A PROOF OF THE INVARIANT SUBSPACE CONJECTURE

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Abstract. A proof is given of the announced existence theorem for invariant subspaces of continuous (linear) transformations of a Hilbert space into itself [5]. If the transformation is not a scalar multiple of the identity transformation, a closed subspace other than the least subspace and the greatest subspace exists which is an invariant subspace for every continuous linear transformation which commutes with the given transformation.

Since a continuous transformation of a Hilbert space into itself is a scalar multiple of a contractive transformation, it is sufficient to prove the existence of invariant subspaces for transformations which are contractive. The existence of invariant subspaces is a theorem of Hilbert for transformations which map a Hilbert space isometrically onto itself. When a contractive transformation A maps a Hilbert space into itself, the set of element f of the space which have the same norm as $A^n f$ for every nonnegative integer n is an invariant subspace for every continuous transformation which commutes with A. The existence of the desired invariant subspace follows when the subspace contains a nonzero element. The existence of invariant subspaces is now obtained when the subspace contains no nonzero element. A canonical model of the transformation is applied. A contractive transformation C onto a coefficient Hilbert $\mathcal{C}$ exists such that the identity

$$\|Af\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2 - |Cf|^2$$

holds for every element f of the Hilbert space $\mathcal{H}$. The absolute value denotes the norm in the coefficient Hilbert space. An element f of $\mathcal{H}$ is uniquely determined by the associated power series

$$Cf + CAfz + CA^2fz^2 + \ldots$$

whose coefficients are taken in the coefficient space. The given transformation is unitarily equivalent to the transformation which takes $f(z)$ into $[f(z) - f(0)]/z$ in a Hilbert space whose elements are power series with coefficients in $\mathcal{C}$. The existence of invariant subspaces is proved for transformations given in this canonical form.

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In the construction of invariant subspaces a vector is an element of the coefficient space. An operator is a continuous transformation of vectors into vectors. If b is a vector, b^- is the linear functional on vectors defined by the scalar product

$$b^- a = \langle a, b \rangle$$

in the coefficient space. If a and b are vectors, the operator ab^- is defined by the associative law

$$(ab^-)c = a(b^- c)$$

for every vector c. Complex numbers are treated as multiplication operators. The absolute value denotes the operator norm of an operator as well as the norm of a vector. A bar denotes the adjoint A^- of an operator A.

A Hilbert space $\mathcal{H}$, whose elements are power series with vector coefficients, is said to satisfy the inequality for difference quotients if $[f(z) - f(0)]/z$ belongs to the space whenever $f(z)$ belongs to the space and if the inequality for difference quotients

$$\|[f(z) - f(0)]/z\|_{\mathcal{H}}^2 \leq \|f(z)\|_{\mathcal{H}}^2 - |f(0)|^2$$

is satisfied. The space is said to satisfy the identity for difference quotients if equality always holds in the inequality for difference quotients.

The space $\mathcal{C}(z)$ of square summable power series is a Hilbert space of power series with vector coefficients which satisfies the identity for difference quotients. The elements of the space are the power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$$

with vector coefficient such that the sum

$$\|f(z)\|^2 = |a_0|^2 + |a_1|^2 + |a_2|^2 + \ldots$$

is finite. A Hilbert space $\mathcal{H}$ of power series with vector coefficients which satisfies the inequality for difference quotients is contained contractively in $\mathcal{C}(z)$. The spaces which are contained isometrically in $\mathcal{C}(z)$ are treated by Beurling [1] in the essential case of the complex numbers as coefficient space. A generalization of the Beurling theory to contractive inclusions is made in joint work with James Rovnyak [6].

The treatment of contractive inclusions applies a generalization of orthogonal complementation [3]. If a Hilbert space $\mathcal{P}$ is contained contractively in a Hilbert space $\mathcal{H}$, a unique Hilbert space $\mathcal{Q}$, which is contained contractively in $\mathcal{H}$, exists such that the inequality

$$\|c\|_{\mathcal{H}}^2 \leq \|a\|_{\mathcal{P}}^2 + \|b\|_{\mathcal{Q}}^2$$

holds whenever

$$c = a + b$$

is the sum of an element a of $\mathcal{P}$ and an element b of $\mathcal{Q}$ and such that every element c of $\mathcal{H}$ admits a decomposition for which equality holds. The space $\mathcal{Q}$ is called the complementary space to $\mathcal{P}$ in $\mathcal{H}$. The minimal decomposition of an element c of $\mathcal{H}$ is unique. The element a of $\mathcal{P}$ is obtained from c under the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$. The element b of $\mathcal{Q}$ is obtained from c under the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$. The intersection of $\mathcal{P}$ and $\mathcal{Q}$ is a Hilbert space $\mathcal{P} \wedge \mathcal{Q}$, which is contained contractively in $\mathcal{H}$, with scalar product determined by the identity

$$\|c\|^2_{\mathcal{P}\wedge\mathcal{Q}} = \|c\|^2_{\mathcal{P}} + \|c\|^2_{\mathcal{Q}}.$$

The Hilbert space $\mathcal{H}$ in which $\mathcal{P}$ and $\mathcal{Q}$ are contained contractively as complementary spaces is denoted $\mathcal{P} \vee \mathcal{Q}$. The inequality

$$\|c\|^2_{\mathcal{P}\vee\mathcal{Q}} \leq \|a\|^2_{\mathcal{P}} + \|b\|^2_{\mathcal{Q}}$$

holds whenever

$$c = a + b$$

is the sum of an element a of $\mathcal{P}$ and an element b of $\mathcal{Q}$. Every element c of $\mathcal{P} \vee \mathcal{Q}$ admits a unique decomposition for which equality holds.

Complementation is preserved under partially isometric transformations. If Hilbert spaces $\mathcal{P}$ and $\mathcal{Q}$ are contained contractively as complementary subspaces of a Hilbert space $\mathcal{P} \vee \mathcal{Q}$ and if T is a partially isometric transformation of $\mathcal{P} \vee \mathcal{Q}$ onto a Hilbert space $\mathcal{H}$, then Hilbert spaces $\mathcal{P}'$ and $\mathcal{Q}'$ exists which are contained contractively as complementary subspaces of the space

$$\mathcal{H} = \mathcal{P}' \vee \mathcal{Q}'$$

such that T acts as a partially isometric transformation of $\mathcal{P}$ onto $\mathcal{P}'$ and of $\mathcal{Q}$ onto $\mathcal{Q}'$.

If a Hilbert space $\mathcal{H}$ of power series with vector coefficients satisfies the inequality for difference quotients, then the space is contained contractively in the space $\mathcal{C}(z)$ of square summable power series. The complementary space to $\mathcal{H}$ in $\mathcal{C}(z)$ is a Hilbert space $\mathcal{M}$, which is contained contractively in $\mathcal{C}(z)$, such that multiplication by z is a contractive transformation of $\mathcal{M}$ into itself. If a given Hilbert space $\mathcal{M}$ is contained contractively in $\mathcal{C}(z)$ and if multiplication by z is a contractive transformation of $\mathcal{M}$ into itself, then the complementary space to $\mathcal{M}$ in $\mathcal{C}(z)$ is a Hilbert space $\mathcal{H}$ which satisfies the inequality for difference quotients.

If multiplication by a power series $W(z)$ with operator coefficients is a contractive transformation of $\mathcal{C}(z)$ into itself, then multiplication by $W(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Hilbert space $\mathcal{M}$ such that multiplication by z is a contractive transformation of $\mathcal{M}$ into itself. The complementary space $\mathcal{H}(W)$ to $\mathcal{M}$ in $\mathcal{C}(z)$ is a Hilbert space of power series with vector coefficients which satisfies the inequality for difference quotients.

If a Hilbert space $\mathcal{H}$ of power series with vector coefficients satisfies the inequality for difference quotients, an augmented space $\mathcal{H}'$ of power series with vector coefficients is constructed which satisfies the inequality for difference quotients. The elements of $\mathcal{H}'$ are the power series $f(z)$ with vector coefficients such that $[f(z) - f(0)]/z$ belongs to $\mathcal{H}$. The scalar product in $\mathcal{H}'$ is determined by the identity for difference quotients

$$\|[f(z) - f(0)]/z\|_{\mathcal{H}}^2 = \|f(z)\|_{\mathcal{H}'}^2 - |f(0)|^2.$$

The given space $\mathcal{H}$ is contained contractively in the augmented space $\mathcal{H}'$. If the dimension of the complementary space to $\mathcal{H}$ in $\mathcal{H}'$ is less than or equal to the dimension of the coefficient space, a partially isometric transformation exists of the coefficient space $\mathcal{C}$ onto the complementary space. A power series $W(z)$ with operator coefficients exists such that multiplication by $W(z)$ acts as a partially isometric transformation of $\mathcal{C}$ onto the complementary space to $\mathcal{H}$ in $\mathcal{H}'$. Multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. The given space $\mathcal{H}$ is isometrically equal to the complementary space $\mathcal{H}(W)$ to the range of multiplication by $W(z)$ in $\mathcal{C}(z)$. The augmented space $\mathcal{H}'$ is isometrically equal to the complementary space to the range of multiplication by $zW(z)$ in $\mathcal{C}(z)$.

The construction of a space $\mathcal{H}(W)$ is possible when a Hilbert space $\mathcal{M}$ is contained contractively in $\mathcal{C}(z)$ and when multiplication by z is an isometric transformation of the space $\mathcal{M}$ into itself. The complementary space $\mathcal{H}$ to $\mathcal{M}$ in $\mathcal{C}(z)$ then satisfies the inequality for difference quotients. The space $\mathcal{H}$ is isometrically equal to a space $\mathcal{H}(W)$ if the dimension of the complementary space to $\mathcal{H}$ in its augmented space $\mathcal{H}'$ is less than or equal to the dimension of the coefficient space $\mathcal{C}$. The complementary space to $\mathcal{H}'$ in $\mathcal{C}(z)$ is a Hilbert space $\mathcal{M}'$, which is contained contractively in $\mathcal{M}$, such that multiplication by z is an isometric transformation of $\mathcal{M}$ onto $\mathcal{M}'$. The complementary space to $\mathcal{H}$ in $\mathcal{H}'$ is isometrically equal to the complementary space to $\mathcal{M}'$ in $\mathcal{M}$.

The dimension estimate is satisfied when the space $\mathcal{H}$ satisfies the identity for difference quotients.

Theorem 1. *A Hilbert space $\mathcal{H}$ of power series with vector coefficients which satisfies the identity for difference quotients is isometrically equal to a space $\mathcal{H}(W)$ for a power series $W(z)$ with operator coefficients such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself whose kernel contains $[f(z)-f(0)]/z$ whenever it contains $f(z)$.*

The theorem is due to Beurling [1] for a space $\mathcal{H}$ which is contained isometrically in $\mathcal{C}(z)$ in the essential case of the complex numbers as coefficient space. The proof of the theorem is prepared by a dimension estimate obtained in joint work with James Rovnyak as a undergraduate at Lafayette College and as a graduate student of Yale University before he came to Purdue University in 1963 in a postdoctoral position. A Hilbert space which is contained contractively in $\mathcal{C}(z)$ and satisfies the inequality for difference quotients has a complementary space $\mathcal{M}$ in $\mathcal{C}(z)$ such that multiplication by z is a contractive transformation of $\mathcal{M}$ into itself.

Lemma. *If a Hilbert space $\mathcal{M}$ is contained contractively in $\mathcal{C}(z)$ and if multiplication by z is an isometric transformation of $\mathcal{M}$ into itself, then the dimension of the orthogonal complement in $\mathcal{M}$ of the range of multiplication by z is less than or equal to the dimension of the coefficient space.*

Proof of Lemma. The dimension estimate is satisfied when the coefficient space has infinite dimension since the dimension of $\mathcal{M}$ is less than or equal to the dimension of $\mathcal{C}(z)$ which is equal to the dimension of $\mathcal{C}$. It remains to obtain the dimension estimate when the coefficient space has finite dimension r for some positive integer r.

