THE RIEMANN HYPOTHESIS FOR JACOBIAN ZETA FUNCTIONS

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Abstract. Jacobian zeta functions are analogues of the Euler zeta function which are generated in Fourier analysis, here advantageously applied in skew–fields. Zeta functions of Fourier analysis are produced in scattering by vibrating strings whose structure is treated [1] as an inverse problem in Hilbert spaces whose elements are entire functions. Scattering produces functions which are analytic and without zeros in the upper half–plane. The Riemann hypothesis [2] identifies special strings whose scattering produces functions which are analytic and without zeros in a larger half–plane. A Radon transformation in Fourier analysis on skew–fields generates these special strings. A proof of the analogue of the Riemann hypothesis follows from Jacobian zeta functions. A proof of the Riemann hypothesis for the Euler zeta function is a corollary.

1. The Inverse Problem for the Vibrating String

The Riemann hypothesis is a conjecture concerning the zeros of a special entire function, not a polynomial, which presumes a relationship to zeros found in polynomials. The Hermite class of entire functions is a class of entire functions which contains the polynomials having a given zero–free half–plane and which maintains the relationship to zeros found in these polynomials.

A nontrivial entire function is said to be of Hermite class if it can be approximated by polynomials whose zeros are restricted to a given half–plane. For applications to the vibrating string the upper half–plane is chosen as the half–plane free of zeros. If an entire function $E(z)$ of z is of Hermite class, then the modulus of $E(x+iy)$ is a nondecreasing function of positive y which satisfies the inequality

$$|E(x-iy)| \leq |E(x+iy)|$$

for every real number x. These necessary conditions are also sufficient. The Hermite class is also known as the Pólya class.

An entire function of Hermite class which has no zero is the exponential

$$\exp F(z)$$

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of an entire function $F(z)$ of z with derivative $F'(z)$ such that the real part of

$$iF'(z)$$

is nonnegative in the upper half–plane. The function

$$iF'(z) = a - ibz$$

is a polynomial of degree less than two by the Poisson representation of functions which are nonnegative and harmonic in the upper half–plane. The constant coefficient a has nonnegative real part and b is nonnegative.

If an entire function $E(z)$ of z is of Hermite class and has a zero w, then the entire function

$$E(z)/(z - w)$$

of z is of Hermite class. A sequence of polynomials $P_n(z)$ exists such that

$$E(z)/P_n(z)$$

is an entire function of Hermite class for every nonnegative integer n and such that

$$E(z) = \lim P_n(z)E_n(z)$$

uniformly on compact subsets of the upper half–plane for entire functions $E_n(z)$ of Hermite class which have no zeros.

An analytic weight function is defined as a function $W(z)$ of z which is analytic and without zeros in the upper half–plane. An entire function of Hermite class is an analytic weight function in the upper half–plane. Hilbert spaces of functions analytic in the upper half–plane were introduced in Fourier analysis by Hardy.

The weighted Hardy space $\mathcal{F}(W)$ is defined as the Hilbert space of functions $F(z)$ of z, which are analytic in the upper half–plane, such that the least upper bound

$$\|F\|^2_{\mathcal{F}(W)} = \sup \int_{-\infty}^{+\infty} |F(x + iy)/W(x + iy)|^2 dx$$

taken over all positive y is finite. The classical Hardy space is obtained when $W(z)$ is identically one. Multiplication by $W(z)$ is an isometric transformation of the classical Hardy space onto the weighted Hardy space with analytic weight function $W(z)$.

An isometric transformation of the weighted Hardy space $\mathcal{F}(W)$ into itself is defined by taking a function $F(z)$ of z into the function

$$F(z)(z - w)/(z - w^-)$$

of z when w is in the upper half–plane. The range of the transformation is the set of elements of the space which vanish at w.

A continuous linear functional on the weighted Hardy space $\mathcal{F}(W)$ is defined by taking a function $F(z)$ of z into its value $F(w)$ at w whenever w is in the upper half–plane. The function

$$W(z)W(w)^-/[2\pi i(w^- - z)]$$

of z belongs to the space when w is in the upper half–plane and acts as reproducing kernel function for function values at w.

