whose closure is itself.

A nonempty convex subset of the algebraic skew–plane is said to be a disk if it is disjoint from the closure of every disjoint convex set. If A is a disk, if a is an element of A, and if b is an element of the algebraic skew–plane, then a positive rational number h exists such that $1-h$ is nonnegative and such that

$$a(1-h)+bh$$

belongs to A. A nonempty convex subset A of the algebraic skew–plane is a disk if for every element a of A and every element b of the algebraic skew–plane the convex combination

$$a(1-h)+bh$$

belongs to A for some positive rational number h such that $1-h$ is nonnegative.

If A is a disk and if B is a convex set, the intersection of A with the closure of B is contained in the closure of the intersection of A with B. The intersection of two disks is a disk if it is nonempty. The algebraic skew–plane is a Hausdorff space whose open set are the unions of disks. A subset of algebraic skew–plane is said to be closed if it is the complement of an open set. Intersections of closed sets are closed. A finite union of closed sets is closed. A convex set is closed if, and only if, it is equal to its closure.

The topology of the algebraic skew–plane is clarified by constructions of convex sets. If B is a nonempty convex set and if s is an element of the algebraic skew–plane, a convex subset $B(s)$ of the algebraic skew–plane is constructed whose closure contains s and the elements of B. The set $B(s)$ is defined as the set of convex combinations

$$s(1-h)+ch$$

of s with elements c of B for positive rational numbers h such that $1-h$ is positive. Convexity of $B(s)$ is proved by showing that a convex combination

$$[s(1-p)+ap](1-h)+[s(1-q)+bq]h$$

of elements

$$s(1-p)+ap$$

and

$$s(1-q)+bq$$

of $B(s)$ is an element
$$s(1-k)+ck$$
of $B(s)$. Since a and b are elements of the convex set B and since p and q are positive rational numbers such that $1-p$ and $1-q$ are positive, an element c of B is obtained as solution of the equation

$$c[p(1-h)+qh]=ap(1-h)+bqh.$$

The positive rational number
$$k=p(1-h)+qh$$
has the desired property that

$$1-k=(1-p)(1-h)+(1-q)h$$

is positive.

The Hahn–Banach theorem admits a formulation for the algebraic skew–plane. A disk A which is disjoint from a nonempty convex set B is contained in a disk which is disjoint from B and whose complement is convex. The proof is an application of the Kuratowski–Zorn lemma.

A nonempty convex set B which is disjoint from a disk A is contained in a maximal convex set which is disjoint from A. It is sufficient to consider the case in which B is a maximal convex set which is disjoint from A. The closure of B is disjoint from A since A is a disk. Since the closure of B is a convex set which contains B, it is equal to B. The Hahn–Banach theorem is proved by showing that the complement of B is convex. The convex set $B(s)$ contains an element of A when s belongs to the complement of B.

If u and v are elements of the complement of B, then an element

$$u(1-p)+ap$$

of $B(u)$ and an element
$$v(1-q)+bq$$
of $B(v)$ belong to A. A convex combination of u and v is a solution s of the equation

$$s[(1-p)(1-h)+(1-q)h]=u(1-p)(1-h)+v(1-q)h$$

for a nonnegative rational number h such that $1-h$ is nonnegative. An element c of B is obtained as a solution of the equation

$$c[p(1-h)+qh]=ap(1-h)+bqh.$$

The element
$$s[(1-p)(1-h)+(1-q)h]+c[p(1-h)+qh]$$

of $B(s)$ is a convex combination

$$[u(1-p)+ap](1-h)+[v(1-q)+bq]h$$

of elements of A. Since A is convex, s belongs to the complement of B.

A center of a convex set is an element c of the set such that $2c-a$ belongs to the set whenever a belongs to the set. A convex set is said to be centered at c if c is a center of the set. A construction of centered convex sets is made using the convexity identity

$$[a(1-h)+bh]^-[a(1-h)+bh]+h(1-h)(b-a)^-(b-a)=(1-h)a^-a+hb^-b$$

which holds for all elements a and b of the algebraic skew–plane when h is a nonnegative rational number such that $1-h$ is nonnegative. An example of a disk which is centered at an element c of the algebraic skew–plane is obtained for every positive rational number ϵ as the set of elements a which satisfy the inequality

$$(a-c)^-(a-c)<\epsilon.$$

The Euclidean skew–plane associated with an algebraic skew–plane is a completion of the algebraic skew–plane which is constructed from Cauchy classes of disks. A class of disks is said to be Cauchy if the intersection of a finite number of members of the class always contains a member of the class and if for every disk U centered at the origin a member C of the class exists such that $b-a$ belongs to U whenever a and b belong to C. Cauchy classes are considered equivalent if they are subclasses of the same Cauchy class. Every Cauchy class is contained in a maximal Cauchy class. The disk completion of an algebraic skew–field is the set whose elements are the maximal Cauchy classes. A Cauchy class is said to be convergent if the intersection of the members of the class is nonempty. A convergent Cauchy class contains only one element of the algebraic skew–field. An element of the algebraic skew–field determines an element of the disk completion consisting of the class of all disks containing the element.