Argue by contradiction assuming that the orthogonal complement of $\mathcal{M}'$ in $\mathcal{M}$ contains $r+1$ linearly independent elements

$$f_0(z), \dots, f_r(z).$$

If $c_0, \dots, c_r$ are corresponding vectors, then the square matrix which has entry

$$c_i^- f_j(z)$$

in the i–th row and j–th column has vanishing determinant. The determinant is a power series with complex coefficients which represents a bounded function in the unit disk. Expansion of the determinant on the row i equal to zero produces the vanishing power series

$$c_0^- f_0(z) g_0(z) + \dots + c_0^- f_r(z) g_r(z)$$

with

$$(-1)^k g_k(z)$$

the determinant of the matrix obtained by deleting the 0–th row and k–th column from the starting matrix. Since the vector c_0 is arbitrary, the element

$$f_0(z) g_0(z) + \dots + f_r(z) g_r(z)$$

of $\mathcal{M}$ vanishes identically. Each power series

$$g_k(z) = a_{k0} + a_{k1} z + a_{k2} z^2 + \dots$$

with complex coefficients is square summable since it represents a bounded function in the unit disk. Since multiplication by z is isometric in $\mathcal{M}$ and since

$$f_0(z), \dots, f_r(z)$$

are orthogonal to the range of the transformation, the elements

$$z^m f_0(z), \dots, z^m f_r(z)$$

of $\mathcal{M}$ are orthogonal to the elements

$$z^n f_0(z), \dots, z^n f_r(z)$$

when m and n are distinct nonnegative integers. The element

$$f_0(z) a_{0n} z^n + \dots + f_r(z) a_{rn} z^n$$

vanishes for every nonnegative integer n. Since the elements

$$f_0(z), \dots, f_r(z)$$

of $\mathcal{M}$ are linearly independent, all coefficients of the power series

$$g_0(z), \dots, g_r(z)$$

vanish identically.

Since the power series are determinants of square matrices of the same form as the starting matrix, the construction can be repeated to produce square matrices of smaller order with vanishing determinant. A contradiction is obtained since all coefficients

$$c_i^- f_j(z)$$

of the starting matrix then vanish identically.

This completes the proof of the theorem.

The space $\mathcal{H}(W)$ is characterized as the state space of a linear system. The main transformation A, which takes $f(z)$ into $[f(z) - f(0)]/z$, maps the state space into itself. The input transformation B, which takes c into $[W(z) - W(0)]c/z$, maps the coefficient space into the state space. The output transformation C, which takes $f(z)$ into $f(0)$, maps the state space into the coefficient space. The external operator D, which takes c into $W(0)c$, maps the coefficient space into itself. The matrix of the linear system

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

maps the Cartesian product of the state space and the coefficient space into itself. The linear system is coisometric since the matrix has an isometric adjoint. The linear system is canonical since a canonical model of the state space is applied. The transfer function $W(z)$ of the linear system is a Schur function since it is characterized as a power series with operator coefficients which represents a function with contractive values in the unit disk.

A convex structure applies to the Hilbert spaces which are contained contractively in a Hilbert space $\mathcal{H}$. If Hilbert spaces $\mathcal{P}$ and $\mathcal{Q}$ are contained contractively in $\mathcal{H}$ and if t is in the interval $[0, 1]$, a unique Hilbert space

$$\mathcal{S} = (1 - t)\mathcal{P} + t\mathcal{Q}$$

exists, which is contained contractively in $\mathcal{H}$, such that the convex combination

$$c = (1 - t)a + tb$$

belongs to $\mathcal{S}$ whenever a belongs to $\mathcal{P}$ and b belongs to $\mathcal{Q}$, such that the inequality

$$\|c\|_{\mathcal{S}}^2 \leq (1 - t)\|a\|_{\mathcal{P}}^2 + t\|b\|_{\mathcal{Q}}^2$$

is satisfied, and such that every element c of $\mathcal{S}$ is a convex combination for which equality holds. The adjoint of the inclusion of $\mathcal{S}$ in $\mathcal{H}$ is the convex combination

$$(1-t)P + tQ$$

of the adjoint P of the inclusion of $\mathcal{P}$ in $\mathcal{H}$ and the adjoint Q of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$.

A topology applies to the convex set of Hilbert spaces which are contained contractively in a Hilbert space $\mathcal{H}$. For the definition of a topology a space is identified with the contractive transformation of $\mathcal{H}$ into itself which extends the adjoint of the inclusion of the space in $\mathcal{H}$. The space of continuous transformations of $\mathcal{H}$ into itself is a Hausdorff space in the weak topology induced by duality with the space of continuous transformations which are of trace class. The convex set of contractive transformations of $\mathcal{H}$ into itself which are nonnegative is compact in the subspace topology. If a Hilbert space $\mathcal{P}$ is contained contractively in $\mathcal{H}$, then the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$ agrees with a nonnegative and contractive transformation P. If P is a nonnegative and contractive transformation, a unique Hilbert space $\mathcal{P}$ exists which is contained contractively in $\mathcal{H}$ and which has P as the adjoint of its inclusion in $\mathcal{H}$. The convex set of Hilbert spaces which are contained contractively in $\mathcal{H}$ is a compact Hausdorff space in the topology of associated nonnegative and contractive transformations.

A convex combination of Hilbert spaces of power series with vector coefficients which satisfy the inequality for difference quotients is a Hilbert space of power series with vector coefficients which satisfies the inequality for difference quotients. The convex set of Hilbert spaces of power series with vector coefficients which satisfy the inequality for difference quotients is a closed subset of the convex set of all Hilbert spaces which are contained contractively in $\mathcal{C}(z)$. The convex set of Hilbert spaces of power series with vector coefficients which satisfy the inequality for difference quotients is the closed convex span of its extreme points by the Krein–Milman theorem.

A Hilbert space $\mathcal{H}$ of power series with vector coefficients which satisfies the identity for difference quotients is an extreme point of the convex set of Hilbert spaces of power series with vector coefficients which satisfy the inequality for difference quotients. If

$$\mathcal{H} = (1-t)\mathcal{P} + t\mathcal{Q}$$

is a convex combination with t in the interval $(0,1)$ of Hilbert spaces $\mathcal{P}$ and $\mathcal{Q}$ of power series with vector coefficients which satisfy the inequality for difference quotients, then every element

$$h(z) = (1-t)f(z) + tg(z)$$

of $\mathcal{H}$ is a convex combination of elements $f(z)$ of $\mathcal{P}$ and $g(z)$ of $\mathcal{Q}$ such that equality holds in the inequality

$$\|h(z)\|^2_{\mathcal{H}} \leq (1-t)\|f(z)\|^2_{\mathcal{P}} + t\|g(z)\|^2_{\mathcal{Q}}.$$

Since the inequality

$$\|[h(z)-h(0)]/z\|^2_{\mathcal{H}} \leq (1-t)\|[f(z)-f(0)]/z\|^2_{\mathcal{P}} + t\|[g(z)-g(0)]/z\|^2_{\mathcal{Q}}$$

holds with

$$\|[f(z) - f(0)]/z\|_{\mathcal{P}}^2 \le \|f(z)\|_{\mathcal{P}}^2 - |f(0)|^2$$

and

$$\|[g(z) - g(0)]/z\|_{\mathcal{Q}}^2 \le \|g(z)\|_{\mathcal{Q}}^2 - |g(0)|^2,$$

the inequality

$$\|[h(z) - h(0)]/z\|_{\mathcal{H}}^2 \le \|h(z)\|_{\mathcal{H}}^2 - (1-t)|f(0)|^2 - t|g(0)|^2$$

is satisfied. The inequality reads

$$\|[h(z) - h(0)]/z\|_{\mathcal{H}}^2 \le \|h(z)\|_{\mathcal{H}}^2 - |h(0)|^2 - t(1-t)|g(0) - f(0)|^2$$

since the convexity identity

$$|(1-t)f(0) + tg(0)|^2 + t(1-t)|g(0) - f(0)|^2 = (1-t)|f(0)|^2 + t|g(0)|^2$$

is satisfied. Since the space $\mathcal{H}$ satisfies the identity for difference quotients, the constant coefficients of $f(z)$, $g(z)$, and $h(z)$ are equal. An inductive argument shows that the n-th coefficients of $f(z)$, $g(z)$, and $h(z)$ are equal for every nonnegative integer n. Since $f(z)$, $g(z)$, and $h(z)$ are always equal, the spaces $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{H}$ are isometrically equal.

The construction of invariant subspaces applies Hilbert spaces of power series with vector coefficients which originate in the construction by David Hilbert of invariant subspaces for an isometric transformation of a Hilbert space onto itself. An approach to the spectral theory of unitary transformations which is close to the methods of Hilbert is preserved in perturbation theory. A canonical model of pairs of unitary transformations acting in the same Hilbert space is applied which is related to the canonical model of contractive transformations. Lawrence Shulman in his thesis [8] applies the model in a computation of scattering operators showing unitary equivalence of components of the spectrum. The domain and range of scattering operators are conjectured to be determined by square summability of power series appearing in the canonical model [7].

A Herglotz space is a Hilbert space, whose elements are power series with vector coefficients, such that a contractive transformation of the space into itself is defined by taking $f(z)$ into $[f(z) - f(0)]/z$, such that the adjoint transformation is isometric, and such that a continuous transformation of the space into the coefficient space is defined by taking $f(z)$ into $f(0)$. If $\mathcal{H}$ is a Herglotz space, a power series $\varphi(z)$ with operator coefficients, which is unique within an added skew–conjugate operator, exists such that the series

$$\tfrac{1}{2}[\varphi(z) + \varphi(0)^-]c$$

belongs to the space for every vector c and such that the identity

$$c^- f(0) = \langle f(z), \tfrac{1}{2}[\varphi(z) + \varphi(0)^-]c\rangle_{\mathcal{H}}$$

holds for every element $f(z)$ of the space. The elements of a Herglotz space are power series which converge in the unit disk. If $f(z)$ is an element of a Herglotz space, elements $f_n(z)$ of the space are defined inductively for nonnegative integers n by

$$f_0(z) = f(z)$$

and by

$$f_{n+1}(z) = [f_n(z) - f_n(0)]/z$$

for every nonnegative integer n. The series

$$[zf(z) - wf(w)]/(z - w) = f_0(z) + wf_1(z) + w^2 f_2(z) + \ldots$$

converges in the metric topology of the space when w is in the unit disk. A continuous transformation of the space into the coefficient space is defined by taking a power series $f(z)$ into the value $f(w)$ at w of the represented function when w is in the unit disk. The power series $\varphi(z)$ converges in the unit disk and represents a function with value $\varphi(w)$ at w. The series

$$\tfrac{1}{2}[\varphi(z) + \varphi(w)^-]c/(1 - w^- z)$$

belongs to the Herglotz space for every vector c when w is in the unit disk. The identity

$$c^- f(w) = \langle f(z), \tfrac{1}{2}[\varphi(z) + \varphi(w)^-]c/(1 - w^- z)\rangle_{\mathcal{H}}$$

holds for every element $f(z)$ of the space. The Herglotz space $\mathcal{H}$ determined by the function $\phi(z)$ is denoted $\mathcal{L}(\phi)$.

The space $\mathcal{C}(z)$ of square summable power series is isometrically equal to a Herglotz space $\mathcal{L}(\phi)$ with

$$\phi(z) = 1.$$

A Herglotz space $\mathcal{L}(\phi)$ is contained contractively in a Herglotz space $\mathcal{L}(\psi)$ if, and only if, a Herglotz space $\mathcal{L}(\theta)$ exists with

$$\theta(z) = \psi(z) - \phi(z).$$

The Herglotz space $\mathcal{L}(\theta)$ is then isometrically equal to the complementary space to the space $\mathcal{L}(\phi)$ in the space $\mathcal{L}(\psi)$. A convex combination

$$(1 - t)\mathcal{L}(\phi) + t\mathcal{L}(\theta)$$

of Herglotz spaces $\mathcal{L}(\phi)$ and $\mathcal{L}(\theta)$ is a Herglotz space $\mathcal{L}(\psi)$ with

$$\psi(z) = (1 - t)\phi(z) + t\theta(z).$$

The Herglotz spaces which are contained contractively in $\mathcal{C}(z)$ form a compact convex set. The convex set is the closed convex span of its extreme points by the Krein–Milman theorem.