A Hilbert space of functions analytic in the upper half–plane which has dimension greater than one is isometrically equal to a weighted Hardy space if an isometric transformation of the space onto the subspace of functions which vanish at w is defined by taking $F(z)$ into

$$F(z)(z - w)/(z - w^-)$$

when w is in the upper half–plane and if a continuous linear functional is defined on the space by taking $F(z)$ into $F(w)$ for w of the upper half–plane.

Examples of weighted Hardy spaces in Fourier analysis are constructed from the Euler gamma function. The gamma function is a function $\Gamma(s)$ of s which is analytic in the complex plane with the exception of singularities at the nonpositive integers and which satisfies the recurrence relation

$$s\Gamma(s) = \Gamma(s + 1).$$

An analytic weight function

$$W(z) = \Gamma(s)$$

is defined by

$$s = \tfrac{1}{2} - iz.$$

A maximal dissipative transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z + i)$ whenever the functions of z belong to the space.

A relation T with domain and range in a Hilbert space is said to be dissipative if the transformation

$$(T - \lambda)/(T + \lambda^-)$$

with domain and range in the Hilbert space is contractive for some, and hence every, complex number λ in the right half–plane. The relation T is said to be maximal dissipative if the domain of the contractive transformation is the whole space for some, and hence every, complex number λ in the right half–plane.

Theorem 1. *A maximal dissipative transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space if, and only if, the function*

$$W(z - \tfrac{1}{2}i)/W(z + \tfrac{1}{2}i)$$

of z admits an extension which is analytic and has nonnegative real part in the upper half–plane.

Proof of Theorem 1. A Hilbert space $\mathcal{H}$ whose elements are functions analytic in the upper half–plane is constructed when a maximal dissipative transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space. The space $\mathcal{H}$ is constructed from the graph of the adjoint of the transformation which takes $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space.

An element

$$F(z) = (F_+(z), F_-(z))$$

of the graph is a pair of analytic functions of z, which belong to the space $\mathcal{F}(W)$, such that the adjoint takes $F_+(z)$ into $F_-(z)$. The scalar product

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{F}(W)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{F}(W)}$$

of elements $F(z)$ and $G(z)$ of the graph is defined formally as the sum of scalar products in the space $\mathcal{F}(W)$. Scalar self–products are nonnegative in the graph since the adjoint of a maximal dissipative transformation is dissipative.

An element $K(w, z)$ of the graph is defined by

$$K_+(w, z) = W(z)W(w - \tfrac{1}{2}i)^-/[2\pi i(w^- + \tfrac{1}{2}i - z)]$$

and

$$K_-(w, z) = W(z)W(w + \tfrac{1}{2}i)^-/[2\pi i(w^- - \tfrac{1}{2}i - z)]$$

when w is in the half–plane

$$1 < iw^- - iw.$$

The identity

$$F_+(w + \tfrac{1}{2}i) + F_-(w - \tfrac{1}{2}i) = \langle F(t), K(w, t)\rangle$$

holds for every element

$$F(z) = (F_+(z), F_-(z))$$

of the graph. An element of the graph which is orthogonal to itself is orthogonal to every element of the graph.

An isometric transformation of the graph onto a dense subspace of $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into the function

$$F_+(z + \tfrac{1}{2}i) + F_-(z - \tfrac{1}{2}i)$$

of z in the half–plane

$$1 < iz^- - iz.$$

The reproducing kernel function for function values at w in the space $\mathcal{H}$ is the function

$$[W(z + \tfrac{1}{2}i)W(w - \tfrac{1}{2}i)^- + W(z - \tfrac{1}{2}i)W(w + \tfrac{1}{2}i)^-]/[2\pi i(w^- - z)]$$

of z in the half–plane when w is in the half–plane.

Assume that a maximal dissipative transformation is defined. Division by $W(z + \frac{1}{2}i)$ is an isometric transformation of the space $\mathcal{H}$ onto a Hilbert space appearing in the Poisson representation of functions which are analytic and have nonnegative real part in the upper half–plane [1]. The function

$$\phi(z) = W(z - \tfrac{1}{2}i)/W(z + \tfrac{1}{2}i)$$

of z admits an analytic extension with nonnegative real part to the upper half–plane. The function

$$[\phi(z) + \phi(w)^-]/[2\pi i(w^- - z)]$$

of z belongs to the space when w is in the upper half–plane and acts as reproducing kernel function for function values at w. Since multiplication by $W(z + \frac{1}{2}i)$ is an isometric transformation of the space onto $\mathcal{H}$, the elements of $\mathcal{H}$ have analytic extensions to the upper half–plane. The function

$$[W(z + \tfrac{1}{2}i)W(w - \tfrac{1}{2}i)^- + W(z - \tfrac{1}{2}i)W(w + \tfrac{1}{2}i)^-]/[2\pi i(w^- - z)]$$

of z belongs to the space when w is in the upper half–plane and acts as reproducing kernel function for function values at w.