The conjugate of a Cauchy class α is a Cauchy class β whose members are constructed from the members of α. If a disk A is of class α, the corresponding disk B of class β is obtained from A under the transformation c into c^- for elements of the algebraic skew–field. The equivalence class of β is determined by the equivalence class of α. If α is a maximal Cauchy class, then β is a maximal Cauchy class. If the Cauchy class α is determined by an element of the algebraic skew–field, then the Cauchy class β is determined by the conjugate element of the algebraic skew–field. If the Cauchy class

$$\beta=\alpha^-$$

is the conjugate of the Cauchy class α, then the Cauchy class

$$\alpha=\beta^-$$

is the conjugate of the Cauchy class β.

A Cauchy class is self–conjugate if, and only if, every member of the class contains a self–conjugate element of the algebraic skew–field. The inequality

$$\alpha \leq \beta$$

for self–conjugate Cauchy classes α and β is defined to mean that for every disk A of class α and for every disk B of class β the inequality

$$a \leq b$$

holds for some self–conjugate element a of A and for some self–conjugate element b of B. When α and γ are equivalent Cauchy classes and when β and δ are equivalent Cauchy classes, the inequality

$$\gamma \leq \delta$$

holds if, and only if, the inequality

$$\alpha \leq \beta$$

is satisfied. Self–conjugate Cauchy classes α and β are equivalent if the inequalities

$$\alpha \leq \beta$$

and

$$\beta \leq \alpha$$

are satisfied. The inequality

$$\gamma \leq \gamma$$

holds for every self–conjugate Cauchy class γ. The inequality

$$\alpha \leq \gamma$$

holds for Cauchy classes α and γ if the inequalities

$$\alpha \leq \beta$$

and

$$\beta \leq \gamma$$

hold for a Cauchy class β. When a Cauchy class α is determined by a self–conjugate element a of the algebraic skew–field and a Cauchy class β is determined by an element b of the algebraic skew–field, the inequality

$$\alpha \leq \beta$$

holds if, and only if, the inequality

$$a \leq b$$

is satisfied.

An open subset of the Euclidean skew–plane is determined by an open set A for the disk topology of the algebraic skew–plane. The open set A' of the Euclidean skew–plane is the set of maximal Cauchy classes whose members are disks having a nonempty intersection with A. Finite intersections of open subsets of the Euclidean skew–plane are open. Unions of open subsets of the Euclidean skew–plane are open. The Euclidean skew–plane is a Hausorff space which contains the algebraic skew–plane as a dense subset. The disk topology of the algebraic skew–plane is the subspace topology of the Euclidean skew–plane. An element of the Euclidean skew–plane is said to be algebraic if it is an element of the algebraic skew–plane.

Since addition is continuous as a transformation of the Cartesian product of the algebraic skew–plane with itself into the algebraic skew–plane in the disk topology, addition has a unique continuous extension as a transformation of the Cartesian product of the Euclidean skew–plane with itself into the Euclidean skew–plane. The associative law

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$$

holds for all elements α, β, and γ of the Euclidean skew–plane. The commutative law

$$\alpha + \beta = \beta + \alpha$$

holds for all elements α and β of the Euclidean skew–plane. The origin of the algebraic skew–plane is the origin of the Euclidean skew–plane since the identity

$$0 + \gamma = \gamma = \gamma + 0$$

holds for every element γ of the Euclidean skew–plane. For every element γ of the Euclidean skew–plane a unique element $-\gamma$ exists which satisfies the identities

$$(-\gamma) + \gamma = 0 = \gamma + (-\gamma).$$

Another completion of the algebraic skew–plane is constructed from the homomorphisms of the ring of integral elements into finite quotient ring. An element of the algebraic plane is said to be integral if it is represented by a polynomial with integral coefficients. Sums and products of integral elements are integral. The conjugate of an integral element is integral. A polynomial with integer coefficients is divisible by a positive integer r, if and only if, its coefficients are divisible by r. The quotient ring of the ring of polynomials with integer coefficients modulo the ideal of polynomial whose coefficients are divisible by r is isomorphic to the ring of polynomials with coefficients in the integers modulo r. Since the generating polynomial of the algebraic field has integer coefficients, it determines a polynomial with coefficients in the integers modulo r. The polynomial obtained with coefficients in the integers modulo r generates an ideal in the ring of polynomials in the integers modulo r. When r is not one, the quotient ring of the ring of polynomials with coefficients in the integers modulo r is a nontrivial finite ring.

An element ω of the algebraic skew–plane is said to be an integral unit if

$$\omega^n = 1$$

for some positive integer n and if for some

$$\omega^{-}\omega = 1.$$

An element of the algebraic skew–plane is said to be integral if it is a finite sum of units. Sums and products of integral elements of the algebraic skew–plane are integral. The conjugate of an integral element of the algebraic skew–plane is integral.

The adic topology of the algebraic skew–plane is a topology with respect to which addition is continuous as a transformation of the Cartesian product of the algebraic skew–plane with itself into the algebraic skew–plane. A basic neighborhood of the origin for the adic topology of the algebraic skew–plane is defined for a positive rational number t as the set of products $t\gamma$ with γ an integral element of the algebraic skew–plane.