A Herglotz space is associated with a power series $W(z)$ with operator coefficients such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself. The adjoint of multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself which takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$. The adjoint of multiplication by $W(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\phi)$ which is contained contractively in $\mathcal{C}(z)$. The complementary space in $\mathcal{C}(z)$ to the Herglotz space $\mathcal{L}(\phi)$ is the Herglotz space $\mathcal{L}(1 - \phi)$.

An element $f(z)$ of $\mathcal{C}(z)$ belongs to the space $\mathcal{L}(1-\phi)$ if, and only if, $W(z)f(z)$ belongs to the space $\mathcal{H}(W)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(1-\phi)} = \|f(z)\|^2 + \|W(z)f(z)\|^2_{\mathcal{H}(W)}$$

holds for every element $f(z)$ of the space $\mathcal{L}(1 - \phi)$. An element $f(z)$ of $\mathcal{C}(z)$ belongs to the space $\mathcal{H}(W)$ if, and only if, the adjoint of multiplication by $W(z)$ maps $f(z)$ into an element $g(z)$ of the space $\mathcal{L}(1 - \phi)$. The identity

$$\|f(z)\|^2_{\mathcal{H}(W)} = \|f(z)\|^2 + \|g(z)\|^2_{\mathcal{L}(1-\phi)}$$

holds for every element $f(z)$ of the space $\mathcal{H}(W)$.

The Herglotz space $\mathcal{L}(1-\phi)$ associated with a space $\mathcal{H}(W)$ was introduced in a graduate course taught at Purdue University in the fall semester 1963. A fundamental property of the space was discovered by Virginia Rovnyak as a graduate student in the course. The space $\mathcal{H}(W)$ satisfies the identity for difference quotients if, and only if, the inclusion of the Herglotz space in $\mathcal{C}(z)$ is isometric on polynomial elements of the space. The present construction of the space applies the Lowdenslager condition for Wiener factorization [9].

Proof of Theorem 1. A Hilbert space $\mathcal{H}$ of power series with vector coefficients which satisfies the inequality for difference quotients is contained contractively in $\mathcal{C}(z)$. Multiplication by z is a contractive transformation of the complementary space $\mathcal{M}$ to $\mathcal{H}$ in $\mathcal{C}(z)$ into $\mathcal{M}$. The space $\mathcal{H}$ is shown isometrically equal to a space $\mathcal{H}(W)$ by showing that multiplication by z is an isometric transformation of $\mathcal{M}$ into itself when $\mathcal{H}$ satisfies the identity for difference quotients. The existence of the power series $W(z)$ is then given by the dimension estimate of the lemma.

It is sufficient to obtain the isometric property of multiplication by z in $\mathcal{M}$ when the space $\mathcal{H}$ is known to be a space $\mathcal{H}(W)$ since the coefficient space can be enlarged. Multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself which acts as a partially isometric transformation of $\mathcal{C}(z)$ onto $\mathcal{M}$. The isometric property of multiplication by z in $\mathcal{M}$ is proved by showing that the kernel of multiplication by $W(z)$ contains $[f(z)-f(0)]/z$ whenever it contains $f(z)$.

The adjoint of multiplication by $W(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\phi)$ which is contained contractively in $\mathcal{C}(z)$ and whose complementary space in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\phi)$.

The space $\mathcal{H}(W)$ is contained contractively in the space $\mathcal{H}(W')$,

$$W'(z) = zW(z).$$

Multiplication by $W'(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself which is the composition of the contractive multiplication by $W(z)$ and the isometric multiplication by z. The composition can be taken in either order. The adjoint of multiplication by $W'(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\phi')$ which is contained contractively in $\mathcal{C}(z)$ and whose complementary space in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\phi')$.

The adjoint of multiplication by z is a partially isometric transformation of $\mathcal{C}(z)$ into itself which takes $f(z)$ into $[f(z)-f(0)]/z$. A partially isometric transformation of the Herglotz space $\mathcal{L}(\phi)$ onto the Herglotz space $\mathcal{L}(\phi')$ is defined by taking $f(z)$ into $[f(z)-f(0)]/z$. A partially isometric transformation of the space $\mathcal{L}(1-\phi)$ onto the Herglotz space $\mathcal{L}(1-\phi')$ is defined by taking $f(z)$ into $[f(z)-f(0)]/z$.

The adjoint of multiplication by $W(z)$ as a transformation of $\mathcal{C}(z)$ into itself takes an element $f(z)$ of the space $\mathcal{H}(W)$ into an element $g(z)$ of the space $\mathcal{L}(1-\phi)$ which satisfies the identity

$$\|f(z)\|^2_{\mathcal{H}(W)} = \|f(z)\|^2 + \|g(z)\|^2_{\mathcal{L}(1-\phi)}.$$

The adjoint of multiplication by $W'(z)$ as a transformation of $\mathcal{C}(z)$ into itself takes $f(z)$ into the element $[g(z)-g(0)]/z$ of the space $\mathcal{L}(1-\phi')$ which satisfies the identity

$$\|f(z)\|^2_{\mathcal{H}(W')} = \|f(z)\|^2 + \|[g(z)-g(0)]/z\|^2_{\mathcal{L}(1-\phi')}.$$

The inclusion of the space $\mathcal{H}(W)$ in the space $\mathcal{H}(W')$ is isometric on $f(z)$ if, and only if, $g(z)$ is orthogonal to elements of the coefficient space which belong to $\mathcal{L}(1-\phi)$.

The adjoint of multiplication by $W(z)$ as a transformation of the Herglotz space $\mathcal{L}(1-\phi)$ into the space $\mathcal{H}(W)$ is the restriction of the adjoint of multiplication by $W(z)$ as a transformation of $\mathcal{C}(z)$ into itself.

The kernel of multiplication by $W(z)$ is the orthogonal complement in the Herglotz space $\mathcal{L}(1-\phi)$ of the elements obtained from the space $\mathcal{H}(W)$ under the adjoint of multiplication by $W(z)$. Since the identity for difference quotients holds in the space $\mathcal{H}(W)$ if, and only if, the space $\mathcal{H}(W)$ is contained isometrically in the space $\mathcal{H}(W')$, the identity for difference quotients holds in the space $\mathcal{H}(W)$ if, and only if, the kernel of multiplication by $W(z)$ contains the elements of the coefficient space which belong to the space $\mathcal{L}(1-\phi)$.

Since the set of elements of the Herglotz space $\mathcal{L}(1-\phi)$ which are obtained from the space $\mathcal{H}(W)$ under the adjoint of multiplication by $W(z)$ contains $[f(z)-f(0)]/z$ whenever it contains $f(z)$, the kernel of multiplication by $W(z)$ is an invariant subspace for the adjoint of the transformation taking $f(z)$ into $[f(z)-f(0)]/z$ of the Herglotz space into itself. Since the inclusion of the Herglotz space in $\mathcal{C}(z)$ is isometric on the kernel of multiplication by $W(z)$, the adjoint agrees with multiplication by z on the kernel of multiplication by $W(z)$.

The orthogonal complement of the kernel of multiplication by $W(z)$ in the Herglotz space $\mathcal{L}(1-\phi)$ is an invariant subspace for the transformation taking $f(z)$ into $[f(z)-f(0)]/z$ on which the transformation is isometric when the space $\mathcal{H}(W)$ satisfies the identity for difference quotients. The inverse transformation is an isometric transformation of the orthogonal complement of the kernel onto itself. Since the orthogonal complement of the kernel is an invariant subspace for the adjoint of the transformation taking $f(z)$ into $[f(z) - f(0)]/z$ of the Herglotz space into itself, the kernel of multiplication by $W(z)$ is an invariant subspace for the transformation.

This completes the proof of the theorem.

If a Hilbert space $\mathcal{H}$ of power series with vector coefficients satisfies the inequality for difference quotients, a greatest subspace of $\mathcal{H}$ exists which is contained isometrically in $\mathcal{H}$ and which satisfies the identity for difference quotients. An element $f(z)$ of $\mathcal{H}$ belongs to the subspace if the elements $f_n(z)$ of $\mathcal{H}$ defined inductively by

$$f_0(z) = f(z)$$

and

$$f_{n+1}(z) = [f_n(z) - f_n(0)]/z$$

satisfy the identity for difference quotients

$$\|f_{n+1}(z)\|_{\mathcal{H}}^2 = \|f_n(z)\|_{\mathcal{H}}^2 - |f_n(0)|^2$$

for every nonnegative integer n. The subspace is isometrically equal to a space $\mathcal{H}(U)$ for a power series $U(z)$ with operator coefficients such that multiplication by $U(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself whose kernel contains $[f(z) - f(0)]/z$ whenever it contains $f(z)$.

An element of the orthogonal complement of $\mathcal{H}(U)$ in $\mathcal{H}$ is a product

$$U(z)f(z)$$

with $f(z)$ an element of $\mathcal{C}(z)$ which is orthogonal to the kernel of multiplication by $U(z)$. A Hilbert space $\mathcal{H}'$ exists which satisfies the inequality for difference quotients and which is orthogonal in $\mathcal{C}(z)$ to the kernel of multiplication by $U(z)$ such that multiplication by $U(z)$ acts as an isometric transformation of $\mathcal{H}'$ onto the orthogonal complement of $\mathcal{H}(U)$ in $\mathcal{H}$. There is no nonzero element in the greatest subspace of $\mathcal{H}'$ which satisfies the identity for difference quotients.

When the space $\mathcal{H}$ is a space $\mathcal{H}(W)$, multiplication by $W(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the complementary space $\mathcal{M}(W)$ to the space $\mathcal{H}(W)$ in $\mathcal{C}(z)$. Multiplication by $U(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the complementary space $\mathcal{M}(U)$ to the space $\mathcal{H}(U)$ in $\mathcal{C}(z)$. Since the space $\mathcal{H}(U)$ is contained contractively in the space $\mathcal{H}(W)$, the space $\mathcal{M}(W)$ is contained contractively in the space $\mathcal{M}(U)$. A contractive transformation of $\mathcal{C}(z)$ into itself is defined by taking $f(z)$ into the solution $g(z)$ of the equation

$$U(z)g(z) = W(z)f(z)$$

which is orthogonal to the kernel of multiplication by $U(z)$. The transformation takes $zf(z)$ into $zg(z)$ whenever it takes $f(z)$ into $g(z)$ since the kernel of multiplication by $U(z)$ contains $[h(z) - h(0)]/z$ whenever it contains $h(z)$. A power series $V(z)$ with operator coefficients exist such that multiplication by $V(z)$ is the contractive transformation of $\mathcal{C}(z)$ into itself taking $f(z)$ into $g(z)$. The space $\mathcal{H}(V)$ is orthogonal in $\mathcal{C}(z)$ to the kernel of multiplication by $U(z)$. Multiplication by $U(z)$ is an isometric transformation of the space $\mathcal{H}(V)$ onto the orthogonal complement in the space $\mathcal{H}(W)$ of the space $\mathcal{H}(U)$. The identity

$$W(z) = U(z)V(z)$$

is satisfied.

The factorization admits a reformulation using Herglotz spaces.