The argument is reversed to construct a maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ when the function $\phi(z)$ of z admits an extension which is analytic and has nonnegative real part in the upper half–plane. The Poisson representation constructs a Hilbert space whose elements are functions analytic in the upper half–plane and which contains the function

$$[\phi(z) + \phi(w)^-]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is in the upper half–plane. Multiplication by $W(z+\frac{1}{2}i)$ acts as an isometric transformation of the space onto a Hilbert space $\mathcal{H}$ whose elements are functions analytic in the upper half–plane and which contains the function

$$[W(z + \tfrac{1}{2}i)W(w - \tfrac{1}{2}i)^- + W(z - \tfrac{1}{2}i)W(w + \tfrac{1}{2}i)^-]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is in the upper half–plane.

A transformation is defined in the space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z + i)$ whenever the functions of z belong to the space. The graph of the adjoint is a space of pairs

$$F(z) = (F_+(z), F_-(z))$$

of elements of the space such that the adjoint takes the function $F_+(z)$ of z into the function $F_-(z)$ of z. The graph contains

$$K(w, z) = (K_+(w, z), K_-(w, z))$$

with

$$K_+(w,z) = W(z)W(w - \tfrac{1}{2}i)^-/[2\pi i(w^- + \tfrac{1}{2}i - z)]$$

and

$$K_-(w,z) = W(z)W(w + \tfrac{1}{2}i)^-/[2\pi i(w^- - \tfrac{1}{2}i - z)]$$

when w is in the half–plane

$$1 < iw^- - iw.$$

The elements $K(w,z)$ of the graph span the graph of a restriction of the adjoint. The transformation in the space $\mathcal{F}(W)$ is recovered as the adjoint of the restricted adjoint.

A scalar product is defined on the graph of the restricted adjoint so that an isometric transformation of the graph of the restricted adjoint into the space $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into

$$F_+(z + \tfrac{1}{2}i) + F_-(z - \tfrac{1}{2}i).$$

The identity

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{F}(W)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{F}(W)}$$

holds for all elements

$$F(z) = (F_+(z), F_-(z))$$

and

$$G(z) = (G_+(z), G_-(z))$$

of the graph of the restricted adjoint. The restricted adjoint is dissipative since scalar self–products are nonnegative in its graph. The adjoint is dissipative since the transformation in the space $\mathcal{F}(W)$ is the adjoint of its restricted adjoint.

The dissipative property of the adjoint is expressed in the inequality

$$\|F_+(t) - \lambda^- F_-(t)\|_{\mathcal{F}(W)} \leq \|F_+(t) + \lambda F_-(t)\|_{\mathcal{F}(W)}$$

for elements

$$F(z) = (F_+(z), F_-(z))$$

of the graph when λ is in the right half–plane. The domain of the contractive transformation which takes the function

$$F_+(z) + \lambda F_-(z)$$

of z into the function

$$F_+(z) - \lambda^- F_-(z)$$

of z is a closed subspace of the space $\mathcal{F}(W)$. The maximal dissipative property of the adjoint is the requirement that the contractive transformation be everywhere defined for some, and hence every, λ in the right half–plane.

Since $K(w,z)$ belongs to the graph when w is in the half–plane

$$1 < iw^- - iw,$$

an element $H(z)$ of the space $\mathcal{F}(W)$ which is orthogonal to the domain satisfies the identity

$$H(w - \tfrac{1}{2}i) + \lambda H(w + \tfrac{1}{2}i) = 0$$

when w is in the upper half–plane. The function $H(z)$ of z admits an analytic extension to the complex plane which satisfies the identity

$$H(z) + \lambda H(z + i) = 0.$$

A zero of $H(z)$ is repeated with period i. Since

$$H(z)/W(z)$$

is analytic and of bounded type in the upper half–plane, the function $H(z)$ of z vanishes everywhere if it vanishes somewhere.