The topology is computable on the ring of integral elements of the algebraic skew–plane. For every positive integer r a basic neighborhood of the origin is the set of products $r\gamma$ with γ an element of the ring. The neighborhood of the origin is an ideal of the ring which is closed under conjugation. Since the quotient ring is finite, it admits a unique topology with respect to which it is a Hausdorff space. Every subset of the quotient ring is both open and closed. Addition and multiplication are continuous as transformations of the Cartesian product of the quotient ring with itself into the quotient ring. Conjugation is continuous as a transformation of the quotient ring into itself. The adic topology of the ring of integral elements of the algebraic skew–plane is the weakest topology with respect to which the projection into the quotient ring is continuous for every positive integer r. Addition and multiplication are continuous as transformations of the Cartesian product of the ring of integral elements with itself into the ring. Conjugation is continuous as a transformation of the ring of integral elements into itself.

The adic skew–plane is an algebra over the rational numbers which contains the algebraic skew–plane as a subalgebra and which has a topology whose subspace topology is the adic topology of the algebraic skew–plane. Addition is continuous as a transformation of the Cartesian product of the adic skew–plane with itself into the adic skew–plane. The algebraic skew–plane is dense in the adic skew–plane. An element of the adic skew–plane is said to be integral if it belongs to the closure of the ring of integral elements of the algebraic skew–plane. The set of integral elements of the adic skew–plane is a compact open subring of the adic skew–plane. Multiplication by a positive rational number is a continuous transformation of the adic skew–plane into itself. Every element of the adic skew–plane is a product

$$t\gamma$$

of a positive rational number t and an integral element γ of the adic skew–plane. Multiplication is continuous as a transformation of the Cartesian product of the ring of integral elements with itself into the ring. Conjugation is continuous as a transformation of the adic skew–plane with itself into the adic skew–plane. An element of the adic skew–plane is said to be algebraic if it belongs to the algebraic skew–plane.

The p–adic skew–plane is a quotient algebra of the adic skew–plane which is defined for every prime p. The p–adic skew–plane is constructed as the completion of a quotient space of the algebraic skew–plane in a topology for which addition is continuous as a transformation of the Cartesian product of the skew–field with itself into the skew–field. A basic neighborhood of the origin for the p–adic topology is an ideal of the ring of integral elements which is generated by a power of the prime p. Since the ideal is invariant under automorphisms and the conjugation, the finite quotient algebra of the ring of integral elements inherits automorphisms and a conjugation. The completion of the ring of integral elements is a compact Hausdorff space since the p–adic topology of the ring of integral elements is the weakest topology with respect to which each projection onto a finite quotient algebra is continuous. An element of the p–adic skew–plane is said to be integral if it belongs to the closure of the ring of integral elements of the algebraic skew–plane in the p–adic topology. The ring of integral elements of the p–adic skew–plane is a compact Hausdorff space which is a neighborhood of the origin for the p–adic topology of the p–adic skew–plane. Addition has a continuous extension as a transformation of the Cartesian product of the p–adic skew–plane with itself into the p–adic skew–plane. The automorphisms and the conjugation of the algebraic skew–field have continuous extensions as automorphisms and a conjugation of the p–adic skew–plane. An element of the p–adic skew–plane is said to be algebraic if it is an element of the algebraic skew–plane. Multiplication by an algebraic element of the p–adic skew–plane has a continuous extension as a transformation of the p–adic skew–plane into itself. An invertible integral element of the p–adic skew–plane is said to be a unit if its inverse is integral. An element of the p–adic skew–plane is invertible if, and only if, it is the product of a unit of the p–adic skew–plane and a nonzero algebraic element of the p–adic skew–plane.

The adic skew–plane is canonically isomorphic to a subalgebra of the Cartesian product of the p–adic skew–planes taken over all primes p. An element of the Cartesian product determines an element of the adic skew–plane if, and only if, it is integral for all but a finite number of primes p.

The compactification of the Euclidean skew–plane is a quotient space of the Cartesian product of the Euclidean skew–plane and the adic skew–plane. An element ξ of the Cartesian product has an Euclidean component ξ_+, which is an element of the Euclidean skew–plane, and an adic component ξ_-, which is an element of the adic skew–plane. An algebraic element of the Euclidean skew–plane is an element of the adic skew–plane. Elements ξ and η of the Cartesian product are considered equivalent if

$$\eta_+ = \xi_+ + \gamma$$

and

$$\eta_- = \xi_- - \gamma$$

for an element γ of the algebraic skew–plane. The sum of elements α and β of the quotient space is the element

$$\gamma = \alpha + \beta$$

of the quotient space whose Euclidean component

$$\gamma_+ = \alpha_+ + \beta_+$$

is the sum of the Euclidean components of α and β and whose adic component

$$\gamma_- = \alpha_- + \beta_-$$

is the sum of the adic components of α and β. The definition does not depend on the choice of representatives in equivalence classes.

The product

$$\eta = \gamma\xi$$

of an element γ of the algebraic skew–plane and an element ξ of the quotient space is the element η of the quotient space whose Euclidean component

$$\eta_+ = \gamma\xi_+$$

is the product of γ with the Euclidean component of ξ and whose adic component

$$\eta_- = \gamma\xi_-$$

is the product of γ with the adic component of ξ. The definition does not depend on the choice of representatives in equivalence classes.

The product

$$\eta = \xi\gamma$$

of an element γ of the algebraic skew–plane and an element ξ of the quotient space is the element η of the quotient space whose Euclidean component

$$\eta_+ = \xi_+\gamma$$

is the product of γ with the Euclidean component of ξ and whose adic component

$$\eta_- = \xi_-\gamma$$

is the product of γ with the adic component of ξ. The definition does not depend on the choice of representatives in equivalence classes.