Theorem 2. *If $W(z)$ is a power series with operator coefficients such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself and if the adjoint of multiplication by $W(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\phi)$, then the orthogonal complement in the space $\mathcal{H}(W)$ of elements $W(z)f(z)$ with $f(z)$ a polynomial element of the Herglotz space $\mathcal{L}(1-\phi)$ is a Hilbert space $\mathcal{H}(U)$ which is contained isometrically in the space $\mathcal{H}(W)$ and which satisfies the identity for difference quotients. A power series $V(z)$ with operator coefficients exists such that multiplication by $V(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself whose range is orthogonal to the kernel of multiplication by $U(z)$ and which satisfies the identity*

$$W(z) = U(z)V(z).$$

Multiplication by $U(z)$ is an isometric transformation of the space $\mathcal{H}(V)$ onto the orthogonal complement of the space $\mathcal{H}(U)$ in the space $\mathcal{H}(W)$. The adjoint of multiplication by $V(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\psi)$ such that the Herglotz space $\mathcal{L}(1-\psi)$ is contained isometrically in the Herglotz space $\mathcal{L}(1-\phi)$ and contains the polynomial elements of $\mathcal{L}(1-\phi)$. The adjoint of multiplication by $U(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto a Herglotz space $\mathcal{L}(\theta)$. The adjoint of multiplication by $V(z)$ acts as an isometric transformation of the Herglotz space $\mathcal{L}(1-\theta)$ onto the Herglotz space $\mathcal{L}(\psi-\phi)$ which is the orthogonal complement in the Herglotz space $\mathcal{L}(1-\psi)$ in the Herglotz space $\mathcal{L}(1-\phi)$. A dense set of elements of the space $\mathcal{H}(V)$ are products $V(z)f(z)$ with $f(z)$ a polynomial element of the space $\mathcal{L}(1-\phi)$.

Proof of Theorem 2. The construction of the space $\mathcal{H}(U)$ is an equivalent construction of the greatest subspace of the space $\mathcal{H}(W)$ which is contained isometrically in the space $\mathcal{H}(W)$ and which satisfies the identity for difference quotients. The construction of the space $\mathcal{H}(V)$ repeats a construction which has previously been made with a statement of consequences for associated Herglotz spaces. Proofs are an application of complementation. Care is taken to avoid the conclusion that the polynomial elements of the Herglotz space $\mathcal{L}(1-\psi)$ are dense in the space. Density is asserted only for images of polynomials in the space $\mathcal{H}(V)$.

This completes the proof of the theorem.

If a Herglotz space $\mathcal{L}(\phi)$ is contained contractively in $\mathcal{C}(z)$ and if the polynomial elements of the space are dense in the space, a partially isometric transformation of $\mathcal{C}(z)$ onto the

Herglotz space $\mathcal{L}(\phi)$ exists which takes $[f(z)-f(0)]/z$ into $[g(z)-g(0)]/z$ whenever it takes $f(z)$ into $g(z)$. Such transformations are not unique. A power series $B(z)$ with operator coefficients exists such that multiplication by $B(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself whose adjoint coincides with the partially isometric transformation of $\mathcal{C}(z)$ onto the Herglotz space.

A partially isometric transformation of $\mathcal{C}(z)$ onto the Herglotz space exists which takes $[f(z) - f(0)/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$ and whose kernel contains $zf(z)$ whenever it contains $f(z)$. Such transformations are essentially unique. A power series $A(z)$ with operator coefficients exists such that multiplication by $A(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself whose adjoint coincides with the partially isometric transformation of $\mathcal{C}(z)$ onto the Herglotz space.

A power series $C(z)$ with operator coefficients exists such that multiplication by $C(z)$ is a partially isometric transformation of $\mathcal{C}(z)$ into itself such that the identity

$$B(z) = A(z)C(z)$$

is satisfied.

Convex decompositions of Hilbert spaces of power series with vector coefficients which satisfy the inequality for difference quotients leave fixed the elements of the greatest subspace which satisfies the identity for difference quotients. The determination of extreme points is reduced to spaces containing no nonzero element in the greatest subspace satisfying the identity for difference quotients.

Theorem 3. *If a Hilbert space $\mathcal{H}$ of power series with vector coefficients which satisfies the inequality for difference quotients is a convex combination*

$$\mathcal{H} = (1-t)\mathcal{H}_+ + t\mathcal{H}_-$$

of Hilbert spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ of power series with vector coefficients which satisfy the inequality for difference quotients, then the greatest subspace $\mathcal{H}(U)$ of $\mathcal{H}$ which satisfies the identity for difference quotients is contained isometrically in the spaces $\mathcal{H}_+$ and $\mathcal{H}_-$. Hilbert spaces $\mathcal{H}'_+, \mathcal{H}'_-$, and $\mathcal{H}'$ which satisfy the inequality for difference quotients and which are orthogonal in $\mathcal{C}(z)$ to the kernel of multiplication by $U(z)$ exist such that multiplication by $U(z)$ acts as an isometric transformation of $\mathcal{H}'_+$ onto the orthogonal complement of $\mathcal{H}(U)$ in $\mathcal{H}_+$, of $\mathcal{H}'_-$ onto the orthogonal complement of $\mathcal{H}(U)$ in $\mathcal{H}_-$, and of $\mathcal{H}'$ onto the orthogonal complement of $\mathcal{H}(U)$ in $\mathcal{H}$. The space

$$\mathcal{H}' = (1-t)\mathcal{H}'_+ + t\mathcal{H}'_-$$

is a convex combination of $\mathcal{H}'_+$ and $\mathcal{H}'_-$.

Proof of Theorem 3. If an element

$$h(z) = (1-t)f(z) + tg(z)$$

of the greatest subspace of $\mathcal{H}$ which satisfies the identity for difference quotients is a minimal convex combination of $f(z)$ in $\mathcal{H}_+$ and $g(z)$ in $\mathcal{H}_-$, then $f(z)$ and $g(z)$ are equal to $h(z)$ with the norm of $f(z)$ in $\mathcal{H}_+$ and the norm of $g(z)$ in $\mathcal{H}_-$ equal to the norm of $h(z)$ in $\mathcal{H}$. The construction of the space $\mathcal{H}'$ from $\mathcal{H}$ applies to the construction of the space $\mathcal{H}'_+$ from $\mathcal{H}_+$ and to the construction of the space $\mathcal{H}'_-$ from $\mathcal{H}_-$. The convex decomposition of the space $\mathcal{H}'$ restates the convex decomposition of the space $\mathcal{H}$.

This completes the proof of the theorem.

The extension space of a Herglotz space $\mathcal{L}(\phi)$ is a Hilbert space $\mathcal{E}(\phi)$ of Laurent series with vector coefficients such that multiplication by z is an isometric transformation of the space onto itself and such that a partially isometric transformation of the space onto the space $\mathcal{L}(\phi)$ is defined by taking a Laurent series into the power series which has the same coefficient of z^n for every nonnegative integer n.

The extension space of the Herglotz space $\mathcal{C}(z)$ of square summable power series is the space ext $\mathcal{C}(z)$ of square summable Laurent series

$$f(z) = \sum a_n z^n$$

with

$$\|f(z)\|^2 = \sum |a_n|^2.$$

If $\mathcal{E}(\phi)$ and $\mathcal{E}(\psi)$ are extension spaces of Herglotz spaces $\mathcal{L}(\phi)$ and $\mathcal{L}(\psi)$, then a Herglotz space

$$\mathcal{L}(\phi + \psi) = \mathcal{L}(\phi) \vee \mathcal{L}(\psi)$$

exists in which the Herglotz spaces $\mathcal{L}(\phi)$ and $\mathcal{L}(\psi)$ are contained contractively as complementary spaces. The extension spaces $\mathcal{E}(\phi)$ and $\mathcal{E}(\psi)$ are contained contractively as complementary spaces in the extension space

$$\mathcal{E}(\phi + \psi) = \mathcal{E}(\phi) \vee \mathcal{E}(\psi).$$

A Herglotz space $\mathcal{L}(\theta)$ exists such that the extension space

$$\mathcal{E}(\theta) = \mathcal{E}(\phi) \wedge \mathcal{E}(\psi)$$

is the intersection space of the extension spaces $\mathcal{E}(\phi)$ and $\mathcal{E}(\psi)$.

A Herglotz space $\mathcal{L}(\phi - \theta)$ exists such that the extension space $\mathcal{E}(\phi - \theta)$ is the complementary space in the extension space $\mathcal{E}(\phi)$ of the extension space $\mathcal{E}(\theta)$. A Herglotz space $\mathcal{L}(\psi - \theta)$ exists such that the extension space $\mathcal{E}(\psi - \theta)$ is the complementary space in the extension space $\mathcal{E}(\psi)$ of the extension space $\mathcal{E}(\theta)$. A Herglotz space $\mathcal{L}(\phi + \theta)$ exists such that the extension space $\mathcal{E}(\phi + \theta)$ is the complementary space in the extension space $\mathcal{E}(\phi + \psi)$ of the extension space $\mathcal{E}(\psi - \theta)$. A Herglotz space $\mathcal{L}(\psi + \theta)$ exists such that the extension space $\mathcal{E}(\psi + \theta)$ is the complementary space in the extension space $\mathcal{E}(\phi + \psi)$ of the extension space $\mathcal{E}(\phi - \theta)$. The extension space

$$\mathcal{E}(\phi) = \tfrac{1}{2}\mathcal{E}(\phi + \theta) + \tfrac{1}{2}\mathcal{E}(\phi - \theta)$$

is a convex combination of the extension spaces $\mathcal{E}(\phi+\theta)$ and $\mathcal{E}(\phi-\theta)$. The extension space

$$\mathcal{E}(\psi) = \tfrac{1}{2}\ \mathcal{E}(\psi+\theta) + \tfrac{1}{2}\mathcal{E}(\psi-\theta)$$

is a convex combination of the extension spaces $\mathcal{E}(\psi+\theta)$ and $\mathcal{E}(\psi-\theta)$.

A partially isometric transformation of the extension space $\mathcal{E}(\phi+\psi)$ onto the Herglotz space $\mathcal{L}(\phi+\psi)$ is defined by taking a Laurent series into the power series which has the same coefficient of z^n for every nonnegative integer n. The restriction acts as a partially isometric transformation of the extension space $\mathcal{E}(\phi)$ onto the Herglotz space $\mathcal{L}(\phi)$, of the extension space $\mathcal{E}(\psi)$ onto the Herglotz space $\mathcal{L}(\psi)$, and of the extension space $\mathcal{E}(\theta)$ onto the Herglotz space $\mathcal{L}(\theta)$. Partially isometric transformations are obtained of the extension spaces $\mathcal{E}(\phi+\theta)$ and $\mathcal{E}(\phi-\theta)$ onto the Herglotz spaces $\mathcal{L}(\phi+\theta)$ and $\mathcal{L}(\phi-\theta)$ and of the extension spaces $\mathcal{E}(\psi+\theta)$ and $\mathcal{E}(\psi-\theta)$ onto the Herglotz spaces $\mathcal{L}(\psi+\theta)$ and $\mathcal{L}(\psi-\theta)$.

The Herglotz space $\mathcal{L}(\theta)$ is contained contractively in the Herglotz spaces $\mathcal{L}(\phi)$ and $\mathcal{L}(\psi)$. The Herglotz space $\mathcal{L}(\phi-\theta)$ is the complementary space in the Herglotz space $\mathcal{L}(\phi)$ of the Herglotz space $\mathcal{L}(\theta)$. The Herglotz space $\mathcal{L}(\psi-\theta)$ is the complementary space in the Herglotz space $\mathcal{L}(\psi)$ of the Herglotz space $\mathcal{L}(\theta)$. The Herglotz space $\mathcal{L}(\phi+\theta)$ is the complementary space in the Herglotz space $\mathcal{L}(\phi+\psi)$ of the Herglotz space $\mathcal{L}(\psi-\theta)$. The Herglotz space $\mathcal{L}(\psi+\theta)$ is the complementary space in the Herglotz space $\mathcal{L}(\phi+\psi)$ of the Herglotz space $\mathcal{L}(\phi-\theta)$. The Herglotz space

$$\mathcal{L}(\phi) = \tfrac{1}{2}\mathcal{L}(\phi+\theta) + \tfrac{1}{2}\mathcal{L}(\phi-\theta)$$

is a convex combination of the Herglotz spaces $\mathcal{L}(\phi+\theta)$ and $\mathcal{L}(\phi-\theta)$. The Herglotz space

$$\mathcal{L}(\psi) = \tfrac{1}{2}\mathcal{L}(\psi+\theta) + \tfrac{1}{2}\mathcal{L}(\psi-\theta)$$

is a convex combination of the Herglotz spaces $\mathcal{L}(\psi+\theta)$ and $\mathcal{L}(\psi-\theta)$.