The space of elements $H(z)$ of the space $\mathcal{F}(W)$ which are solutions of the equation

$$H(z) + \lambda H(z + i) = 0$$

for some λ in the right half–plane has dimension zero or one. The dimension is independent of λ.

If τ is positive, multiplication by

$$\exp(i\tau z)$$

is an isometric transformation of the space $\mathcal{F}(W)$ into itself which takes solutions of the equation for a given λ into solutions of the equation with λ replaced by

$$\lambda \exp(\tau).$$

A solution $H(z)$ of the equation for a given λ vanishes identically since the function

$$\exp(-i\tau z)H(z)$$

of z belongs to the space for every positive number τ and has the same norm as the function $H(z)$ of z.

The transformation which takes $F(z)$ into $F(z + i)$ whenever the functions of z belong to the space $\mathcal{F}(W)$ is maximal dissipative since it is the adjoint of its adjoint, which is maximal dissipative.

This completes the proof of the theorem.

An example of an analytic weight function which satisfies the hypotheses of the theorem is obtained when

$$W(z) = \Gamma(\tfrac{1}{2} - iz)$$

since

$$W(z + \tfrac{1}{2}i)/W(z - \tfrac{1}{2}i) = -iz$$

is analytic and has nonnegative real part in the upper half–plane by the recurrence relation for the gamma function. The weight functions which satisfy the hypotheses of the theorem are generalizations of the gamma function in which an identity is replaced by an inequality.

The theorem has another formulation. A maximal dissipative transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ for some real number h by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space if, and only if, the function

$$W(z + \tfrac{1}{2}ih)/W(z - \tfrac{1}{2}ih)$$

of z admits an extension which is analytic and has nonnegative real part in the upper half–plane.

Another theorem is obtained in the limit of small h. A maximal dissipative transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $iF'(z)$ whenever the functions of z belong to the space if, and only if, the function

$$iW'(z)/W(z)$$

of z has nonnegative real part in the upper half–plane.

The proof of the theorem is similar to the proof of Theorem 1. A relationship to the Hermite class of entire functions is indicated in another formulation. A maximal dissipative transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $iF'(z)$ whenever the functions of z belong to the space if, and only if, the modulus of $W(x + iy)$ is a nondecreasing function of positive y for every real number x.

An Euler weight function is defined as an analytic weight function $W(z)$ such that a maximal dissipative transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ whenever h is in the interval $[-1, 1]$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space.

If a nontrivial function $\phi(z)$ of z is analytic and has nonnegative real part in the upper half–plane, a logarithm of the functions is defined continuously in the half–plane with values in the strip of width π centered on the real line. The inequalities

$$-\pi \le i \log \phi(z)^- - i \log \phi(z) \le \pi$$

are satisfied. A function $\phi_h(z)$ of z which is analytic and has nonnegative real part in the upper half–plane is defined when h is in the interval $(-1, 1)$ by the integral

$$\log \phi_h(z) = \sin(\pi h) \int_{-\infty}^{+\infty} \frac{\log \phi(z - t)dt}{\cos(2\pi it) + \cos(\pi h)}.$$

An application of the Cauchy formula in the upper half–plane shows that the function

$$\frac{\sin(\pi h)}{\cos(2\pi iz)+\cos(\pi h)} = \int_{-\infty}^{+\infty} \exp(2\pi itz) \, \frac{\exp(\pi ht - \exp(-\pi ht)}{\exp(\pi t) - \exp(-\pi t)} \, dt$$

of z is the Fourier transform of a function

$$\frac{\exp(\pi ht) - \exp(-\pi ht)}{\exp(\pi t) - \exp(-\pi t)}$$

of positive t which is square integrable with respect to Lebesgue measure and bounded by h when h is in the interval $(0, 1)$.