The quotient space inherits a conjugation. The quotient space is a compact Hausdorff space in the quotient topology of the Cartesian product space. A computation of topology results from the construction of a fundamental domain for the equivalence relation on the Cartesian product space. If ξ is an element of the Euclidean skew–plane, a nearest integral element η of the Euclidean skew–plane is found by minimizing the nonnegative number

$$\text{spur } [(\xi - \eta)^-(\xi - \eta)].$$

The fundamental domain for the Cartesian product space is the set of elements with integral adic component for which the integral element of the Euclidean skew–plane nearest the Euclidean component is unique and is the origin. The fundamental domain is an open set whose closure is compact. Addition by an element of the Cartesian product space equivalent to the origin maps the fundamental region onto an open set containing the element. Open set containing distinct elements equivalent to the origin are disjoint. The Cartesian product space is the union of the closures of the open sets.

The Euclidean skew–plane, the adic skew–plane, the Cartesian product, and the quotient space are examples of locally compact additive groups. In Fourier analysis a character is defined as a continuous homomorphism of the group into the multiplicative group of complex numbers of absolute value one. A character for the Euclidean skew–plane is a function

$$\exp[\pi i \operatorname{spur}(\xi^- \eta + \eta^- \xi)]$$

of ξ in the Euclidean skew–plane. The trace

$$\operatorname{spur}(\xi^- \eta + \eta^- \xi)$$

is a rational number when ξ and η are elements of the algebraic skew–plane. The trace is an integer when ξ and η are integral.

If η is an element of the algebraic skew–plane, the function

$$\exp[\pi i \operatorname{spur}(\xi^- \eta + \eta^- \xi)]$$

of elements ξ of the algebraic skew–plane is continuous with respect to the adic topology. The function has a unique continuous extension as a function

$$\exp[\pi i \operatorname{spur}(\xi^- \eta + \eta^- \xi)]$$

of elements ξ of the adic skew–plane. If η is an element of the adic skew–plane, the function

$$\exp[\pi i \operatorname{spur}(\xi^- \eta + \eta^- \xi)]$$

of elements ξ of the algebraic skew–plane is continuous with respect to the adic topology. The function has a unique continuous extension as a function

$$\exp[\pi i \operatorname{spur}(\xi^- \eta + \eta^- \xi)]$$

of ξ in the adic skew–plane.

A Fourier character for the Cartesian product space is determined by an element η of the Cartesian product space as a function

$$\exp[\pi i \operatorname{spur}(\xi_+^- \eta_+ + \eta_+^- \xi_+)] \exp[\pi i \operatorname{spur}(\xi_-^- \eta_- + \eta_-^- \xi_-)]$$

of ξ in the Cartesian product space. A Fourier character for the quotient space is then determined when the Euclidean and adic components of η are equal elements of the algebraic skew–plane.

A dense set of elements of the fundamental domain in the Cartesian product space have integral algebraic elements as adic component. Since these elements of the fundamental domain are equivalent to elements of the Cartesian product space with vanishing adic component, they are determined by elements of the Euclidean skew–plane. The quotient space is isomorphic as a locally compact additive group to the completion of the Euclidean skew–plane in a topology with respect to which addition is continuous as a transformation of the Cartesian product of the Euclidean skew–plane with itself into the Euclidean skew–plane. The topology is the weakest topology with respect to which

$$\exp[\pi i \,\mathrm{spur}(\xi^- \eta + \eta^- \xi)]$$

is a continuous function of ξ in the Euclidean skew–plane for every element η of the algebraic skew–plane.

Haar measure for a locally compact additive group is an essentially unique nonnegative measure on the Baire subsets of the group which is invariant under translation, has finite values on compact sets, and has positive values on nonempty open sets. The measure is unique within a constant factor. Since the Euclidean skew–plane is a vector space of finite dimension over the real numbers, Haar measure is a normalization of Lebesgue measure. The measure is normalized so that measure one is assigned to the set of elements of the Euclidean skew–plane which have the origin as nearest integral element. Haar measure for the adic skew–plane is normalized so that measure one is assigned to the set of integral elements. Haar measure for the Cartesian product space is the Cartesian product of Haar measure for the Euclidean skew–plane and Haar measure for the adic skew–plane. The fundamental domain of the Cartesian product space has measure one. Haar measure for the quotient space is determined by Haar measure on the fundamental domain.

The Fourier transformation for the Euclidean skew–plane is an isometric transformation of the space of square integrable functions with respect to Haar measure onto itself. The transformation takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into a function $g(\eta)$ of η in the Euclidean skew–plane when the integral

$$g(\eta) = \int \exp[\pi i \,\mathrm{spur}(\xi^- \eta + \eta^- \xi)]$$

is absolutely convergent. The inverse transformation takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into a function $g(\eta)$ of η in the Euclidean skew–plane when the Fourier transformation takes the conjugate function $f(\xi)^-$ of ξ in the Euclidean skew–plane into the conjugate function $g(\eta)^-$ of η in the Euclidean skew–plane.