An application of the convex decomposition is the determination of the extreme points of the convex set of Herglotz spaces which are contained contractively in the space $\mathcal{C}(z)$ of square summable power series. A Herglotz space $\mathcal{L}(\phi)$, which belongs to the convex set, is an extreme point of the convex set if, and only if, the extension space $\mathcal{E}(\phi)$ is contained isometrically in the space ext $\mathcal{C}(z)$ of square summable Laurent series.

An existence theorem for invariant subspaces for isometric transformations of a Hilbert space onto itself is an application of extension spaces of Herglotz spaces. Such a transformation is unitarily equivalent to multiplication by z in the extension space $\mathcal{E}(\phi)$ of some Herglotz space $\mathcal{L}(\phi)$. A decomposition

$$\phi(z) = \phi_+(z) + \phi_-(z)$$

holds for Herglotz spaces $\mathcal{L}(\phi_+)$ and $\mathcal{L}(\phi_-)$ such that the operator valued function represented by $\phi_+(z)$ admits an analytic extension to the upper half–plane and the operator valued function represented by $\phi_-(z)$ admits an analytic extension to the lower half–plane.

The intersection space $\mathcal{E}(\phi_+) \wedge \mathcal{E}(\phi_-)$ is an extension space whose elements are sums

$$a(z-1)^{-1} + b(z+1)^{-1}$$

for vectors a and b. The decomposition can be made so that the intersection space contains no nonzero element. This can be done for example so that the space $\mathcal{E}(\phi_-)$ contains every element

$$a(z+1)^{-1}$$

of the space $\mathcal{E}(\phi)$ and the space $\mathcal{E}(\phi_+)$ contains every element

$$b(z-1)^{-1}$$

of the space $\mathcal{E}(\phi)$. The spaces $\mathcal{E}(\phi_+)$ and $\mathcal{E}(\phi_-)$ are then contained isometrically in the space $\mathcal{E}(\phi)$ and are invariant subspaces of every continuous transformation of the space $\mathcal{E}(\phi)$ into itself which commute with multiplication by z.

The decomposition of Herglotz functions is an adaptation to the unit disk of the Poisson representation of functions which are analytic and have nonnegative real part in the upper half–plane. The transition from complex valued functions to operator valued functions causes no difficulty in the Poisson representation.

The original proof of existence of invariant subspaces for isometric transformations of a Hilbert space onto itself was presented by Hilbert in lectures at the Göttingen mathematical institute. Although the lecture notes taken by his student Ernst Hellinger have been lost, his subsequent research indicates a formulation of the existence theorem related to Herglotz spaces. These spaces are named after a member of the Göttingen institute who was clearly influenced by Hilbert.

The extension space of a Hilbert space $\mathcal{H}$ of power series with vector coefficients which satisfies the identity for difference quotients is a Hilbert space ext $\mathcal{H}$ which is contained contractively in ext $\mathcal{C}(z)$ and which contains the space $\mathcal{H}$ isometrically. The orthogonal complement of $\mathcal{H}$ in ext $\mathcal{H}$ is contained isometrically in ext $\mathcal{C}(z)$ and coincides with the orthogonal complement of $\mathcal{C}(z)$ in ext $\mathcal{C}(z)$. The complementary space to ext $\mathcal{H}$ in ext $\mathcal{C}(z)$ is isometrically equal to the complementary space to $\mathcal{H}$ in $\mathcal{C}(z)$. When there is no nonzero element in the greatest subspace of $\mathcal{H}$ satisfying the identity for difference quotients, the orthogonal complement of $\mathcal{H}$ in ext $\mathcal{H}$ is the set of elements of ext $\mathcal{H}$ on which the inclusion in ext $\mathcal{C}(z)$ is isometric.

A convex decomposition applies to a Hilbert space of power series with vector coefficients which satisfies the inequality for difference quotients when the identity for difference quotients is not satisfied.

Theorem 4. *An extreme point of the convex set of Hilbert spaces of power series with vector coefficients which satisfy the inequality for difference quotients is a space which satisfies the identity for difference quotients.*

Proof of Theorem 4. A convex decomposition is made of a Hilbert space $\mathcal{H}$ of power series with vector coefficients which satisfies the inequality for difference quotients. The decomposition is nontrivial when the space does not satisfy the identity for difference quotients. It can be assumed without loss of generality that the greatest subspace of $\mathcal{H}$ which satisfies the identity for difference quotients contains no nonzero element.

Division by z is a contractive transformation of ext $\mathcal{H}$ into itself which acts as an isometric transformation of ext $\mathcal{H}$ onto a Hilbert space which is contained contractively in ext $\mathcal{H}$ and whose complementary space in ext $\mathcal{H}$ is a Hilbert space $\mathcal{B}$ which is contained contractively in ext $\mathcal{H}$. The intersection space of $\mathcal{B}$ and its complementary space in ext $\mathcal{H}$ is a Hilbert space Δ which is contained contractively in $\mathcal{B}$ and in the complementary space to $\mathcal{B}$ in ext $\mathcal{H}$. The complementary space to Δ in $\mathcal{B}$ is a Hilbert space $\mathcal{B}_+$ which is contained contractively in $\mathcal{B}$. A Hilbert space $\mathcal{B}_-$, which is contained contractively in ext $\mathcal{C}(z)$ is defined by its complementary space in ext $\mathcal{C}(z)$, which is equal to the complementary space to Δ in the space which is complementary to $\mathcal{B}$ in ext $\mathcal{C}(z)$. The space $\mathcal{B}$ is contained contractively in the space $\mathcal{B}_-$. The space Δ is isometrically equal to the complementary space to $\mathcal{B}$ in $\mathcal{B}_-$. The convex decomposition

$$\mathcal{B} = (1-t)\mathcal{B}_- + t\mathcal{B}_+$$

applies with t and $1-t$ equal.

Multiplication by z^n acts as an isometric transformation of $\mathcal{B}$ onto a Hilbert space $\mathcal{B}^n$ which is contained contractively in ext $\mathcal{H}$ for every nonpositive integer n. If elements c_n of $\mathcal{B}^n$ are chosen for nonpositive integers n with convergent sum

$$\sum \|c_n\|^2_{\mathcal{B}^n},$$

then the sum

$$c = \sum c_n$$

converges in the metric topology of ext $\mathcal{H}$ and represents an element c which satisfies the inequality

$$\|c\|^2_{\text{ext } \mathcal{H}} \leq \sum \|c_n\|^2_{\mathcal{B}^n}.$$

Every element c of ext $\mathcal{H}$ admits a representation for which equality holds.

Multiplication by z^n acts as an isometric transformation of $\mathcal{B}_+$ onto a Hilbert space $\mathcal{B}^n_+$ which is contained contractively in ext $\mathcal{H}$ for every nonpositive integer n. If elements a_n of $\mathcal{B}^n_+$ are chosen for nonpositive integers n with convergent sum

$$\sum \|a_n\|^2_{\mathcal{B}^n_+}$$

then the sum

$$a = \sum a_n$$

converges in the metric topology of ext $\mathcal{H}$ and represents an element a which satisfies the inequality

$$\|a\|^2_{\text{ext } \mathcal{H}} \leq \sum \|a_n\|^2_{\mathcal{B}^n_+}.$$

The represented elements of ext $\mathcal{H}$ form a Hilbert space ext $\mathcal{H}_-$, which is contained contractively in ext $\mathcal{H}$, such that the inequality

$$\|a\|^2_{\text{ext } \mathcal{H}_+} \leq \sum \|a_n\|^2_{\mathcal{B}^n_+}$$

is always satisfied and such that equality holds in some representation. Division by z is a contractive transformation of ext $\mathcal{H}_+$ into itself.

Multiplication by z^n acts as an isometric transformation of $\mathcal{B}_-$ onto a Hilbert space $\mathcal{B}^n_-$ which is contained contractively in ext $\mathcal{C}(z)$ for every nonpositive integer n. If element b_n of $\mathcal{B}^n_-$ are chosen for nonpositive integers n with convergent sum

$$\sum \|b_n\|^2_{\mathcal{B}^n_-},$$

then the sum

$$b = \sum b_n$$

converges in the metric topology of ext $\mathcal{C}(z)$ and represents an element b which satisfies the inequality

$$\|b\|^2_{\text{ext } \mathcal{C}(z)} \leq \sum \|b_n\|^2_{\mathcal{B}^n_-}.$$

The represented elements of ext $\mathcal{C}(z)$ form a Hilbert space ext $\mathcal{H}_-$, which is contained contractively in ext $\mathcal{C}(z)$, such that the inequality

$$\|b\|^2_{\text{ext } \mathcal{H}_-} \leq \sum \|b_n\|^2_{\mathcal{B}^n_-}$$

is always satisfied and such that equality holds in some representation. Division by z is a contractive transformation of ext $\mathcal{H}_-$ into itself.

The convex combination

$$\text{ext } \mathcal{H} = (1-t) \text{ ext } \mathcal{H}_+ + t \text{ ext } \mathcal{H}_-$$

applies with t and $1-t$ equal. The elements of ext $\mathcal{H}$ on which the inclusion of ext $\mathcal{H}$ in ext $\mathcal{C}(z)$ is isometric are elements of ext $\mathcal{H}_+$ on which the inclusion of ext $\mathcal{H}_+$ in ext $\mathcal{C}(z)$ is isometric and are elements of ext $\mathcal{H}_-$ on which the inclusion of ext $\mathcal{H}_-$ in ext $\mathcal{C}(z)$ is isometric. The orthogonal complement of these elements in ext $\mathcal{H}_+$ is a Hilbert space $\mathcal{H}_+$ of power series with vector coefficients which is contained isometrically in ext $\mathcal{H}_+$ and which satisfies the inequality for difference quotients. The orthogonal complement of these elements in ext $\mathcal{H}_-$ is a Hilbert space $\mathcal{H}_-$ of power series with vector coefficients which is contained isometrically in the space ext $\mathcal{H}_-$ and which satisfies the inequality for difference quotients. The convex combination

$$\mathcal{H} = (1-t)\mathcal{H}_+ + t\mathcal{H}_-$$

applies with t and $1-t$ equal.

When the space $\mathcal{H}$ does not satisfy the identity for difference quotients, the space Δ contains a nonzero element, the spaces $\mathcal{B}_+$ and $\mathcal{B}_-$ are not isometrically equal, the spaces ext $\mathcal{H}_+$ and ext $\mathcal{H}_-$ are not isometrically equal, and the spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ are not isometrically equal.

This completes the proof of the theorem.

