THE HAHN–BANACH THEOREM FOR WEIERSTRASS ALGEBRAS

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Abstract. The Hahn–Banach theorem is formulated as a construction of invariant subspaces for algebras of transformations of a complex vector space into itself when the transformations are adjoints of transformations of a conjugate dual space into itself. A two–sided representation of the algebra is constructed when the vector space and its conjugate dual space are completions of a vector space in duality with itself with a scalar product which is symmetric and positive. The representation is assumed to admit a topology compatible with a convex structure constructed from the representation. The Hahn–Banach theorem separates a convex set from an open convex set by a linear functional with convex kernel. The construction of invariant subspaces is applied to a problem of polynomial approximation underlying the Riemann mapping theorem.

A Weierstrass algebra is an algebra of transformations of a complex vector space $\mathcal{H}$ into itself whose adjoints are transformations of the conjugate dual space $\mathcal{H}^*$ into itself. The scalar product

$$\langle a, b\rangle$$

of an element a of $\mathcal{H}$ and an element b of $\mathcal{H}^*$ is a complex number which is a linear function of a for fixed b and a conjugate linear function of b for fixed a. If b is an element of $\mathcal{H}^*$, b^- is the linear functional on $\mathcal{H}$ defined by the scalar product

$$b^- a = \langle a, b\rangle$$

for every element a of $\mathcal{H}$. The origin is assumed to be the only element of $\mathcal{H}$ which is annihilated by b^- for every element b of $\mathcal{H}^*$. The scalar product

$$\langle b, a\rangle = \langle a, b\rangle^-$$

of an element b of $\mathcal{H}^*$ and an element a of $\mathcal{H}$ is the conjugate of the scalar product of the element a of $\mathcal{H}$ and the element b of $\mathcal{H}^*$. If a is an element of $\mathcal{H}$, a^- is the linear functional on $\mathcal{H}^*$ defined by the scalar product

$$a^- b = \langle b, a\rangle$$

for every element b of $\mathcal{H}^*$. The origin is assumed to be the only element of $\mathcal{H}^*$ which is annihilated by a^- for every element a of $\mathcal{H}$.

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The weak topology of $\mathcal{H}$ induced by $\mathcal{H}^*$ is the weakest topology with respect to which b^- is continuous for every element b of $\mathcal{H}^*$. The adjoint of a continuous transformation T of $\mathcal{H}$ into itself is the transformation T^- of $\mathcal{H}^*$ into itself defined by the identity

$$\langle Ta, b\rangle = \langle a, T^- b\rangle$$

for every element a of $\mathcal{H}$ and every element b of $\mathcal{H}^*$. The weak topology of $\mathcal{H}^*$ induced by $\mathcal{H}$ is the weakest topology with respect to which a^- is continuous for every element a of $\mathcal{H}$. A transformation of $\mathcal{H}^*$ into itself is weakly continuous if, and only if, it is the adjoint of a weakly continuous transformation of $\mathcal{H}$ into itself. The adjoint of a weakly continuous transformation S of $\mathcal{H}^*$ into itself is the transformation S^- of $\mathcal{H}$ into itself defined by the identity

$$\langle S^- a, b\rangle = \langle a, Sb\rangle$$

for every element a of $\mathcal{H}$ and every element b of $\mathcal{H}^*$. A transformation of $\mathcal{H}$ into itself is weakly continuous if, and only if, it is the adjoint of a weakly continuous transformation of $\mathcal{H}^*$ into itself. If T is a transformation of $\mathcal{H}$ into itself and if S is a transformation of $\mathcal{H}^*$ into itself, then

$$S = T^-$$

is the adjoint of T if, and only if,

$$T = S^-$$

is the adjoint of S.

If a is an element of $\mathcal{H}$ and if b is an element of $\mathcal{H}^*$, a continuous transformation ab^- of $\mathcal{H}$ into itself is defined by

$$(ab^-)c = a(b^- c)$$

for every element c of $\mathcal{H}$. The adjoint is the transformation ba^- of $\mathcal{H}^*$ into itself defined by

$$(ba^-)c = b(a^- c)$$

for every element c of $\mathcal{H}^*$. A continuous transformation of $\mathcal{H}$ into itself with finite–dimensional range is a finite sum of transformations ab^- with a in $\mathcal{H}$ and b in $\mathcal{H}^*$. A continuous transformation of $\mathcal{H}^*$ into itself with finite–dimensional range is a finite sum of transformations ba^- with a in $\mathcal{H}$ and b in $\mathcal{H}^*$. A continuous transformation of $\mathcal{H}$ into itself has finite–dimensional range if, and only if, its adjoint is a continuous transformation of $\mathcal{H}^*$ into itself which has finite–dimensional range.

The trace of a continuous transformation T of $\mathcal{H}$ into itself with finite–dimensional range is a complex number

$$spur(T)$$

which is a linear function of T and which is equal to

$$b^- a$$

when

$$T = ab^-$$

for an element a of $\mathcal{H}$ and an element b of $\mathcal{H}^*$. The trace of a continuous transformation S of $\mathcal{H}^*$ into itself with finite–dimensional range is a complex number

$$spur(S)$$

which is a linear function of S and which is equal to

$$a^- b$$

when

$$S = ba^-$$

for an element a of $\mathcal{H}$ and an element b of $\mathcal{H}^*$. The identity

$$spur(S) = spur(T)^-$$

holds when

$$S = T$$

is the adjoint of a continuous transformation T of $\mathcal{H}$ into itself with finite–dimensional range.