The Fourier transformation for the adic skew–plane is an isometric transformation of the space of square integrable functions with respect to Haar measure onto itself. The transformation takes a function $f(\xi)$ of ξ in the adic skew–plane into a function $g(\eta)$ of η in the adic skew–plane when the integral

$$g(\eta) = \int \exp[\pi i \,\mathrm{spur}(\xi^- \eta + \eta^- \xi)]$$

is absolutely convergent. The inverse transformation takes a function $f(\xi)$ of ξ in the adic skew–plane into a function $g(\eta)$ of η in the adic skew–plane when the Fourier transformation takes the conjugate function $f(\xi)^-$ of ξ in the adic skew–plane into the conjugate function $g(\eta)^-$ of η in the adic skew–plane.

The Fourier transformation for the Cartesian product space is an isometric transformation of the space of square integrable functions with respect to Haar measure onto itself. The transformation takes a function $f(\xi)$ of ξ in the Cartesian product space into a function $g(\eta)$ of η in the Cartesian product space when the integral

$$g(\eta) = \int \exp[\pi i \operatorname{spur}(\xi_+^- \eta_+ + \eta_+^- \xi_+) \exp[\pi i \operatorname{spur}(\xi_-^- \eta_- + \eta_-^- \xi_-)]$$

is absolutely convergent. The inverse transformation takes a function $f(\xi)$ of ξ in the Cartesian product space into a function $g(\eta)$ of η in the Cartesian product space when the Fourier transformation takes the conjugate function $f(\xi)^-$ of ξ in the Cartesian product space into the conjugate function $g(\eta)^-$ of η in the Cartesian product space.

The Fourier transformation for the quotient space is an isometric transformation of the space of square integrable functions with respect to Haar measure for the quotient space onto the space of square summable functions of algebraic elements of the Euclidean skew–plane. The transformation takes a function $f(\xi)$ of ξ in the quotient space into a function $g(\eta)$ of elements of the algebraic skew–plane defined by the integral

$$g(\eta) = \int \exp[\pi i \operatorname{spur}(\xi_+^- \eta_+ + \eta_+^- \xi_+)] \exp[\pi i \operatorname{spur}(\xi_-^- \eta_- + \eta_-^- \xi_-)]$$

with respect to Haar measure for the quotient space. The identity

$$\int |f(\xi)|^2 d\xi = \sum |g(\eta)|^2$$

holds with integration with respect to Haar measure for the quotient space and with summation over the elements of the algebraic skew–plane. The inverse transformation takes a function $f(\xi)$ of elements of the algebraic skew–plane into a function $g(\eta)$ of η in the quotient space defined by the orthogonal expansion

$$g(\eta)^- = \sum \exp[\pi i \operatorname{spur}(\xi_+^- \eta_+ + \eta_+^- \xi_-)] \exp[\pi i \operatorname{spur}(\xi_-^- \eta_- + \eta_-^- \xi_-)]$$

of conjugate functions.

Poisson summation constructs an integrable function with respect to Haar measure on the quotient space from an integrable function with respect to Haar measure on the Cartesian product space. The Poisson sum

$$f'(\xi) = \sum f(\xi + \eta)$$

of a function $f(\xi)$ of ξ in the Cartesian product space is a function $f'(\xi)$ of ξ in the quotient space defined by summation over the elements η of the Cartesian product space which are equivalent to the origin. The inequality

$$\int |f'(\xi)| d\xi \leq \int |f(\xi)| d\xi$$

holds with integration on the left with respect to Haar measure for the quotient space and with integration on the right with respect to Haar measure for the Cartesian product space. If γ is an element of the Cartesian product space whose Euclidean component γ_+ and whose adic component γ_- are equal elements of the algebraic skew–plane, then the integral

$$\int \exp[\pi i \operatorname{spur}(\xi_+^- \gamma_+ + \gamma_+^- \xi_+)] \exp[\pi i \operatorname{spur}(\xi_-^- \gamma_- + \gamma_-^- \xi_-)]$$

with respect to Haar measure for the quotient space is equal to the integral

$$\int \exp[\pi i \operatorname{spur}(\gamma_+^- \xi_+ + \xi_+^- \gamma_+)] \exp[\pi i \operatorname{spur}(\gamma_-^- \xi_- + \xi_- \gamma_-)]$$

with respect to Haar measure for the Cartesian product space.

The Poisson formula

$$\sum f(\eta) = \sum g(\eta)$$

states that the Poisson sum of a bounded integrable function $f(\xi)$ of ξ in the Cartesian product space has the same value at the origin as the Poisson sum of its Fourier transform when the Fourier transform is a bounded integrable function $g(\xi)$ of ξ in the Cartesian product space. Boundedness and integrability imply the square integrability condition under which the Fourier transformation has been defined. The function values applied in the Poisson formula are meaningful since the Fourier transform of an integrable function is continuous. The Poisson formula is obtained by Fourier analysis on the quotient space. The Fourier coefficients of a function of ξ on the quotient space are determined by the values $g(\eta)$ when η is equivalent to zero since this condition means that the Euclidean component η_+ and the adic component η_- are algebraic elements with sum zero. The sum

$$\sum g(\eta)$$

taken over the elements η of the Cartesian product space which are equivalent to zero is the value at the origin of a function

$$\sum \exp[\pi i \operatorname{spur}(\xi_+^- \eta_+ + \eta_+^- \xi_+)] \exp[-\pi i \operatorname{spur}(\xi_-^- \eta_- - \eta_-^- \xi_-)]$$

of ξ in the Cartesian product space which has the same value at the origin as $f'(\xi)$.