If $W(z)$ is a power series with operator coefficients such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, then multiplication by $W(z)$ has a unique extension as a contractive transformation of ext $\mathcal{C}(z)$ into itself which takes $f(z)$ into $g(z)$ whenever it takes $zf(z)$ into $zg(z)$. Multiplication by $W(z)$ is a contractive transformation of ext $\mathcal{C}(z)$ into itself which commutes with multiplication by z. The conjugate power series

$$W^*(z) = W_0^- + W_1^- z + W_2^- z^2 + \ldots$$

is defined with operator coefficients which are adjoints of the operator coefficients of the power series

$$W(z) = W_0 + W_1 z + W_2 z^2 + \ldots$$

Multiplication by $W^*(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself which extends to a contractive transformation of the ext $\mathcal{C}(z)$ into itself. The adjoint of multiplication by $W(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself takes $f(z)$ into $g(z)$ when multiplication by $W^*(z)$ takes $z^{-1}f(z^{-1})$ into $z^{-1}g(z^{-1})$. If the adjoint of multiplication by $W(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself takes $f(z)$ into $g(z)$ and if the coefficient of z^n in $f(z)$ vanishes for every nonnegative integer n, then the coefficient of z^n in $g(z)$ vanishes for every nonnegative integer n. The adjoint of multiplication by $W(z)$ as a transformation of $\mathcal{C}(z)$ into itself takes an element $f(z)$ of $\mathcal{C}(z)$ into the power series $g(z)$ which has the same coefficient of z^n for every nonnegative integer n as the Laurent series obtained from $f(z)$ under the adjoint of multiplication by $W(z)$ as a transformation of ext $\mathcal{C}(z)$ into itself.

Invariant subspaces for contractive transformations of a Hilbert space into itself are constructed by factorization of transfer functions in canonical models. If $A(z)$ and $B(z)$ are power series with operator coefficients such that multiplication by $A(z)$ and multiplication by $B(z)$ are contractive transformations of $\mathcal{C}(z)$ into itself and if

$$B(z) = A(z)C(z)$$

for a power series $C(z)$ with operator coefficients such that multiplication by $C(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, then the space $\mathcal{H}(A)$ is contained contractively in the space $\mathcal{H}(B)$. Multiplication by $A(z)$ is a contractive transformation of the space $\mathcal{H}(C)$ onto the complementary space to the space $\mathcal{H}(A)$ in the space $\mathcal{H}(B)$.

A converse result [4] applies when the identity for difference quotients is satisfied.

Theorem 5. *If a space $\mathcal{H}(A)$ is contained contractively in a space $\mathcal{H}(B)$, then the equation*

$$B(z) = A(z)C(z)$$

has as solution a power series $C(z)$ with operator coefficients such that multiplication by $C(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself which annihilates elements of the coefficient space annihilated on multiplication by $B(z)$ and whose adjoint annihilates elements of the coefficient space annihilated on multiplication by $A(z)$.

Proof of Theorem 5. If r is a nonnegative integer, multiplication by the power series

$$A_r(z) = z^r A(z)$$

and by the power series

$$B_r(z) = z^r B(z)$$

is a contractive transformation of $\mathcal{C}(z)$ into itself. The set of polynomial elements of $\mathcal{C}(z)$ of degree less than r is a Hilbert space which is contained isometrically in $\mathcal{C}(z)$ and which is contained isometrically in the spaces $\mathcal{H}(A_r)$ and $\mathcal{H}(B_r)$. Multiplication by z^r acts as an isometric transformation of the space $\mathcal{H}(A)$ onto the orthogonal complement in the space $\mathcal{H}(A_r)$ of the polynomial elements of degree less than r and of the space $\mathcal{H}(B)$ onto the orthogonal complement in the space $\mathcal{H}(B_r)$ of the polynomial elements of degree less than r.

An isometric transformation of the space $\mathcal{H}(A_r)$ into the space ext $\mathcal{H}(A)$ is defined by taking $f(z)$ into $z^{-r} f(z)$ for every nonnegative integer r. The union of the images of the spaces $\mathcal{H}(A_r)$ is dense in the space ext $\mathcal{H}(A)$.

An isometric transformation of the space $\mathcal{H}(B_r)$ into the space ext $\mathcal{H}(B)$ is defined by taking $g(z)$ into $z^{-r} g(z)$ for every nonnegative integer r. The union of the images of the spaces $\mathcal{H}(B_r)$ is dense in the space ext $\mathcal{H}(B)$.

The space ext $\mathcal{H}(A)$ is contained contractively in the space ext $\mathcal{H}(B)$ since the space $\mathcal{H}(A_r)$ is contained contractively in the space $\mathcal{H}(B_r)$ for every nonnegative integer r.

The space ext $\mathcal{H}(A)$ is contained contractively in ext $\mathcal{C}(z)$. Multiplication by $A(z)$ acts as a partially isometric transformation of ext $\mathcal{C}(z)$ onto the complementary space to the space ext $\mathcal{H}(A)$ in ext $\mathcal{C}(z)$. An element $h(z)$ of ext $\mathcal{C}(z)$ admits a minimal decomposition

$$h(z) = h(z) - A(z)f(z) + A(z)f(z)$$

with

$$h(z) - A(z)f(z)$$

in ext $\mathcal{H}(A)$ and $f(z)$ in ext $\mathcal{C}(z)$. Equality holds in the inequality

$$\|h(z)\|^2 \leq \|h(z) - A(z)f(z)\|^2_{\text{ext } \mathcal{H}(A)} + \|f(z)\|^2.$$

The element $f(z)$ of ext $\mathcal{C}(z)$ is obtained from $h(z)$ on multiplication by $A^*(z^{-1})$.

The space ext $\mathcal{H}(B)$ is contained contractively in ext $\mathcal{C}(z)$. Multiplication by $B(z)$ acts as a partially isometric transformation of ext $\mathcal{C}(z)$ onto the complementary space to the space ext $\mathcal{H}(B)$ in ext $\mathcal{C}(z)$. An element $h(z)$ of ext $\mathcal{C}(z)$ admits a minimal decomposition

$$h(z) = h(z) - B(z)g(z) + B(z)g(z)$$

with

$$h(z) - B(z)g(z)$$

in ext $\mathcal{H}(B)$ and $g(z)$ in ext $\mathcal{C}(z)$. Equality holds in the inequality

$$\|h(z)\|^2 \leq \|h(z) - B(z)g(z)\|^2_{\text{ext } \mathcal{H}(B)} + \|g(z)\|^2.$$

The element $g(z)$ of ext $\mathcal{C}(z)$ is obtained from $h(z)$ under multiplication by $B^*(z^{-1})$.

Since the element

$$h(z) - A(z)f(z)$$

of the space ext $\mathcal{H}(A)$ is obtained from $h(z)$ under the adjoint of the inclusion of the space in ext $\mathcal{C}(z)$, since the element

$$h(z) - B(z)g(z)$$

of the space ext $\mathcal{H}(B)$ is obtained from $h(z)$ under the adjoint of the inclusion of the space in ext $\mathcal{C}(z)$, and since the space ext $\mathcal{H}(A)$ is contained contractively in the space ext $\mathcal{H}(B)$, the element

$$h(z) - A(z)f(z)$$

of the space ext $\mathcal{H}(A)$ is obtained from the element

$$h(z) - B(z)g(z)$$

of the space ext $\mathcal{H}(B)$ under the adjoint of the inclusion of the space ext $\mathcal{H}(A)$ in the space ext $\mathcal{H}(B)$. The inequality

$$\|h(z) - A(z)f(z)\|_{\text{ext } \mathcal{H}(A)} \leq \|h(z) - B(z)g(z)\|_{\text{ext } \mathcal{H}(B)}$$

holds since the adjoint of a contractive transformation is contractive. This completes the proof of the inequality

$$\|g(z)\| \leq \|f(z)\|.$$

The contractive transformation taking $f(z)$ into $g(z)$ admits a continuous extension as a contractive transformation of the orthogonal complement in ext $\mathcal{C}(z)$ of the kernel of multiplication by $A(z)$ into the orthogonal complement in ext $\mathcal{C}(z)$ of the kernel of multiplication by $B(z)$. The domain of the transformation contains $z^{-1}f(z)$ whenever it contains $f(z)$. The transformation takes $z^{-1}f(z)$ into $z^{-1}g(z)$ whenever it takes $f(z)$ into $g(z)$.

The transformation has a contractive extension which takes $zf(z)$ into $zg(z)$ whenever it takes $f(z)$ into $g(z)$. The extension is unique as a transformation of the orthogonal complement in ext $\mathcal{C}(z)$ of the elements whose coefficients are in the kernel of multiplication by $A(z)$ into the orthogonal complement in ext $\mathcal{C}(z)$ of the elements whose coefficients are in the kernel of multiplication by $B(z)$.

A power series $C(z)$ with operator coefficients exists such that multiplication by $\mathcal{C}(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself and such that multiplication by $C^*(z^{-1})$

extends the contractive transformation of the orthogonal complement in ext $\mathcal{C}(z)$ of the elements whose coefficients lie in the kernel of multiplication by $A(z)$ into the orthogonal complement in ext $\mathcal{C}(z)$ of the elements whose coefficients lie in the kernel of multiplication by $B(z)$. The power series is unique when the operator coefficients of $C(z)$ annihilate elements of the coefficient space in the kernel of multiplication by $B(z)$ and the adjoints of operator coefficients of $C(z)$ annihilate elements of the coefficient space in the kernel of multiplication by $A(z)$.

The factorization

$$B(z) = A(z)C(z)$$

follows from the identity

$$B^*(z^{-1}) = C^*(z^{-1})A^*(z^{-1}).$$

This completes the proof of the theorem.

A contractive inclusion of a space $\mathcal{H}(A)$ in a space $\mathcal{H}(B)$ is an example of a transformation which takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$. When

$$A(z) = D(z)B(z)$$

with $D(z)$ a power series with operator coefficients such that multiplication by $D(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, the adjoint of multiplication by $D(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a contractive transformation of the space $\mathcal{H}(A)$ into the space $\mathcal{H}(B)$ which takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$.

An application of commutant lifting [4] is made to the structure of contractive transformations of a space $\mathcal{H}(A)$ into a space $\mathcal{H}(B)$ which take $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever they take $f(z)$ into $g(z)$.

Theorem 6. *If a contractive transformation T of a space $\mathcal{H}(A)$, which satisfies the identity for difference quotients, into a space $\mathcal{H}(B)$ takes $[f(z)-f(0)]z$ into $[g(z)-g(0)]/z$ whenever it takes $f(z)$ into $g(z)$, then a contractive transformation T' of the space $\mathcal{H}(A')$,*

$$A'(z) = zA(z),$$

into the space $\mathcal{H}(B')$,

$$B'(z) = zB(z),$$

exists which extends T and which takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$.

Proof of Theorem 6. The construction of T' is an application of complementation. The coefficient space $\mathcal{C}$ is contained isometrically in the spaces $\mathcal{H}(A')$ and $\mathcal{H}(B')$. Multiplication by z is an isometric transformation of the space $\mathcal{H}(A)$ into the orthogonal complement of $\mathcal{C}$ in the space $\mathcal{H}(A')$ and of the space $\mathcal{H}(B)$ onto the orthogonal complement of $\mathcal{C}$ in the space $\mathcal{H}(B')$. Since T' is defined to agree with T on the space $\mathcal{H}(A)$, it remains to define T' on the complementary space to the space $\mathcal{H}(A)$ in the space $\mathcal{H}(A')$, which is an orthogonal complement since the space $\mathcal{H}(A)$ satisfies the identity for difference quotients by hypothesis.

A contractive transformation of the complementary space to the space $\mathcal{H}(A)$ in the space $\mathcal{H}(A')$ into the orthogonal complement of $\mathcal{C}$ in the space $\mathcal{H}(B')$ is defined by taking $f(z)$ into $g(z)$ when T takes $[f(z)-f(0)]/z$ into $[g(z)-g(0)]/z$ with $g(0)$ equal to zero. The transformation acts as a partially isometric transformation of the complementary space to the space $\mathcal{H}(A)$ in the space $\mathcal{H}(A')$ onto a Hilbert space $\mathcal{P}$ which is contained contractively in the orthogonal complement of $\mathcal{C}$ in the space $\mathcal{H}(B')$.

A contractive transformation of the space $\mathcal{H}(A)$ into the orthogonal complement of $\mathcal{C}$ in the space $\mathcal{H}(B')$ is defined by taking $f(z)$ into $g(z)$ when T takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ with $g(0)$ equal to zero. The transformation acts as a partially isometric transformation of the space $\mathcal{H}(A)$ onto a Hilbert space $\mathcal{Q}$ which is contained contractively in the orthogonal complement of $\mathcal{C}$ in the space $\mathcal{H}(B')$.