The compositions ST and TS of a continuous transformation S of $\mathcal{H}$ into itself with finite–dimensional range and a continuous transformation T of $\mathcal{H}$ into itself are continuous transformations of $\mathcal{H}$ into itself which have finite–dimensional range and which have equal trace. The compositions ST and TS of a continuous transformation S of $\mathcal{H}^*$ into itself with finite–dimensional range and a continuous transformation T of $\mathcal{H}^*$ into itself are continuous transformations of $\mathcal{H}^*$ into itself which have finite–dimensional range and which have equal trace.

A continuous transformation T of $\mathcal{H}$ into itself vanishes identically if

$$spur(ST) = 0$$

for every continuous transformation S of $\mathcal{H}$ into itself with finite–dimensional range. The space of continuous transformations of $\mathcal{H}$ into itself with finite–dimensional range is considered in the weakest topology with respect to which

$$spur(ST) = spur(TS)$$

is a continuous function of S in the space for every continuous transformation T of $\mathcal{H}$ into itself. Addition is continuous as a transformation of the Cartesian product of the space with itself into the space. Continuous transformations of the space into itself are defined by taking S into ST and S into TS for every continuous transformation T of $\mathcal{H}$ into itself. A continuous transformation of $\mathcal{H}$ into itself is said to be of trace class if it belongs to the closure of the subspace of transformations of finite–dimensional range. The compositions ST and TS of a trace–class transformation S of $\mathcal{H}$ into itself and a continuous transformation T of $\mathcal{H}$ into itself are trace–class transformations of $\mathcal{H}$ into

itself. The trace has a continuous extension to the trace class of transformations of $\mathcal{H}$ into itself. The identity
$$spur(ST) = spur(TS)$$
holds when S is a trace–class transformation of $\mathcal{H}$ into itself and T is a continuous transformation of $\mathcal{H}$ into itself.

A continuous transformation T of $\mathcal{H}^*$ into itself vanishes identically if
$$spur(ST) = 0$$
for every continuous transformation S of $\mathcal{H}^*$ into itself with finite–dimensional range. The space of continuous transformations of $\mathcal{H}^*$ into itself with finite–dimensional range is considered in the weakest topology with respect to which
$$spur(ST) = spur(TS)$$
is a continuous function of S in the space for every continuous transformation T of $\mathcal{H}^*$ into itself. Addition is continuous as a transformation of the Cartesian product of the space with itself into the space. Continuous transformations of the space into itself are defined by taking S into ST and S into TS for every continuous transformation T of $\mathcal{H}^*$ into itself. A continuous transformation of $\mathcal{H}^*$ into itself is said to be of trace class if it belongs to the closure of the subspace of transformations of finite–dimensional range. The compositions ST and TS of a trace-class transformation S of $\mathcal{H}^*$ into itself and a continuous transformation T of $\mathcal{H}^*$ into itself are trace–class transformations of $\mathcal{H}^*$ into itself. The trace has a continuous extension to the trace class of transformations of $\mathcal{H}^*$ into itself. The identity
$$spur(ST) = spur(TS)$$
holds when S is a trace–class transformation of $\mathcal{H}^*$ into itself and T is a continuous transformation of $\mathcal{H}^*$ into itself.

A continuous transformation T of $\mathcal{H}^*$ into itself is of trace class if, and only if, it is the adjoint
$$T = S^*$$
of a continuous transformation S of $\mathcal{H}^*$ into itself. The identity
$$spur(T) = spur(S)^-$$
is then satisfied.

Additional hypotheses are imposed for the definition of a Weierstrass algebra. The intersection of $\mathcal{H}$ and $\mathcal{H}^*$ is assumed to be a vector space on which the vector space operations of $\mathcal{H}$ and $\mathcal{H}^*$ agree. The intersection is then a vector space whose inclusion in $\mathcal{H}$ and whose inclusion in $\mathcal{H}^*$ are linear transformations. The intersection is assumed to be a dense vector subspace of $\mathcal{H}$ and a dense vector subspace of $\mathcal{H}^*$. The scalar product
$$\langle a, b\rangle$$

of a as an element of $\mathcal{H}$ and of b as an element of $\mathcal{H}^*$ is assumed to be equal to the scalar product of a as an element of $\mathcal{H}^*$ and b as an element of $\mathcal{H}$ when a and b belong to the intersection of $\mathcal{H}$ and $\mathcal{H}^*$. The scalar self–product

$$\langle c, c\rangle$$

is assumed to be nonnegative when c belongs to the intersection of $\mathcal{H}$ and $\mathcal{H}^*$.

A Weierstrass algebra is an algebra of continuous linear transformations of $\mathcal{H}$ into itself which contains the identity transformation and which is closed in the weak topology induced by the trace–class transformations of $\mathcal{H}$ into itself. The algebra of adjoints of transformations in the Weierstrass algebra is then an algebra of continuous transformations of $\mathcal{H}^*$ into itself which contains the identity transformation and which is closed in the weak topology induced by the trace–class transformations of $\mathcal{H}^*$ into itself. The transformations in a Weierstrass algebra are assumed to take the intersection of $\mathcal{H}$ and $\mathcal{H}^*$ into itself and to have continuous extensions as transformations of $\mathcal{H}^*$ into itself. The adjoints of the transformations of a Weierstrass algebra are assumed to take the intersection of $\mathcal{H}$ and $\mathcal{H}^*$ into itself and to have continuous extensions as transformations of $\mathcal{H}$ into itself.