A function of

$$\xi = t + ix + jy + kz$$

is an Euclidean skew–field is said to be a homogeneous polynomial of degree ν if it is a linear combination of monomials

$$x^a y^b z^c t^d$$

whose exponents a, b, c, and d are nonnegative integers with sum ν. The space of homogeneous polynomials of degree ν is a Hilbert space which admits the monomials as an orthogonal basis. The scalar self–product of the monomial with exponents a, b, c, and c is

$$\frac{a!\; b!\; c!}{\nu!}.$$

Isometric transformations of the space of homogeneous polynomials of degree ν into itself are defined by taking a function $f(\xi)$ of ξ in the Euclidean skew–field into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the Euclidean skew–field for every unit ω of the Euclidean skew–field.

The Laplacian
$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial t^2}$$
acts as a linear transformation of the space of homogeneous polynomials of degree ν into the space of homogeneous polynomials of degree $\nu-2$ when ν is greater than one and annihilates homogeneous polynomials of smaller degree. The Laplacian commutes with the transformations which take a function $f(\xi)$ of ξ in the Euclidean skew–field into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the Euclidean skew–field for every unit ω of the Euclidean skew–field.

A homogeneous polynomial of degree ν is said to be harmonic if it is annihilated by the Laplacian. The homogeneous harmonic polynomials of degree ν form an invariant subspace of dimension $1+2\nu$ for the transformation which takes $f(\xi)$ into $f(\omega^-\xi\omega)$ for every unit ω of the Euclidean skew–field.

A harmonic polynomial of degree ν is a homogeneous polynomial of degree ν which is annihilated by the Laplacian. The harmonic polynomials of degree ν form an invariant subspace for the transformations which take a function $f(\xi)$ of ξ in the Euclidean skew–field into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the Euclidean skew–field for every unit ω of the Euclidean skew–field. The dimension of the space of homogeneous polynomials of degree ν is
$$(\nu+1)(\nu+2)(\nu+3)/6.$$
The dimension of the space of harmonic polynomials of degree ν is
$$(\nu+1)^2.$$

Commuting self–adjoint transformations $\Delta(n)$ are defined in the space of harmonic polynomials of degree ν for every positive integer n. The transformation $\Delta(n)$ takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into the function $g(\xi)$ of ξ in the Euclidean skew–plane defined by the summations
$$n^{\frac{1}{2}\nu}g(\xi)=\sum f(\omega^-\xi)$$
over the integral elements ω of the algebraic skew–field such that
$$n=\omega^-\omega.$$
The identity
$$\Delta(m)\Delta(n)=\sum\Delta(mn/k^2)\Delta(1)$$

holds for all positive integers m and n with summation over the common divisors k of m and n. Each transformation $\Delta(n)$ commutes with the transformation which takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into the function

$$f^*(\xi) = f(\xi^-)^-$$

of ξ in the Euclidean skew–plane.

The transformations $\Delta(n)$ take harmonic polynomials of degree ν into harmonic polynomials of degree ν. The transformations are self–adjoint in the space of harmonic polynomials of degree v. The space is the orthogonal sum of simultaneous invariant subspaces for the transformations $\Delta(n)$. Each invariant subspace is invariant under the transformation $f(\xi)$ into $f^*(\xi)$. The nonzero elements of an invariant subspace are characterized as eigenfunctions of $\Delta(n)$ for a real eigenvalue $\tau(n)$ for every positive integer n. The identity

$$\tau(m)\tau(n) = \sum \tau(mn/k^2)$$

holds for all positive integers m and n with summation over the common divisors k of m and n.

A nonzero harmonic polynomial of degree ν, which is an eigenfunction of the transformation $\Delta(n)$ for some eigenvalue $\tau(n)$ for every positive integer n, is said to be primitive modulo ν if no nonzero harmonic polynomial of smaller degree exists which is an eigenfunction of the transformation $\Delta(n)$ for the same eigenvalue $\tau(n)$ for every positive integer n.

The Poisson summation formula is applied to functions of ξ in the Cartesian product of the Euclidean skew–plane and the adic skew–plane which vanish when the adic component of ξ is not integral and whose value is independent of the adic component of ξ when the adic component of ξ is not integral. Such a function on the Cartesian product space is determined by the function $f(\xi_+)$ of ξ_+ in the Euclidean skew–plane which is obtained when the adic component of ξ is integral. The function of ξ in the Cartesian product space is square integrable with respect to Haar measure for the Cartesian product space if, and only if, the function $f(\xi_+)$ of ξ_+ in the Euclidean skew–plane is square integrable with respect to Haar measure for the Euclidean skew–plane. The Fourier transform for the Cartesian product space of the square integrable function of ξ in the Cartesian product space is a function of ξ in the Cartesian product space which vanishes when the adic component of ξ is not integral and whose value is independent of ξ when the adic component of ξ is integral. The function obtained on the Cartesian product is determined by the function $g(\xi_+)$ of ξ_+ in the Euclidean skew–plane obtained when the adic component of ξ is integral. The function $g(\xi)$ of ξ in the Euclidean skew–plane is the Fourier transform for the Euclidean skew–plane of the function $f(\xi)$ of ξ in the Euclidean skew–plane. The functions $f(\xi)$ and $g(\xi)$ of ξ in the Euclidean skew–plane are integrable with respect to Haar measure for the Euclidean skew–plane if, and only if, the functions determined on the Cartesian product space are integrable with respect to Haar measure on the Cartesian product space. The Poisson summation formula

$$\sum f(\eta) = \sum g(\eta)$$

then applies with summation over the integral elements η of the algebraic skew–field.