A Hilbert space $\mathcal{P} \vee \mathcal{Q}$ exists which is contained contractively in the orthogonal complement of $\mathcal{C}$ in the space $\mathcal{H}(B')$ and which contains the spaces $\mathcal{P}$ and $\mathcal{Q}$ contractively as complementary spaces. A contractive transformation of the space $\mathcal{H}(A')$ into the orthogonal complement of $\mathcal{C}$ in the space $\mathcal{H}(B')$ is defined by taking $f(z)$ into $g(z)$ when T takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ with $g(0)$ equal to zero. The transformation acts as a partially isometric transformation of the space $\mathcal{H}(A')$ onto the space $\mathcal{P} \vee \mathcal{Q}$.

A Hilbert space $\mathcal{M}$ is defined as the set of elements $f(z)$ of the space $\mathcal{H}(B')$ such that $f(z) - f(0)$ belongs to $\mathcal{P} \wedge \mathcal{Q}$ with scalar product determined by the identity

$$\|f(z)\|^2_{\mathcal{M}} = \|f(z) - f(0)\|^2_{\mathcal{P} \vee \mathcal{Q}} + |f(0)|^2.$$

The space $\mathcal{M}$ is contained contractively in the space $\mathcal{H}(B')$.

Since T is a contractive transformation of the space $\mathcal{H}(A)$ into the space $\mathcal{H}(B)$, T acts as a partially isometric transformation of the space $\mathcal{H}(A)$ onto a Hilbert space $\mathcal{Q}'$ which is contained contractively in the space $\mathcal{H}(B)$. Since the space $\mathcal{Q}'$ is contained contractively in the space $\mathcal{M}$, the complementary space to $\mathcal{Q}'$ in $\mathcal{M}$ is a Hilbert space $\mathcal{P}'$ which is contained contractively in $\mathcal{M}$.

A partially isometric transformation of the space $\mathcal{M}$ onto the space $\mathcal{P} \vee \mathcal{Q}$ is defined by taking $f(z) - f(0)$. Since a partially isometric transformation of $\mathcal{Q}'$ onto $\mathcal{Q}$ is defined by taking $f(z)$ into $f(z) - f(0)$, a partially isometric transformation of $\mathcal{P}'$ onto $\mathcal{P}$ is defined by taking $f(z)$ into $f(z) - f(0)$.

The transformation T' is defined to take an element $f(z)$ of the complementary space to the space $\mathcal{H}(A)$ in the space $\mathcal{H}(A')$ into the element $g(z)$ of the space $\mathcal{P}'$ of least norm such that T takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$. Since the complementary space to the space $\mathcal{H}(A)$ is an orthogonal complement, the transformation T' admits a unique linear extension to the space $\mathcal{H}(A')$. The transformation T' is contractive as a transformation of the space $\mathcal{H}(A')$ into the space $\mathcal{H}(B')$. The transformation T' extends T and takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$.

This completes the proof of the theorem.

A contractive transformation of a space $\mathcal{H}(A)$ into a space $\mathcal{H}(B)$ taking $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$ is the composition of an inclusion and the adjoint of multiplication by a power series with operator coefficients when the space $\mathcal{H}(A)$ satisfies the identity for difference quotients.

Theorem 7. *If a contractive transformation of a space $\mathcal{H}(A)$ which satisfies the identity for difference quotients into a space $\mathcal{H}(B)$ takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$, then power series $C(z)$ and $D(z)$ with operator coefficients exist which define contractive multiplications to $\mathcal{C}(z)$ into itself such that the given transformation coincides on the space $\mathcal{H}(A)$ with the adjoint of multiplication by $D(z)$ and such that the identity*

$$A(z)C(z) = D(z)B(z)$$

is satisfied.

Proof of Theorem 7. Since the space $\mathcal{H}(A)$ satisfies the identity for difference quotients by hypothesis, the space $\mathcal{H}(A_r)$ satisfies the identity for difference quotients for every nonnegative integer r. A contractive transformation T_r of the space $\mathcal{H}(A_r)$ into the space $\mathcal{H}(B_r)$ which takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$ is constructed inductively for every nonnegative integer r. The transformation T_0 is the given transformation of the space $\mathcal{H}(A)$ into the space $\mathcal{H}(B)$. The transformation T_{r+1} is constructed from the transformation T_r by commutant lifting so as to agree with T_r on the space $\mathcal{H}(A_r)$.

A transformation of the union of the spaces $\mathcal{H}(A_r)$ into the union of the spaces $\mathcal{H}(B_r)$ is defined which agrees with T_r on the space $\mathcal{H}(A_r)$ for every nonnegative integer r. The transformation takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$. Since the union of the spaces $\mathcal{H}(A_r)$ is dense in $\mathcal{C}(z)$, the transformation admits a unique continuous extension as a contractive transformation of $\mathcal{C}(z)$ into itself. Since the extension takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$, it is the adjoint of multiplication by $D(z)$ for a power series $D(z)$ with operator coefficients such that multiplication by $D(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself.

The given transformation of the space $\mathcal{H}(A)$ into the space $\mathcal{H}(B)$ coincides on the space $\mathcal{H}(A)$ with the adjoint of multiplication by $D(z)$. A space $\mathcal{H}(D)$ exists. Since the space $\mathcal{H}(D)$ is contained contractively in the space $\mathcal{H}(DB)$, multiplication by $D(z)$ acts as a contractive transformation of the space $\mathcal{H}(B)$ onto the complementary space to the space $\mathcal{H}(D)$ in the space $\mathcal{H}(DB)$. An element $h(z)$ of the space $\mathcal{H}(DB)$ admits a minimal decomposition

$$h(z) = h(z) - D(z)g(z) + D(z)g(z)$$

with

$$h(z) - D(z)g(z)$$

in the space $\mathcal{H}(D)$ and $g(z)$ in the space $\mathcal{H}(B)$. Equality holds in the inequality

$$\|h(z)\|^2_{\mathcal{H}(DB)} \leq \|h(z) - D(z)g(z)\|^2_{\mathcal{H}(DB)} + \|g(z)\|^2.$$

The element $g(z)$ of the space $\mathcal{H}(B)$ is obtained from $h(z)$ under the adjoint of multiplication by $g(z)$.

Since the adjoint of multiplication by $D(z)$ takes an element $h(z)$ of the space $\mathcal{H}(A)$ into an element $f(z)$ of the space $\mathcal{H}(B)$, the space $\mathcal{H}(A)$ is contained in the space $\mathcal{H}(DB)$. Equality holds in the inequality

$$\|h(z)\|^2_{\mathcal{H}(DB)} \leq \|h(z) - D(z)f(z)\|^2_{\mathcal{H}(D)} + \|f(z)\|^2_{\mathcal{H}(B)}.$$

Since the inequality

$$\|f(z)\|_{\mathcal{H}(B)} \leq \|h(z)\|_{\mathcal{H}(A)}$$

is satisfied, the space $\mathcal{H}(A)$ is contained contractively in the space $\mathcal{H}(DB)$. The identity

$$A(z)C(z) = D(z)B(z)$$

holds with $C(z)$ a power series with operator coefficients such that multiplication by $\mathcal{C}(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself.

This completes the proof of the theorem.

The relationship between factorization and invariant subspaces is applied in the construction of invariant subspaces. Nontrivial factorizations exist when the existence of invariant subspaces is not given by the Hilbert theorem.

Theorem 8. *If the transformation taking $f(z)$ into $[f(z) - f(0)]/z$ of a space $\mathcal{H}(B)$ into itself is not a scalar multiple of an isometric transformation of the space onto itself, then a Hilbert space $\mathcal{H}$ exists, which satisfies the inequality for difference quotients, which is contained contractively in the space $\mathcal{H}(B)$ and which is not a convex combination of the least and the greatest subspace of the space $\mathcal{H}(B)$, such that every contractive transformation of the space $\mathcal{H}(B)$ into itself, which takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$, acts as a contractive transformation of the space $\mathcal{H}$ into itself.*

Proof of Theorem 8. The space $\mathcal{H}$ is the range of the transformation taking $f(z)$ into $[f(z) - f(0)]/z$ of the space $\mathcal{H}(B)$ into itself. The scalar product of the space $\mathcal{H}$ is determined by the identity

$$\|g(z)\|^2_{\mathcal{H}} = \|g(z)\|^2_{\mathcal{H}(B)} + \inf \|f(z)\|^2_{\mathcal{H}(B)}$$

with the greatest lower bound taken over the elements $f(z)$ of the space $\mathcal{H}(B)$ such that

$$g(z) = [f(z) - f(0)]/z.$$

The space $\mathcal{H}$ satisfies the identity for difference quotients since the space $\mathcal{H}(B)$ satisfies the identity for difference quotients. A contractive transformation of the space $\mathcal{H}(B)$ into itself which takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$ acts as a contractive transformation of the space $\mathcal{H}$ into itself.

The space $\mathcal{H}$ contains a nonzero element since the transformation taking $f(z)$ into $[f(z)-f(0)]/z$ of the space $\mathcal{H}(B)$ into itself is not a scalar multiple of the identity transformation. The space $\mathcal{H}$ is not a convex combination of the least subspace and the greatest subspace of the space $\mathcal{H}(B)$ when the range of the transformation does not contain every element of the space $\mathcal{H}(B)$ or when the kernel of the transformation contains a nonzero element. When the range of the transformation contains every element of the space $\mathcal{H}(B)$ and the kernel of the transformation contains no nonzero element, the space $\mathcal{H}$ is not a convex combination of the least subspace and the greatest subspace of the space $\mathcal{H}(B)$ since the transformation is not a constant multiple of an isometric transformation of the space $\mathcal{H}(B)$ onto itself.

This completes the proof of the theorem.

An invariant subspace is constructed from a factorization when the identity for difference quotients is satisfied. Spaces which satisfy the identity for difference quotients are the extreme points of a compact convex set of spaces which satisfy the inequality for difference quotients. The compact convex set is the closed convex span of its extreme points by the Krein–Milman theorem.

Theorem 9. *If a space $\mathcal{H}(B)$ satisfies the identity for difference quotients and if the transformation of the space into itself which takes $f(z)$ into $[f(z)-f(0)]/z$ is not a scalar multiple of an isometric transformation, then a space $\mathcal{H}(A)$, which satisfies the identity for difference quotients, which is contained contractively in the space $\mathcal{H}(B)$, and which is neither the least nor the greatest subspace, exists such that every contractive transformation of the space $\mathcal{H}(B)$ into itself, which takes $[f(z)-f(0)]/z$ into $[g(z)-g(0)]/z$ whenever it takes $f(z)$ into $g(z)$, acts as a contractive transformation of the space $\mathcal{H}(A)$ into itself.*

Proof of Theorem 9. The convex set consists of the Hilbert spaces, which are contained contractively in the space $\mathcal{H}(B)$ and which satisfy the inequality for difference quotients, such that every contractive transformation of the space $\mathcal{H}(B)$ into itself, which takes $[f(z)-f(0)]/z$ into $[g(z)-g(0)]/z$ whenever it takes $f(z)$ into $g(z)$, acts as a contractive transformation of the subspace into itself.

The convex set of all Hilbert spaces which are contained contractively in $\mathcal{C}(z)$ is compact. Compactness of the convex set of spaces which satisfy the inequality for difference quotients is verified by showing that the set is closed since such spaces are contained contractively in $\mathcal{C}(z)$. If J is the inclusion of the space in $\mathcal{C}(z)$ and if J^* is the adjoint transformation of $\mathcal{C}(z)$ into the space, then the nonnegative self–adjoint transformation JJ^* is contractive when the space is contained contractively in $\mathcal{C}(z)$. If S is multiplication by z as a transformation of $\mathcal{C}(z)$ into itself and if S^* is the adjoint transformation taking $f(z)$ into $[f(z)-f(0)]/z$, then the inequality

$$S^*(1-JJ^*)S \leq 1-JJ^*$$

is equivalent to the inequality for difference quotients since it declares the adjoint of multiplication by z a contractive transformation of the complementary space into itself. The identity implies that the convex set is compact.