A weakly continuous linear transformation T of $\mathcal{H}$ into itself is said to be nonnegative with respect to the Weierstrass algebra if it belongs to the weak closure of finite sums of transformations of the form c into $\xi c \xi^-$ for an element ξ of the Weierstrass algebra. The adjoint of a nonnegative transformation is denoted by the same symbol. Sums and products of nonnegative transformations are nonnegative. The inequality

$$S \leq T$$

for weakly continuous linear transformations S and T means that

$$T - S$$

is a nonnegative transformation. Multiplication by a nonnegative real number is a nonnegative transformation which is identified with the number. The inequality

$$0 \leq T$$

for a weakly continuous linear transformation T means that T is a nonnegative transformation. A nonnegative transformation T is said to be positive if the inequality

$$\epsilon \leq T$$

holds for a positive number ϵ. The inequality

$$S < T$$

for weakly continuous transformations S and T means that the transformation $T - S$ is positive.

Properties of positive transformations are required as hypotheses for the preparation and proof of the Hahn–Banach theorem as well as its applications.

A positive transformation T is assumed to have an inverse, which is a nonnegative transformation satisfying the inequality

$$T^{-1} \leq 1$$

when T satisfies the inequality

$$1 \leq T.$$

If P and Q are positive transformations, the transformation

$$(1-T)P + TQ$$

is then positive whenever T is a nonnegative transformation such that $1-T$ is nonnegative.

A convex combination

$$(1-T)a + Tb$$

of elements a and b of the representation space is defined by a nonnegative transformation T such that $1-T$ is nonnegative. A subset of the representation space is said to be convex if it contains the convex combinations of every pair of elements.

A Hausdorff topology for the representation space of a Weierstrass algebra is said to be locally convex if addition is continuous as a transformation of the Cartesian product of the space with itself into the space and if every open set has an absorption property: Whenever a is an element of the set and b is an element of the space, a positive transformation T exists such that $1-T$ is nonnegative and such that the convex combination

$$(1-T)a + Tb$$

belongs to the space.

The Cartesian product of the Weierstrass algebra with itself is assumed to have a topology which is locally convex when the Cartesian product is treated as a representation space over the algebra. The transformation of the Cartesian product of the algebra with itself into the representation space which takes a pair of elements ξ and η of the algebra into the element $\xi c \eta^-$ of the representation space is assumed to be continuous for every element c of the representation space. A nonnegative transformation T of the algebra into itself determines a nonnegative transformation, also denoted T_1 of the representation space into itself. If an element ξ of the algebra exists such that T takes a into $\xi a \xi^-$ for every element a of the algebra, then T takes b into $\xi b \xi^-$ for every element b of the representation space.

If a locally convex topology of a representation space of a Weierstrass algebra is given, then a related locally convex topology is constructed using the concept of a hyperdisk. A nonempty convex subset of the space is said to be a disk if it is disjoint from the closure of every disjoint convex set.

If A is a disk and if B is a convex set, then the intersection of A with the closure of B is contained in the closure of the intersection of A with B. For an element of A which does not belong to the closure of the intersection of A with B belongs to a convex open set C whose intersection with A is disjoint from C. Since the intersection of B with C is a convex set which is disjoint from A, the disk A is disjoint from the closure of the intersection of B with C. Since C is an open set, the intersection of C with the closure of B is contained in the closure of the intersection of B with C. It follows that the intersection of A with C is disjoint from the closure of B.

The intersection of disks A and B is a disk if it is nonempty. For the intersection of A and B is a convex set. If a convex set C is disjoint from the intersection of A and B, then the intersection of B and C is a convex set which is disjoint from A. Since A is a disk, A is disjoint from the closure of the intersection of B and C. Since B is a disk, the intersection of B with the closure of C is contained in the closure of the intersection of B with C. It follows that the intersection of A and B is disjoint from the closure of C.

The disk topology of a locally convex space is the locally convex topology whose open sets are the unions of disks. The disk topology has the same closed convex sets as the given topology. Since every nonempty convex set which is open for the given topology is a disk, every convex set which is closed for the given topology is closed for the disk topology. If a convex set B is closed for the disk topology, then an element of the space which does not belong to B belongs to a disk A which is disjoint from B. Since A is disjoint from the closure of B, an element of the space which does not belong to B does not belong to the closure of B. The nonnegative transformations for the disk topology are identical with the nonnegative transformations for the given locally convex topology.

The closure of a convex set B with respect to a locally convex topology is convex. For if u and v are elements of the closure of B and if A is a convex open set containing the origin, then elements a and b of B exist such that $u-a$ and $v-b$ belong to A. An element of the convex span of u and v is a convex combination

$$(1-T)u+Tv$$

with T a nonnegative transformation such that $1-T$ is nonnegative. Since B is convex, the convex combination

$$(1-T)a+Tb$$

belongs to B. Since A is convex, the difference

$$[(1-T)u+Tv]-[(1-T)a+Tb]=(1-T)(u-a)+T(v-b)$$

belongs to A.