A computation of Fourier transforms is made when the Euclidean skew–plane is the Euclidean skew–field of hypercomplex numbers with real coordinates. If z is in the upper half–plane, the function

$$\varphi(\xi)\exp(\pi iz\xi^-\xi)$$

of ξ in the Euclidean skew–plane is square integrable with respect to Haar measure for the Euclidean skew–plane whenever $\varphi(\xi)$ is a harmonic polynomial of degree ν. The Fourier transform for the Euclidean skew–plane is a function

$$(i/z)^{2+4\nu}\psi(\xi)\exp(-\pi iz^{-1}\xi^-\xi)$$

of ξ in the Euclidean skew–plane for a harmonic polynomial $\psi(\xi)$ of degree ν. If the harmonic polynomial $\varphi(\xi)$ is an eigenfunction of $\Delta(n)$ for the eigenvalue $\tau(n)$ for every positive integer n, then the harmonic polynomial $\psi(\xi)$ is an eigenfunction of $\Delta(n)$ for the eigenvalue $\tau(n)$ for every positive integer n. If the harmonic polynomial $\varphi(\xi)$ is primitive for the given eigenvalues, then the polynomials $\varphi(\xi)$ and $\psi(\xi)$ are linearly dependent. The identity

$$\psi(\xi) = \sigma\varphi(\xi)$$

holds for a real number σ of absolute value one.

The Poisson summation formula

$$(i/z)^{2+\nu}\sum\phi(\xi)\exp(-\pi iz^{-1}\xi^-\xi) = \sigma\sum\phi(\xi)\exp(\pi iz\xi^-\xi)$$

holds with summation over the integral elements ξ of the algebraic skew–field when z is in the upper half–plane. Since the identity

$$\tau(n)n^{\frac{1}{2}\nu}\phi(1) = \sum\varphi(\xi^-\xi)$$

holds for every positive integer n with summation over the integral elements ξ of the algebraic skew–field such that

$$n = \xi^-\xi,$$

the Poisson summation formula reads

$$(i/z)^{2+\nu}\sum\theta 6au(n)n^{\frac{1}{2}\nu}\exp(-\pi inz^{-1}) = \sigma\sum\tau(n)n^{\frac{1}{2}\nu}\exp(\pi inz)$$

with summation over the positive integers n when z is in the upper half–plane.

The modular group is the set of matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with integer entries and determinant one. The signature for the modular group is a homomorphism of the group into the fourth roots of unity whose action on the matrix is written

$$\operatorname{sgn}\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

has signature minus one. The matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has signature i. The identity

$$\theta(z) = \operatorname{sgn}^\nu \begin{pmatrix} A & B \\ C & D \end{pmatrix} \frac{1}{(Cz+D)^{2+\nu}} \; \theta\left(\frac{Az+B}{Cz+D}\right)$$

holds for every element

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of the modular group when

$$\theta(z) = \sum \tau(n) n^{\frac{1}{2}\nu} \exp(\pi i n z).$$

The zeta function

$$\zeta(s) = \sum n^{-\frac{1}{2}} \tau(n) n^{-s}$$

is defined as a sum over the positive integers n in the half–plane $\mathcal{R}s > 1$. The zeta function has an analytic extension as a function of s in the complex plane which satisfies the functional identity

$$\begin{aligned} &\pi^{-\frac{1}{2}\nu-\frac{1}{2}+s-1}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}+1-s)\zeta(1-s) \\ &= \sigma\pi^{-\frac{1}{2}\nu-\frac{1}{2}-s}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}+s)\zeta(s). \end{aligned}$$

The Euler product

$$\zeta(s)^{-1} = \prod(1 - p^{-\frac{1}{2}}\tau(p)p^{-s}),$$

taken over the primes p, converges in the half–plane $\mathcal{R}s > 1$.

The domain of the Laplace transformation of order ν and harmonic ϕ for the Euclidean skew–plane is the space of functions $f(\xi)$ of ξ in the Euclidean skew–plane which satisfy the identity

$$\phi(\xi) f(\xi\omega) = \phi(\xi\omega) f(\xi)$$

for every unit ω of the Euclidean skew–plane and which are square integrable with respect to Haar measure for the Euclidean skew–plane. The Laplace transform of the function $f(\xi)$ of ξ in the Euclidean skew–plane is the analytic function

$$g(z) = \int \phi(\xi)^- f(\xi) \exp(\pi i z \xi^- \xi) d\xi$$

of z in the upper half–plane defined by integration with respect to Haar measure for the Euclidean skew–plane. When w is in the upper half–plane, the Laplace transform of the function

$$\phi(\xi) \exp(-\pi i w^- \xi^- \xi)$$

of ξ in the upper half–plane is the function

$$\pi \frac{\|\phi\|^2 \Gamma(2+\nu)}{(\pi i w^- - \pi i z)^{2+\nu}}$$

of z in the upper half–plane with the norm of the harmonic polynomial ϕ taken in the Hilbert space of homogeneous polynomials of degree ν. An analytic function $g(z)$ of z in the upper half–plane is a Laplace transform of order ν and harmonic ϕ if, and only if, the least upper bound

$$\sup \int_0^\infty \int_{-\infty}^{+\infty} |g(x+iy)|^2 y^\nu dx dy,$$

taken over all positive numbers y, is finite. The least upper bound is then equal to the integral

$$(2\pi)^{-\nu} \Gamma(1+\nu) \|\phi\|^2 \int |f(\xi)|^2 d\xi$$

with respect to Haar measure for the Euclidean skew–plane.