The identity

$$\phi_{-h}(z) = \phi_h(z)^{-1}$$

is satisfied. The function

$$\phi(z) = \lim \phi_h(z)$$

of z is recovered in the limit as h increases to one. The identity

$$\phi_{a+b}(z) = \phi_a(z - \tfrac{1}{2}ib)\phi_b(z + \tfrac{1}{2}ia)$$

when a, b, and $a + b$ belong to the interval $(-1, 1)$ is a consequence of the trigonometric identity

$$\frac{\sin(\pi a + \pi b)}{\cos(2\pi iz) + \cos(\pi a + \pi b)}$$
$$= \frac{\sin(\pi a)}{\cos(2\pi iz + \pi b) + \cos(\pi a)} + \frac{\sin(\pi b)}{\cos(2\pi iz - \pi a) + \cos(\pi b)}.$$

An Euler weight function $W(z)$ is defined within a constant factor by the limit

$$iW'(z)/W(z) = \lim \, \frac{\log \phi_h(z)}{h} = \pi \int_{-\infty}^{+\infty} \frac{\log \phi(z-t)dt}{1 + \cos(2\pi it)}.$$

The identity

$$W(z + \tfrac{1}{2}ih) = W(z - \tfrac{1}{2}ih)\phi_h(z)$$

applies when h is in the interval $(-1, 1)$. The identity reads

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z)$$

in the limit as h increases to one.

An Euler weight function $W(z)$ has been constructed which satisfies the identity

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z)$$

for a given nontrivial function $\phi(z)$ of z which is analytic and has nonnegative real part in the upper half–plane.

If a maximal dissipative transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space, then the identity

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z)$$

holds for a function $\phi(z)$ of z which is analytic and has nonnegative real part in the upper half–plane. The analytic weight function $W(z)$ is the product of an Euler weight function and an entire function which is periodic of period i and has no zeros.

If $W(z)$ is an Euler weight function, the maximal dissipative transformation defined for h in the interval $[0, 1]$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space $\mathcal{F}(W)$ is subnormal: The transformation is the restriction to an invariant subspace of a normal transformation in the larger Hilbert space $\mathcal{H}$ of (equivalence classes of) Baire functions $F(x)$ of real x for which the integral

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt$$

converges. The passage to boundary value functions maps the space $\mathcal{F}(W)$ isometrically into the space $\mathcal{H}$.

For given h the function

$$\phi(z) = W(z + \tfrac{1}{2}ih)/W(z - \tfrac{1}{2}ih)$$

of z is analytic and has nonnegative real part in the upper half–plane. The transformation T is defined to take $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space $\mathcal{F}(W)$. The transformation S is defined to take $F(z)$ into

$$\varphi(z + \tfrac{1}{2}ih)F(z)$$

whenever the functions of z belong to the space $\mathcal{F}(W)$. The adjoint

$$T^* = S^{-1}TS^*$$

of T is computable from the adjoint S^* of S on a dense subset of the space $\mathcal{F}(W)$ containing the reproducing kernel functions for function values. The transformation $T^{-1}ST$ takes $F(z)$ into

$$\varphi(z - \tfrac{1}{2}ih)^{-1}F(z)$$

on a dense set of elements $F(z)$ of the space $\mathcal{F}(W)$. The transformation $T^{-1}T^*$ is the restriction of a contractive transformation of the space $\mathcal{F}(W)$ into itself. The transformation takes $F(z)$ into $G(z)$ when the boundary value function $G(x)$ is the orthogonal projection of

$$F(x)\phi(x - \tfrac{1}{2}ih)^-/\phi(x - \tfrac{1}{2}ih)$$

in the image of the space $\mathcal{F}(W)$.

An isometric transformation of the space $\mathcal{H}$ onto itself is defined by taking a function $F(x)$ of real x into the function

$$F(x)\phi(x - \tfrac{1}{2}ih)^- / \varphi(x - \tfrac{1}{2}ih)$$

of real x. A dense set of elements of $\mathcal{H}$ are products

$$\exp(-iax)F(x)$$

for nonnegative numbers a with $F(x)$ the boundary value function of a function $F(z)$ of z which belongs to the space $\mathcal{F}(W)$. A normal transformation is defined in the space $\mathcal{H}$ as the closure of a transformation which is computed on such elements. The transformation takes the function

$$\exp(-iax)F(x)$$

of real x into the function

$$\exp(-iah)\exp(-iax)G(x)$$

of real x for every nonnegative number a when $F(z)$ and

$$G(z) = F(z + ih)$$

are functions of z in the upper half–plane which belong to the space $\mathcal{F}(W)$.