If B is a nonempty convex set and if s is an element of the locally convex space which does not belong to B, then a convex set $B(s)$ is constructed so that B is contained in $B(s)$ and so that s belongs to the closure of $B(s)$. The set $B(s)$ is the set of convex combinations

$$(1-T)s+Tc$$

with c an element of B and T a positive transformation such that $1 - T$ is nonnegative. Every convex open set which contains s contains an element of $B(s)$ by the definition of a locally convex topology. It is sufficient by a translation to verify convexity of $B(s)$ when s is the origin. A convex combination

$$(1 - T)Pa + TQb$$

of elements Pa and Qb of $B(s)$ is constructed from elements a and b of B with T a nonnegative transformation such that $1 - T$ is nonnegative and with P and Q positive transformations such that $1 - P$ and $1 - Q$ are nonnegative. Then

$$R = (1 - T)P + TQ$$

is a nonnegative transformation such that

$$1 - P = (1 - T)(1 - P) + T(1 - Q)$$

is nonnegative. Since P and Q are positive, R is positive. A nonnegative transformation S such that $1 - S$ is nonnegative is obtained as a solution of the equations

$$RS = TQ$$

and

$$R(1 - S) = (1 - T)P.$$

Since the set B is convex,

$$c = (1 - S)a + Sb$$

is an element of B. The convex combination

$$(1 - T)Pa + TQb = Rc$$

of elements Pa and Qb of the set $B(s)$ is then an element Rc of the set $B(s)$.

The Hahn–Banach theorem is a construction of continuous linear functionals with convex kernel.

Theorem 1. *If a disk A of a locally convex space $\mathcal{H}$ is disjoint from a convex subset B of the space, then an element b of the dual space $\mathcal{H}^*$ exists such that the kernel of b^- is convex and such that b^- maps A and B into disjoint subsets of the real line.*

Proof of Theorem 1. It can be assumed that the set B is nonempty. A maximal convex set which contains B and is disjoint from A exists by the Zorn lemma. It is sufficient to give a proof of the theorem with B is a maximal convex set which is disjoint from A. Since the closure of B is convex and is disjoint from A, B is a closed convex set. It will be shown that the complement of B is convex.

If u belongs to the complement of B, a convex set $B(u)$ is constructed as the set of convex combinations

$$(1-P)u+Pa$$

of u and elements a of B with P a positive transformation such that $1-P$ is positive. Since the closure of $B(u)$ contains u and every element of B, the element of $B(u)$ can be chosen in A.

If v belongs to the complement of B, a convex set $B(v)$ is obtained as the set of convex combinations

$$(1-Q)v+Qb$$

of v and elements b of B with Q a positive transformation such that $1-Q$ is positive. Since the closure of $B(v)$ contains v and every element of B, the element of $B(v)$ can be chosen in A.

A convex combination

$$(1-V)u+Vv$$

of u and v is defined using a nonnegative transformation V such that $1-V$ is nonnegative. Since the transformation

$$(1-P)V+(1-Q)(1-V)$$

is positive, a nonnegative transformation T such that $1-T$ is nonnegative exists which satisfies the equation

$$T(1-Q)(1-V)=(1-T)(1-P)V.$$

The transformations

$$R=(1-T)P+TQ$$

and

$$1-R=(1-T)(1-P)+T(1-Q)$$

are positive. A nonnegative transformation U such that $1-U$ is nonnegative exists which satisfies the identities

$$R(1-U)=(1-T)P$$

and

$$RU=TQ.$$

The identities

$$(1-R)(1-V)=(1-T)(1-P)$$

and

$$(1-R)V=T(1-Q)$$

are satisfied. Since the identity

$$\begin{aligned}&(1-T)[(1-P)u+Pa]+T[(1-Q)u+Qb]\\&=(1-R)[(1-V)u+Vv]+R[(1-U)a+Ub]\end{aligned}$$

is satisfied, the convex combination of elements

$$(1 - P)u + Pa$$

and

$$(1 - Q)v + Qb$$

of elements of A is an element of A which is a convex combination of

$$(1 - V)u + Vv$$

and the element

$$(1 - U)a + Ub$$

of B.

This completes the proof that the complement of B is convex. Since the convex set B is closed, the complement of B is open. A continuous linear functional exists which maps B and its complement into disjoint convex subsets of the real line and which has convex kernel. The linear functional is represented by an element of $\mathcal{H}^*$.

This completes the proof of the theorem.

A locally convex space admits a strongest locally convex topology. A convex set is open for the strongest locally convex topology if for every element a of the set and for every element b of the space, a convex combination

$$(1 - T)a + Tb$$

belongs to the set with T a positive transformation such that $1 - T$ is nonnegative.

A characterization of disks is an application of the proof of the Hahn–Banach theorem. A nonempty convex set, which is open for the strongest locally convex topology, is a disk if, and only if, every linear functional with convex kernel which maps the set into a proper subset of the real line is continuous.

The dual space of the representation space of a Weierstrass algebra is assumed to be the vector span of the elements of the space which define linear functionals with convex kernel. Every weakly open set is then a union of weakly open convex sets. If a is an element of a weakly open set and if b is an element of the representation space, then the convex combination

$$(1 - T)a + Tb$$

belongs to the set for some positive transformation T such that $1 - T$ is nonnegative. The representation space is locally convex in its weak topology.