A compact convex set is the set of all Hilbert spaces which are contained contractively in the space $\mathcal{H}(B)$ and which satisfy the inequality for difference quotients such that every contractive transformation of the space $\mathcal{H}(B)$ into itself which takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$ acts as a contractive transformation of the Hilbert space into itself. If T is the transformation and if J is the inclusion of the Hilbert space in the space $\mathcal{H}(B)$, the constraint place on the Hilbert space is equivalent to the inequality

$$|\langle TJJ^*f, JJ^*g\rangle_{\mathcal{H}(B)}|^2 \leq \langle JJ^*f, f\rangle_{\mathcal{H}(B)}\langle JJ^*g, g\rangle_{\mathcal{H}(B)}$$

for all elements $f(z)$ and $g(z)$ of the space $\mathcal{H}(B)$. The convex set is compact since it is closed.

A Hilbert space

$$\mathcal{H} = (1-t)\mathcal{H}_+ + t\mathcal{H}_-$$

of power series with vector coefficients which satisfies the inequality for difference quotients is a convex combination of Hilbert spaces, with t and $1-t$ equal, of Hilbert spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ of power series with vector coefficients which satisfy the inequality for difference quotients. The spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ are not isometrically equal when the space $\mathcal{H}$ does not satisfy the identity for difference quotients.

The space $\mathcal{H}$ is dense in the space $\mathcal{H}_-$, which is contained contractively in the augmented space $\mathcal{H}'$. The elements of $\mathcal{H}'$ are the elements $f(z)$ of $\mathcal{C}(z)$ such that $[f(z) - f(0)]/z$ belongs to $\mathcal{H}$. The scalar product in the augmented space is determined by the identity for difference quotients

$$\|[f(z) - f(0)]/z\|^2_{\mathcal{H}} = \|f(z)\|^2_{\mathcal{H}'} - |f(0)|^2.$$

The spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ are contained contractively in the space $\mathcal{H}(B)$ when the space $\mathcal{H}$ is contained contractively in the space $\mathcal{H}(B)$ since the space $\mathcal{H}(B)$ satisfies the identity for difference quotients by hypothesis.

The extension space

$$\text{ext } \mathcal{H} = (1-t)\text{ext } \mathcal{H}_+ + t \text{ ext } \mathcal{H}_-$$

is a convex combination, with t and $1-t$ equal, of the extension space ext $\mathcal{H}_+$ and ext $\mathcal{H}_-$. Division by z acts as a contractive transformation of the space ext $\mathcal{H}$ into itself, of the space ext $\mathcal{H}_+$ into itself, and of the space ext $\mathcal{H}_-$ into itself. Each transformation acts as an isometric transformation of the domain Hilbert space onto a Hilbert space which is contained contractively in the domain Hilbert space.

The convex decomposition

$$\mathcal{B} = (1-t)\mathcal{B}_+ + t\mathcal{B}_-$$

applies with $\mathcal{B}$ the complementary space to the range of division by z in the space ext $\mathcal{H}$, $\mathcal{B}_+$ the complementary space to the range of division by z in the space $\mathcal{H}_+$, and $\mathcal{B}_-$ the complementary space to the range of division by z in the space ext $\mathcal{H}_-$.

Division by z acts as an isometric transformation of the space ext $\mathcal{H}$ onto the complementary space to $\mathcal{B}$ in ext $\mathcal{H}$. The intersection space of $\mathcal{B}$ and its complementary space in ext $\mathcal{H}$ is a Hilbert space Δ which is contained contractively in $\mathcal{B}$ and in its complementary space in ext $\mathcal{H}$. The space $\mathcal{B}_+$ is the complementary space to Δ in $\mathcal{B}$. The space Δ is contained contractively in the space $\mathcal{B}_-$. The space $\mathcal{B}$ is the complementary space to Δ in $\mathcal{B}_-$.

A contractive transformation of the space $\mathcal{H}(B)$ into itself which takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ to $g(z)$ is the restriction of a contractive transformation of $\mathcal{C}(z)$ into itself which takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$ since the space $\mathcal{H}(B)$ satisfies the identity for difference quotients by hypothesis. A power series $W(z)$ with operator coefficients exists such that multiplication by $W(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself whose adjoint extends the transformation of the space $\mathcal{H}(B)$ into itself. Multiplication by $W^*(z^{-1})$ acts as a contractive transformation of the space ext $\mathcal{H}(B)$ into itself. If the adjoint of multiplication by $W(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a contractive transformation of the space $\mathcal{H}$ into itself, then multiplication by $W^*(z^{-1})$ acts as a contractive transformation of the space ext $\mathcal{H}$ into itself.

The extension space of the Hilbert space $\mathcal{B}(z)$ of square summable power series with coefficients in $\mathcal{B}$ is the Hilbert space ext $\mathcal{B}(z)$ of square summable Laurent series with coefficients in $\mathcal{B}$. The set of elements of ext $\mathcal{B}(z)$ whose coefficient of z^n vanishes when n is positive is a Hilbert space $\mathcal{B}(z^{-1})$ which is contained isometrically in ext $\mathcal{B}(z)$. If

$$\sum h_n z^n$$

is an element of $\mathcal{B}(z^{-1})$, the series

$$\sum z^n h_n(z)$$

converges to an element of ext $\mathcal{H}$ which satisfies the inequality

$$\| \sum z^n h_n(z) \|^2_{\text{ext } \mathcal{H}} \leq \sum \| h_n(z) \|^2_{\mathcal{B}}.$$

Every element of ext $\mathcal{H}$ admits a representation as

$$\sum z^n h_n(z)$$

for which equality holds.

The Hilbert space $\mathcal{B}_+(z^{-1})$ of square summable power series with coefficients in $\mathcal{B}_+$ and vanishing coefficient of z^n for positive n is applied to the structure of the space ext $\mathcal{H}_+$. If

$$\sum f_n z^n$$

is an element of $\mathcal{B}_+(z^{-1})$, the series

$$\sum z^n f_n(z)$$

converges to an element of ext $\mathcal{H}_+$ which satisfies the inequality

$$\| \sum z^n f_n(z) \|^2_{\text{ext } \mathcal{H}_+} \leq \sum \| f_n(z) \|^2_{\mathcal{B}_+}.$$

Every element of ext $\mathcal{H}_+$ admits a representation as

$$\sum z^n f_n(z)$$

for which equality holds.

The Hilbert space $\mathcal{B}_-(z^{-1})$ of square summable power series with coefficients in $\mathcal{B}_-$ and vanishing coefficient of z^n for positive n is applied to the structure of the space ext $\mathcal{H}_-$. If

$$\sum g_n z^n$$

is an element of $\mathcal{B}_-(z^{-1})$, the series

$$\sum z^n g_n(z)$$

converges to an element of ext $\mathcal{H}_-$ which satisfies the inequality

$$\| \sum z^n g_n(z) \|^2_{\text{ext } \mathcal{H}_-} \leq \sum \| g_n(z) \|^2_{\mathcal{B}_-}.$$

Every element of ext $\mathcal{H}_-$ admits a representation as

$$\sum z^n g_n(z)$$

for which equality holds.

The space $\mathcal{B}_+(z^{-1})$ is the image of a space contained isometrically in the space $\mathcal{B}(z^{-1})$ which is the orthogonal complement of the elements of $\mathcal{B}(z^{-1})$ which represent the origin of ext $\mathcal{H}$. The orthogonal projection of the space $\mathcal{B}(z^{-1})$ onto the space $\mathcal{B}_+(z^{-1})$ takes $z^{-1} f(z)$ into $z^{-1} g(z)$ whenever it takes $f(z)$ into $g(z)$. The adjoint of the inclusion of the space ext $\mathcal{H}_+$ in the space ext $\mathcal{H}$ takes

$$B^*(z^{-1}) \sum z^n f_n(z)$$

into

$$B^*(z^{-1}) \sum z^n g_n(z)$$

whenever it takes

$$\sum z^n f_n(z)$$

into

$$\sum z^n g_n(z).$$

Multiplication by $B^*(z^{-1})$ is a contractive transformation of the space ext $\mathcal{H}_+$ into itself. Multiplication by $B^*(z^{-1})$ is a contractive transformation of the space ext $\mathcal{H}_-$ into itself.

The adjoint of multiplication by $B(z)$ as a transformation of $\mathcal{C}(z)$ into itself acts as a contractive transformation of the space $\mathcal{H}_+$ into itself and of the space $\mathcal{H}_-$ into itself.

This completes the proof of the theorem.

The computation of extreme points completes the proof of existence of invariant subspaces.

Theorem 10. *A continuous transformation of a Hilbert space into itself which is not a scalar multiple of the identity transformation admits a closed invariant subspace other than the least and the greatest subspace which is an invariant subspace for every continuous transformation commuting with the given transformation.*

Proof of Theorem 10. The invariant subspace is essentially due to Hilbert when the transformation is the scalar multiple of an isometric transformation. The invariant subspace can be found by applying the present constructions in the context of Herglotz spaces. When the transformation is not a scalar multiple of an isometric transformation, the transformation can be assumed contractive with iterates which converge to the origin in their action on every element of the Hilbert space. The transformation can be assumed to take $f(z)$ into $[f(z) - f(0)]/z$ in a space $\mathcal{H}(B)$ which is contained isometrically in $\mathcal{C}(z)$ for some coefficient Hilbert space $\mathcal{C}$.

A Hilbert space $\mathcal{H}(A)$, which is contained contractively in the space $\mathcal{H}(B)$, which satisfies the identity for difference quotients, which contains a nonzero element, and which is not isometrically equal to the space $\mathcal{H}(B)$, such that every contractive transformation of the space $\mathcal{H}(B)$ into itself, which takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$, acts as a contractive transformation of the space $\mathcal{H}(A)$ into itself.

A power series $C(z)$ with operator coefficients exists such that multiplication by $C(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself, such that the identity

$$B(z) = A(z)C(z)$$

is satisfied, such that multiplication by $C(z)$ annihilates every element of the coefficient space in the kernel of multiplication by $B(z)$, and such that multiplication by $C^*(z)$ annihilates elements of the coefficient space in the kernel of multiplication by $A(z)$.

Since the space $\mathcal{H}(B)$ is contained isometrically in $\mathcal{C}(z)$, multiplication by $B(z)$ is a partially isometric transformation of $\mathcal{C}(z)$ into itself. The kernel of multiplication by $B(z)$ contains $[f(z) - f(0)]/z$ whenever it contains $f(z)$ since the orthogonal complement of the kernel contains $zf(z)$ whenever it contains $f(z)$.

Since the space $\mathcal{H}(A)$ satisfies the identity for difference quotients, multiplication by $A(z)$ is a contractive transformation of $\mathcal{C}(z)$ into itself whose kernel contains $[f(z)-f(0)]/z$ whenever it contains $f(z)$. Since the range of multiplication by $C(z)$ as a transformation of $\mathcal{C}(z)$ into itself is orthogonal to the kernel of multiplication by $A(z)$, multiplication by $C(z)$ is a partially isometric transformation of $\mathcal{C}(z)$ into itself whose kernel is the kernel of multiplication by $B(z)$. Since the space $\mathcal{H}(C)$ is orthogonal to the kernel of multiplication by $A(z)$, multiplication by $A(z)$ acts as an isometric transformation of the space $\mathcal{H}(C)$ into the space $\mathcal{H}(B)$. The space $\mathcal{H}(A)$ is contained isometrically in the space $\mathcal{H}(B)$ as the orthogonal complement of the image of the space $\mathcal{H}(C)$.

This completes the proof of the theorem.

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