Entire functions of Hermite class are examples of analytic weight functions which are limits of polynomials having no zeros in the upper half–plane. Such polynomials appear in the Stieltjes representation of positive linear functionals on polynomials.

A linear functional on polynomials with complex coefficients is said to be nonnegative if it has nonnegative values on polynomials whose values on the real axis are nonnegative. A positive linear functional on polynomials is a nonnegative linear functional on polynomials which does not vanish identically. A nonnegative linear functional on polynomials is represented as an integral with respect to a nonnegative measure μ on the Baire subsets of the real line. The linear functional takes a polynomial $F(z)$ into the integral

$$\int F(t)d\mu(t).$$

Stieltjes examines the action of a positive linear functional on polynomials of degree less than r for a positive integer r. A polynomial which as nonnegative values on the real axis is a product

$$F(z)F^*(z)$$

of a polynomial $F(z)$ and the conjugate polynomial

$$F^*(z) = F(z^-)^-.$$

If the positive linear functional does not annihilate

$$F(z)F^*(z)$$

for any nontrivial polynomial $F(z)$ of degree less than r, then a Hilbert space exists whose elements are the polynomials of degree less than r and whose scalar product

$$\langle F(t), G(t)\rangle$$

is defined as the action of the positive linear functional on the polynomial

$$G^*(z)F(z).$$

Stieltjes shows that the Hilbert space of polynomials of degree less than r is contained isometrically in a weighted Hardy space $\mathcal{F}(W)$ whose analytic weight function $W(z)$ is a polynomial of degree r having no zeros in the upper half–plane.

Examples of such spaces are applied by Legendre in quadratic approximations of periodic motion and motivate the application to number theory made precise by the Riemann hypothesis.

An axiomatization of the Stieltjes spaces is stated in a general context [2]. Hilbert spaces are examined whose elements are entire functions and which have these properties:

(H1) Whenever an entire function $F(z)$ of z belongs to the space and has a nonreal zero w, the entire function

$$F(z)(z - w^-)/(z - w)$$

of z belongs to the space and has the same norm as $F(z)$.

(H2) A continuous linear functional on the space is defined by taking a function $F(z)$ of z into its value $F(w)$ at w for every nonreal number w.

(H3) The entire function

$$F^*(z) = F(z^-)^-$$

of z belongs to the space whenever the entire function $F(z)$ of z belongs to the space, and it has the same norm as $F(z)$.

An example of a Hilbert space of entire functions which satisfies the axioms is obtained when an entire function $E(z)$ of z satisfies the inequality

$$|E(x - y)| < |E(x + iy)|$$

for all real x when y is positive. A weighted Hardy space $\mathcal{F}(W)$ is defined with analytic weight function

$$W(z) = E(z).$$

A Hilbert space $\mathcal{H}(E)$ which is contained isometrically in the space $\mathcal{F}(W)$ is defined as the set of entire functions $F(z)$ of z such that the entire functions $F(z)$ of z and $F^*(z)$ of z belong to the space $\mathcal{F}(W)$. The entire function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

of z belongs to the space $\mathcal{H}(E)$ for every complex number w and acts as reproducing kernel function for function values at w.

A Hilbert space $\mathcal{H}$ of entire functions which satisfies the axioms (H1), (H2), and (H3) is isometrically equal to a space $\mathcal{H}(E)$ if it contains a nonzero element. The proof applies reproducing kernel functions which exist by the axiom (H2).

For every nonreal number w a unique entire function $K(w,z)$ of z exists which belongs to the space and acts as reproducing kernel function for function values at w. The function does not vanish identically since the axiom (H1) implies that some element of the space has a nonzero value at w when some element of the space does not vanish identically. The scalar self–product $K(w,w)$ of the function $K(w,z)$ of z is positive. The axiom (H3) implies the symmetry

$$K(w^-,z) = K(w,z^-)^-.$$

If λ is a nonreal number, the set of elements of the space which vanish at λ is a Hilbert space of entire functions which is contained isometrically in the given space. The function

$$K(w,z) - K(w,\lambda)K(\lambda,\lambda)^{-1}K(\lambda,z)$$

of z belongs to the subspace and acts as reproducing kernel function for function values at λ. The identity

$$[K(w,z) - K(w,\lambda)K(\lambda,\lambda)^{-1}K(\lambda,z)](z-\lambda^-)(w^- - \lambda)$$
$$= [K(w,z) - K(w,\lambda^-)K(\lambda^-,\lambda^-)^{-1}K(\lambda^-,z)](z-\lambda)(w^- - \lambda^-)$$

is a consequence of the axiom (H1).