A center for a subset of a locally convex space, or of its dual space, is an element a of the set such that

$$(1 - T)b + T(2a - b)$$

belongs to the set whenever b belongs to the set and T is a nonnegative transformation such that $1-T$ is nonnegative. A set is said to be centered at an element a if a is a center of the set.

A disk which is centered at the origin of a locally convex space is determined by a centered weakly compact set of elements of the dual space which define linear functionals with convex kernel.

Theorem 2. *If a disk A of a locally convex space $\mathcal{H}$ is centered at the origin, then the set B of elements b of the dual space $\mathcal{H}^*$ such that b^- maps A into the interval $(-1,1)$ and such that the kernel of b^- is convex is a weakly compact set which is centered at the origin and which contains every element of its convex span which represents a linear functional with convex kernel. The set A is the set of elements a of $\mathcal{H}$ such that a^- maps B into the interval $(-1,1)$.*

Proof of Theorem 2. The set B is centered at the origin and contains every element of its convex span which represents a linear functional with convex kernel. If c is an element of $\mathcal{H}$, a positive transformation T exists such that $1-T$ is nonnegative and such that

$$a = Tc$$

belongs to A. If an element b of $\mathcal{H}^*$ represents a linear functional with convex kernel, the action of b^- on the convex span of c and $-c$ is determined by the action of b^- on the convex span of a and $-a$. Since the interval $[-1,1]$ is a compact Hausdorff space I, the set I^A of all functions defined on A with values in I is a compact Hausdorff space in the Cartesian product topology. An element b of $\mathcal{H}^*$, which represents a linear functional with convex kernel, belongs to B if, and only if, the restriction of b^- to A belongs to I^A. Since A is a disk, these elements of B determine a closed subset of I^A. Since I^A is compact and since the mapping of B into I^A is a homeomorphism for the weak topology, the set B is weakly compact.

If an element a of $\mathcal{H}$ belongs to A, a^- maps B into the interval $(-1,1)$. It will be shown that a^- does not map B into the interval $(-1,1)$ when an element a of $\mathcal{H}$ does not belong to A. An element b of $\mathcal{H}^*$ exists by the Hahn–Banach theorem such that the kernel of b^- is convex and such that b^-a does not belong to the image of A under b^-. Since A is centered at the origin, the image of A is a convex open subset of the real line which is centered at the origin. Since b^-a does not belong to the set, the choice of b can be made so that b^- maps A into the interval $(-1,1)$ and does not a into the interval. Then b is an element of B such that a^- does not b into the interval $(-1,1)$.

This completes the proof of the theorem.

A construction of disks of a locally convex space is made from weakly compact sets of elements of the dual space which represent linear functionals with convex kernel.

Theorem 3. *if a weakly compact subset B of the dual space $\mathcal{H}^*$ of a locally convex space is centered at the origin and contains every element of its convex span which represents a* *linear functional with convex kernel, then the set A of elements a of $\mathcal{H}$ such that a^- maps B into the interval $(-1, 1)$ is a disk which is centered at the origin. The set B is the set of elements b of $\mathcal{H}^*$ such that b^- has convex kernel and maps A into the interval $(-1, 1)$.*

Proof of Theorem 3. It will first be shown that B is the set of elements b of $\mathcal{H}^*$ such that b^- has convex kernel and maps A into the interval $(-1, 1)$. It is sufficient to make the verification when the Weierstrass algebra is the field of real numbers, in which case $\mathcal{H}^*$ is treated as a locally convex space with $\mathcal{H}$ as dual space. An element b of $\mathcal{H}^*$ which does not belong to B belongs to a weakly open convex set which is disjoint from B. An element a of $\mathcal{H}$ exists by the Hahn–Banach theorem such that a^- maps B into a set which does not contain the image of b. Since B is a weakly compact convex set which is centered at the origin, the element a of $\mathcal{H}$ can be chosen in A such that a^-b does not belong to the interval $(-1, 1)$. Then b^- does not map A into the interval $(-1, 1)$.

It remains to show that the set A, which is convex and centered at the origin, is a disk. A proof is first given when the space $\mathcal{H}$ is considered in its strongest locally convex topology. If a is an element of A and if c is an element of $\mathcal{H}$, a positive number t exists such that $1 - t$ is nonnegative and such that

$$(1 - t)a + tc$$

belongs to A. The construction of t is an application of the weak compactness of B. A positive number κ exists such that c^- maps B into the interval $(-\kappa, \kappa)$. Since a^- maps B into the interval $(\epsilon - 1, 1 - \epsilon)$ for some positive number ϵ, it is sufficient to choose t so that the inequality

$$t\kappa < \epsilon$$

is satisfied.

The set A is shown to be a disk for the given locally convex topology by showing that an element b of the dual space of $\mathcal{H}$ for the strongest locally convex topology belongs to $\mathcal{H}^*$ if b^- has convex kernel and maps A into a proper subset of the real line. Since A is centered at the origin, it can be assumed that b^- maps A into the interval $(-1, 1)$. The desired conclusion holds since b belongs to B.

This completes the proof of the theorem.