An isometric transformation of the range of the Laplace transformation onto itself is defined by taking an analytic function $g(z)$ of z in the upper half–plane into the analytic function

$$(i/z)^{2+\nu} g(-1/2)$$

of z in the upper half–plane. The Hankel transformation of order ν and harmonic ϕ for the Euclidean skew–plane is an isometric transformation of the domain of the Laplace transformation onto itself which takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into a function $g(\xi)$ of ξ in the Euclidean skew–plane when the identity

$$\int \phi(\xi)^- g(\xi) \exp(\pi i z \xi^- \xi) d\xi$$
$$= (i/z)^{2+\nu} \int \phi(\xi)^- f(\xi) \exp(-\pi i z^{-1} \xi^- \xi) d\xi$$

holds for z in the upper half–plane with integration with respect to Haar measure for the Euclidean plane. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to Haar measure for the Euclidean skew–plane. The function $f(\xi)$ of ξ in the Euclidean skew–plane is the Hankel transform of order ν and harmonic ϕ for the Euclidean skew–plane of the function $g(\xi)$ of ξ in the Euclidean skew–plane.

The Mellin transformation of order ν and harmonic ϕ for the Euclidean skew–plane is a spectral theory for the Laplace transformation of order ν and harmonic ϕ for the Euclidean skew–plane. The domain of the Mellin transformation of order ν and harmonic ϕ for the Euclidean skew–plane is the space of functions which belong to the domain of the Laplace transformation of order ν and harmonic ϕ for the Euclidean skew–plane and which vanish in a neighborhood of the origin. The Laplace transform of a function $f(\xi)$ of ξ in the Euclidean skew–plane is an analytic function

$$g(z) = \int \phi(\xi)^- f(\xi) \exp(\pi i z \xi^- \xi) d\xi$$

of z in the upper half–plane defined by integration with respect to Haar measure for the Euclidean skew–plane. The Mellin transform of order ν and harmonic ϕ for the Euclidean skew–plane of the function $f(\xi)$ of ξ in the Euclidean skew–plane is the analytic function

$$F(z) = \int_0^\infty g(it) t^{\frac{1}{2}\nu - iz} dt$$

of z in the upper half–plane. Since the function

$$W(z) = \pi^{-\frac{1}{2}\nu - \frac{1}{2} + iz} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz)$$

admits the integral representation

$$W(z) = (\xi^- \xi)^{\frac{1}{2}\nu + 1 - iz} \int_0^\infty \exp(-\pi t \xi^- \xi) t^{\frac{1}{2}\nu - iz} dt$$

when z is in the upper half–plane, the identity

$$F(z)/W(z) = \int \phi(\xi)^- f(\xi) (\xi^- \xi)^{iz - 1 - \frac{1}{2}\nu} d\xi$$

holds when z is in the upper half–plane with integration with respect to Haar measure for the Euclidean skew–plane. If $f(\xi)$ vanishes when

$$\xi^- \xi < a$$

for some positive number a, then the least upper bound

$$\sup \int_{-\infty}^{+\infty} a^y |F(x+iy)/W(x+iy)|^2 dN$$

taken over all positive numbers y is equal to the integral

$$2\pi^2 \|\phi\|^2 \int |f(\xi)|^2 d\xi$$

with respect to Haar measure for the Euclidean skew–plane.

The Radon transformation of order ν and harmonic ϕ for the Euclidean skew–plane is a maximal dissipative transformation in the domain of the Laplace transformation of order ν and harmonic ϕ for the Euclidean skew–plane. The transformation takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into a function $g(\xi)$ of ξ in the Euclidean skew–plane when the identity

$$g(\xi) = \int f(\xi + \eta)|\eta|^{-\frac{1}{2}} d\eta$$

holds formally with integration with respect to a normalization of Haar measure for the space of elements η of the Euclidean skew–plane such that

$$\eta^- \xi + \xi^- \eta = 0.$$

Haar measure is normalized so that the set of elements η such that $\eta^- \eta < 1$ has measure $4\pi/3$. The integral is accepted as the definition when

$$f(\xi) = \phi(\xi) \exp(\pi i z \xi^- \xi)$$

with z in the upper half–plane, in which case

$$g(\xi = (2i/z) f(\xi).$$

The adjoint of the Radon transformation of order ν and harmonic ϕ for the Euclidean skew–plane takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into a function $g(\xi)$ of ξ in the Euclidean skew–plane when the identity

$$\int \phi(\xi)^- g(\xi) \exp(\pi i z \xi^- \xi) d\xi$$
$$= (2i/z) \int \phi(\xi)^- f(\xi) \exp(\pi i z \xi^- \xi) d\xi$$

holds when z is in the upper half–plane with integration with respect to Haar measure for the Euclidean skew–plane.