An entire function $E(z)$ of z exists such that the identity

$$K(w,z) = [E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

holds for all complex z when w is not real. The entire function can be chosen with a zero at λ when λ is in the lower half–plane and with a zero at λ^- when λ is in the upper half–plane. The function is then unique within a constant factor of absolute value one. A space $\mathcal{H}(E)$ exists and is isometrically equal to the given space $\mathcal{H}$.

Examples of Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) are constructed from the analytic weight function

$$W(z) = a^{iz}\Gamma(\tfrac{1}{2} - iz)$$

for every positive number a. The space is contained isometrically in the weighted Hardy space $\mathcal{F}(W)$ and contains every entire function $F(z)$ such that the functions $F(z)$ and $F^*(z)$ of z belong to the space $\mathcal{F}(W)$. The space of entire functions is isometrically equal to a space $\mathcal{H}(E)$ whose defining function $E(z)$ is a confluent hypergeometric series [1]. Properties of the space define a class of Hilbert spaces of entire functions.

An Euler space of entire functions is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) such that a maximal dissipative transformation is defined in the space for every h in the interval $[-1,1]$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space.

Theorem 2. *A maximal dissipative transformation is defined in a Hilbert space $\mathcal{H}(E)$ of entire functions for a real number h by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space if, and only if, a Hilbert space $\mathcal{H}$ of entire functions exists which contains the function*

$$\begin{aligned}&[E(z+\tfrac{1}{2}ih)E(w-\tfrac{1}{2}ih)^- - E^*(z+\tfrac{1}{2}ih)E(w^- + \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]\\ &+ [E(z-\tfrac{1}{2}ih)E(w+\tfrac{1}{2}ih)^- - E^*(z-\tfrac{1}{2}ih)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]\end{aligned}$$

of z as reproducing kernel function for function values at w for every complex number w.

Proof of Theorem 2. The space $\mathcal{H}$ is constructed from the graph of the adjoint of the transformation which takes $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space. An element

$$F(z) = (F_+(z), F_-(z))$$

of the graph is a pair of entire functions of z, which belong to the space $\mathcal{H}(E)$, such that the adjoint takes $F_+(z)$ into $F_-(z)$. The scalar product

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{H}(E)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{H}(E)}$$

of elements $F(z)$ and $G(z)$ of the graph is defined as a sum of scalar products in the space $\mathcal{H}(E)$. Scalar self–products are nonnegative since the adjoint of a maximal dissipative transformation is dissipative.

An element

$$K(w,z) = (K_+(w,z), K_-(w,z))$$

of the graph is defined for every complex number w by

$$K_+(w,z) = [E(z)E(w-\tfrac{1}{2}ih)^- - E^*(z)E(w^- + \tfrac{1}{2}ih)]/[2\pi i(w^- + \tfrac{1}{2}ih - z)]$$

and

$$K_-(w,z) = [E(z)E(w+\tfrac{1}{2}ih)^- - E^*(z)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - \tfrac{1}{2}ih - z)].$$

The identity

$$F_+(w+\tfrac{1}{2}ih) + F_-(w-\tfrac{1}{2}ih) = \langle F(t), K(w,t)\rangle$$

holds for every element

$$F(z) = (F_+(z), F_-(z))$$

of the graph. An element of the graph which is orthogonal to itself is orthogonal to every element of the graph.

A partially isometric transformation of the graph onto a dense subspace of the space $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into the entire function

$$F_+(z + \tfrac{1}{2}ih) + F_-(z - \tfrac{1}{2}ih)$$

of z. The reproducing kernel function for function values at w in the space $\mathcal{H}$ is the function

$$[E(z + \tfrac{1}{2}ih)E(w - \tfrac{1}{2}ih)^- - E^*(z + \tfrac{1}{2}ih)E(w^- + \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$
$$+ [E(z - \tfrac{1}{2}ih)E(w + \tfrac{1}{2}ih)^- - E^*(z - \tfrac{1}{2}ih)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$

of z for every complex number w.