The completion of a locally convex space $\mathcal{H}$ in its disk topology is a locally convex space $\mathcal{H}^\wedge$ over the same Weierstrass algebra. The dual space of the completion coincides as a set with the dual space $\mathcal{H}^*$ of $\mathcal{H}$. The inclusion of $\mathcal{H}^*$ in itself is continuous as a transformation from the weak topology induced by $\mathcal{H}^\wedge$ into the weak topology induced by $\mathcal{H}$. If B is a set of elements of $\mathcal{H}^*$ which represent linear functionals with convex kernel, if B is centered at the origin, and if B contains every element of its convex span with convex kernel, then B is compact in the weak topology induced by $\mathcal{H}^\wedge$ if, and only if, B is compact in the weak topology induced by $\mathcal{H}$. If a disk A of $\mathcal{H}$ is centered at the origin, then the closure of A in $\mathcal{H}^\wedge$ contains a disk $A^\wedge$ of $\mathcal{H}^\wedge$ which is centered at the origin and whose intersection with $\mathcal{H}$ is A. A linear functional on $\mathcal{H}^*$ is represented by an element of $\mathcal{H}^\wedge$ if, and only if, it is weakly continuous on every weakly compact subset of $\mathcal{H}^*$ which is centered at the origin, whose elements represent linear functionals with convex kernel, and which contains every element of its convex span representing a linear functional with convex kernel.

A subset of the dual space $\mathcal{H}^*$ of a locally convex space $\mathcal{H}$ is said to be bounded if its image under a^- is a bounded set of real numbers for every element a of $\mathcal{H}$. Boundedness is applied to subsets of $\mathcal{H}^*$ whose elements represent linear functionals with convex kernel and which contain every element of their convex span representing a linear functional with convex kernel. The set is bounded if, and only if, its closure in the dual space of $\mathcal{H}$ for the strongest locally convex topology is weakly compact. Weak compactness of such bounded sets is a hypothesis in the closed graph theorem. An equivalent hypothesis is that a nonempty convex subset of $\mathcal{H}$ is a disk whenever the set is open for the strongest locally convex topology and its closure is equal to its closure for the strongest locally convex topology.

The Krein-Šmulyan property is another hypothesis in the closed graph theorem. The Krein-Šmulyan property states that certain subsets of the dual space of a locally convex space are weakly closed. A set tested consists of elements of the dual space which represent linear functionals with convex kernel and contains the elements of its convex span which represent linear functionals with convex kernel. The Krein-Šmulyan property is the assertion that a test set is weakly closed if its intersection with every weakly compact test set is weakly compact.

A locally convex space has the Krein–Milman property if a countable basis exists for the neighborhoods of the origin in the disk topology. For then weakly compact test sets C_n exist such that C_n is contained in C_{n+1} for every positive integer n and such that every weakly compact test set is contained in some set C_n. A test set B is to be shown weakly closed if its intersection with every set C_n is weakly compact. It needs to be shown that an element of the dual space does not belong to B if for every positive integer n the element does not belong to the intersection of B with C_n. It can by a translation be assumed that the element is the origin. The sets C_n can be assumed centered at the origin. Since the origin does not belong to the intersection of B with C_n, a weakly open set U_n exists which contains the origin and is disjoint from the intersection of $B$with C_n. The sets U_n are constructed inductively so that the intersection of U_n with B_n is contained in U_{n+1}. A weakly open set U then exists which is disjoint from $\mathcal{C}$ and which contains the intersection of U_n with B_n for every n.

The construction of the sets U_n is an application of the Hahn–Banach theorem to the dual space $\mathcal{H}^*$ treated as a locally convex space with dual space $\mathcal{H}$. The Weierstrass algebra is then the real numbers. The weakly closed convex space of each set C_n is weakly compact. The origin does not belong to the intersection of the weakly closed convex span of B and the weakly closed convex span of C_n. The set U_n is constructed as a weakly open convex set such that the intersection of the weak closure of U_n with the weakly closed convex span of C_n is disjoint from the weakly closed convex span of B.

The continuity of a linear transformation T of a locally convex space $\mathcal{P}$ into a locally convex space $\mathcal{Q}$ implies continuity in weak and disk topologies when T maps convex sets into convex sets and when the inverse image of every convex set is convex. The locally convex space are assumed to be defined over the same Weierstrass algebra. The adjoint transformation T^* takes an element a of the dual space $\mathcal{Q}^*$ of $\mathcal{Q}$ into an element b of the dual space $\mathcal{P}^*$ of $\mathcal{P}$ when the identity

$$\langle c, b\rangle = \langle Tc, a\rangle$$

holds for every element c of $\mathcal{P}$. Continuity of T implies that the domain of T^* contains every element of $\mathcal{Q}^*$. Since the inverse image of every convex subset of $\mathcal{Q}$ is a convex subset of $\mathcal{P}$, the adjoint transformation T^* maps every element of $\mathcal{Q}^*$ which represents a linear functional with convex kernel into an element of $\mathcal{P}^*$ which represents a linear functional with convex kernel. The transformation T is continuous from the weak topology of $\mathcal{P}$ into the weak topology of $\mathcal{Q}$. It follows that the transformation is continuous from the disk topology of $\mathcal{P}$ into the disk topology of $\mathcal{Q}$. For if A is a disk of $\mathcal{Q}$, the inverse image of A in $\mathcal{P}$ is convex. If a convex subset B of $\mathcal{P}$ is disjoint from the inverse image of A, then the image of B in $\mathcal{Q}$ is a convex set which is disjoint from A. Since A is a disk, the closure of the image of B in $\mathcal{Q}$ is a convex set C which is disjoint from A. Since T is continuous from the weak topology of $\mathcal{P}$ into the weak topology of $\mathcal{Q}$, the inverse image of C in $\mathcal{P}$ is a closed convex set. Since B is contained in the inverse image of C and since the inverse image of C is disjoint from the inverse image of A, the closure of B is disjoint from the inverse image of A. This completes the verification that the inverse image of A is a disk.

The closed graph theorem concludes that a linear transformation of a locally convex space into a locally convex space is continuous if it has a closed graph and if the spaces and the transformation are well–related to convexity.

Theorem 4. *A linear transformation T of a locally convex space $\mathcal{P}$ into a locally convex space $\mathcal{Q}$ over the same Weierstrass algebra is continuous from the disk topology of $\mathcal{P}$ into the disk topology of $\mathcal{Q}$ if the graph of the transformation is closed in the Cartesian product of the weak topology of $\mathcal{P}$ and the weak topology of $\mathcal{Q}$ and if three hypotheses are satisfied:*

1) The transformation maps convex sets into convex sets and the inverse image of every convex set is convex. 2) A nonempty convex subset of $\mathcal{P}$ is a disk whenever it is open for the strongest locally convex topology and its closure is equal to its closure for the strongest locally convex topology. 3) The space $\mathcal{Q}$ has the Krein–Šmulyan property.

Proof of Theorem 4. Continuity of T is proved by showing that the domain of T^* is $\mathcal{Q}^*$. The adjoint has a closed graph in the Cartesian product of the weak topology of $\mathcal{Q}^*$ and the weak topology of $\mathcal{P}^*$. The adjoint is a transformation since the domain of T is $\mathcal{P}$. Since T has a closed graph in the Cartesian product of the weak topology of $\mathcal{P}$ and the weak topology of $\mathcal{Q}$, T is the adjoint of T^*. Since T is a transformation the domain of T^* is dense in $\mathcal{Q}^*$. The proof of the theorem is completed by showing that the domain of T^* is weakly closed in $\mathcal{Q}^*$.

The domain of T^* is the convex span of elements which represent linear functionals with convex kernel. Since the space $\mathcal{Q}$ has the Krein–Šmulyan property, it is sufficient to show that the domain of T^* has a weakly compact intersection with every weakly compact set A

whose elements represent linear functionals with convex kernel and which contains every element of its convex span representing a linear functional with convex kernel. The image of A in $\mathcal{P}^*$ consists of elements which represent linear functionals with convex kernel. The set contains every element of its convex span which represents a linear functional with convex kernel. The set is bounded since the identity

$$\langle c, T^* a\rangle = \langle Tc, a\rangle$$

holds for every element a of A when c is in $\mathcal{P}$. Since the closure B of the image of A is weakly compact by hypothesis, the Cartesian product of A and B is a compact subset of the Cartesian product of $\mathcal{Q}^*$ and $\mathcal{P}^*$. Since the graph of T^* is a closed subset of the Cartesian product, it has a compact intersection with the Cartesian product of A and B. It follows that the intersection of A with the domain of T^* is weakly compact.

This completes the proof of the theorem.

An example of a Weierstrass algebra is the set $\mathcal{C}(\mathcal{S})$ of all continuous real valued functions on a Hausdorff space $\mathcal{S}$. The space $\mathcal{S}$ is considered with the weakest topology with respect to which every element of $\mathcal{C}(\mathcal{S})$ is continuous. The topology of pointwise convergence on $\mathcal{S}$ is a locally convex topology of $\mathcal{C}(\mathcal{S})$ when $\mathcal{C}(\mathcal{S})$ is treated as a representation space of itself as Weierstrass algebra. The space $\mathcal{S}$ is a subspace of an essentially unique Hausdorff space $\mathcal{S}^\wedge$ such that every function $f(s)$ of s in $\mathcal{S}$ admits a continuous extension as a function $f(s)$ of s in $\mathcal{S}^\wedge$ and such that every homomorphism of $\mathcal{C}(\mathcal{S})$ onto the real numbers is of the form f into $f(s)$ for a unique element s of $\mathcal{S}^\wedge$. The space $\mathcal{S}^\wedge$ is considered with the weakest topology with respect to which every element f of $\mathcal{C}(\mathcal{S})$ is continuous as a function $f(s)$ of s in $\mathcal{S}^\wedge$. The topology of pointwise convergence on $\mathcal{S}^\wedge$ is a locally convex topology of $\mathcal{C}(\mathcal{S})$. The determination of the Hausforff completion $\mathcal{S}^\wedge$ of $\mathcal{S}$ and of the locally convex topologies of the equal spaces $\mathcal{C}(\mathcal{S})$ and $\mathcal{C}(\mathcal{S}^\wedge)$ is a fundamental problem of analysis.

The Weierstrass algebra $\mathcal{C}(\mathcal{S})$ is determined within an isomorphism by the cardinality of $\mathcal{S}$ when $\mathcal{S}$ is a discrete space. If a cardinality of $\mathcal{S}$ exists such that $\mathcal{S}^\wedge$ is not equal to $\mathcal{S}$, then there is a least such cardinality. The Hausdorff completion of a discrete space is itself when the cardinality of the space is sufficiently small.

A determination of the locally convex topologies of a Weierstrass algebra $\mathcal{C}(\mathcal{S})$ is due to Shirota [6] when $\mathcal{S}$ is due to Shirota [6] when $\mathcal{S}$ is a complete uniform space whose discrete subsets have sufficiently small cardinalities.

Theorem 6. *The strongest locally convex topology of the Weierstrass algebra $\mathcal{C}(\mathcal{S})$ is the topology of uniform convergence on compact subsets of $\mathcal{S}$ when $\mathcal{S}$ is a complete uniform space whose discrete subsets are Hausdorff complete.*

Proof of Theorem 11. A defining pseudo–metric for the uniform space is a function $\rho(a, b)$ of elements a and b of the space with nonnegative values which satisfies the identity

$$\rho(a, b) = \rho(b, a)$$

for all elements a and b of the space and which satisfies the inequality

$$\rho(a,c) \leq \rho(a,b) + \rho(b,c)$$

for all elements a, b, and c of the space. Continuity of a function $f(s)$ of s in $\mathcal{S}$ means that for every element a of $\mathcal{S}$ and every positive number ϵ a defining pseudo–metric ρ exists such that the inequality

$$|f(a) - f(b)| < \epsilon$$

holds whenever the inequality

$$\rho(a,b) < 1$$

is satisfied. The space of all continuous functions on a uniform space $\mathcal{S}$ forms a Weierstrass algebra $\mathcal{C}(\mathcal{S})$. The space $\mathcal{S}$ is a subspace of a Hausdorff space $\mathcal{S}^\wedge$ such that every continuous function $f(s)$ of s in $\mathcal{S}$ has a unique continuous extension as a function $f(s)$ of s in $\mathcal{S}^\wedge$ and such that every homomorphism of the Weierstrass algebra $\mathcal{C}(\mathcal{S})$ onto the complex numbers is of the form f into $f(s)$ for a unique element s of $\mathcal{S}^\wedge$. It will be shown that every defining pseudo–metric $\rho(a,b)$ of a and b in $\mathcal{S}$ admits an extension as a pseudo–metric $\rho(a,b)$ of a and b in $\mathcal{S}^\wedge$ such that every element of the Weierstrass algebra $\mathcal{C}(\mathcal{S})$ is continuous on the resulting uniform space $\mathcal{S}^\wedge$ and such that $\mathcal{S}$ is dense in $\mathcal{S}^\wedge$. It is sufficient to show that for every defining pseudo–metric ρ and for every element s of $\mathcal{S}^\wedge$ an element a of $\mathcal{S}$ exists such that the inequality

$$\rho(a,s) < 1$$

is satisfied.

A well–ordering of the space $\mathcal{S}$ is assumed for the construction of functions from a given pseudo–metric ρ. An element b of the space is said to be generated by an element a of the space if a is the least element of the space which satisfies the inequality

$$\rho(a,b) < 1.$$

The inequality

$$\rho(a,b) \leq 1 - 2^{-n}$$

then holds when n is sufficiently large. The inequality

$$\rho(a',b) \geq 1$$

holds when a' is less than a. If an element s of the space satisfies the inequality

$$\rho(b,s) < 2^{-n-1},$$

then the inequality

$$\rho(a,s) < 1 - 2^{-n-1}$$

is satisfied and the inequality

$$\rho(a',s) > 1 - 2^{-n-1}$$

holds when a' is less than a. A function

$$\delta_n(a, s') = \inf \rho(s, s')$$

of s' is defined as a greatest lower bound taken over the elements s such that either the inequality

$$\rho(a, s) < 1 - 2^{-n-1}$$

is violated or the inequality

$$\rho(a', s) > 1 - 2^{-n-1}$$

is violated for some element a' less than a. The inequality

$$\delta_n(a, b) \geq 2^{-n-1}$$

then holds when b is generated by a and the inequality

$$\rho(a, b) \leq 1 - 2^{-n}$$

is satisfied. When a and a' are distinct generators, the set of elements s such that $\delta_n(a, s)$ is positive is disjoint from the set of elements s such that $\delta_n(a', s)$ is positive.

For every positive integer n the sum

$$\sum \delta_n(a, s)$$

taken over all generators a is a continuous function of s in $\mathcal{S}$ which has a unique continuous extension as a function of s in $\mathcal{S}^\wedge$. The sum has a positive limit in the limit of large n for every element s of $\mathcal{S}^\wedge$. If $k(a)$ is a function of generators a, then the sum

$$\sum k(a)\delta_n(a, s)$$

taken over all generators a is a continuous function of s in $\mathcal{S}$ which has a unique continuous extension as a function of s in $\mathcal{S}^\wedge$. The Weierstrass algebra of all functions on the discrete space of generators admits a unique locally hyperconvex topology. Since the taking of function values at an element s of $\mathcal{S}^\wedge$ is a hyperlinear functional on the algebra, it determines a generator a such that

$$\delta_n(a, s) > 0$$

and hence such that

$$\rho(a, s) < 1.$$

This completes the proof of the theorem.

References

1. L. de Branges, *The Stone-Weierstrass theorem*, Proceedings of the American Mathematical Society **10** (1959), 822-824.
2. ———, *The Riemann mapping theorem*, Journal of Mathematical Analysis and Applications **66** (1978), 60–81.
3. ———, *Vectorial topology*, Journal of Mathematical Analysis and Applications **69** (1979), 443–454.
4. ———, *A construction of invariant subspaces*, Mathematische Nachrichten **163** (1993), 163–175.
5. ———, *The Cantor construction*, Journal of Mathematical Analysis and Applications **76** (1980), 623–630.
6. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.

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