This completes the construction of a Hilbert space $\mathcal{H}$ of entire functions with the desired reproducing kernel functions when the maximal dissipative transformation exists in the space $\mathcal{H}(E)$. The argument is reversed to construct the maximal dissipative transformation in the space $\mathcal{H}(E)$ when the Hilbert space of entire functions with the desired reproducing kernel functions exists.

A transformation is defined in the space $\mathcal{H}(E)$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space. The graph of the adjoint is a space of pairs

$$F(z) = (F_+(z), F_-(z))$$

such that the adjoint takes the function $F_+(z)$ of z into the function $F_-(z)$ of z. The graph contains

$$K(w, z) = (K_+(w, z), K_-(w, z))$$

with

$$K_+(w, z) = [E(z)E(w - \tfrac{1}{2}ih)^- - E^*(z)E(w^- + \tfrac{1}{2}ih)]/[2\pi i(w^- + \tfrac{1}{2}ih - z)]$$

and

$$K_-(w, z) = [E(z)E(w + \tfrac{1}{2}ih)^- - E^*(z)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - \tfrac{1}{2}ih - z)]$$

for every complex number w. The elements $K(w, z)$ of the graph span the graph of a restriction of the adjoint. The transformation in the space $\mathcal{H}(E)$ is recovered as the adjoint of its restricted adjoint.

A scalar product is defined on the graph of the restricted adjoint so that an isometric transformation of the graph of the restricted adjoint into the space $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into

$$F_+(z + \tfrac{1}{2}ih) + F_-(z - \tfrac{1}{2}ih).$$

The identity

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{H}(E)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{H}(E)}$$

holds for all elements

$$F(z) = (F_+(z), F_-(z))$$

of the graph of the restricted adjoint. The restricted adjoint is dissipative since scalar self–products are nonnegative in its graph. The adjoint is dissipative since the transformation in the space $\mathcal{H}(E)$ is the adjoint of its restricted adjoint.

The dissipative property of the adjoint is expressed in the inequality

$$\|F_+(t) - \lambda^- F_-(t)\|_{\mathcal{H}(E)} \leq \|F_+(t) + \lambda F_-(t)\|_{\mathcal{H}(E)}$$

for elements

$$F(z) = (F_+(z), F_-(z))$$

of the graph when λ is in the right half–plane. The domain of the contractive transformation which takes the function

$$F_+(z) + \lambda F_-(z)$$

of z into the function

$$F_+(z) - \lambda^- F_-(z)$$

of z is a closed subspace of the space $\mathcal{H}(E)$. The maximal dissipative property of the adjoint is the requirement that the contractive transformation be everywhere defined for some, and hence every, λ in the right half–plane.

Since $K(w,z)$ belongs to the graph for every complex number w, an entire function $H(z)$ of z which belongs to the space $\mathcal{H}(E)$ and is orthogonal to the domain is a solution of the equation

$$H(z) + \lambda H(z+i) = 0.$$

The function vanishes identically if it has a zero since zeros are repeated periodically with period i and since the function

$$H(z)/E(z)$$

of z is of bounded type in the upper half–plane. The space of solutions has dimension zero or one. The dimension is zero since it is independent of λ.

The transformation which takes $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space $\mathcal{H}(E)$ is maximal dissipative since it is the adjoint of its adjoint, which is maximal dissipative.

This completes the proof of the theorem.

The defining function $E(z)$ of an Euler space of entire functions is of Hermite class since the function

$$E(z - \tfrac{1}{2}ih)/E(z + \tfrac{1}{2}ih)$$

of z is of bounded type and of nonpositive mean type in the upper half–plane when h is in the interval $(0,1)$. Since the function is bounded by one on the real axis, it is bounded by one in the upper half–plane. The modulus of $E(x+iy)$ is a nondecreasing function of positive y for every real x. An entire function $F(z)$ of z which belongs to the space $\mathcal{H}(E)$ is of Hermite class if it has no zeros in the upper half–plane and if the inequality

$$|F(x-iy)| \leq |F(x+iy)|$$

holds for all real x when y is positive.

A construction of Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) appears in the spectral theory of ordinary differential equations of second order which are formally self–adjoint. The spectral theory is advantageously reformulated as a spectral theory of first order differential equations for pairs of scalar functions. The resulting canonical form is the classical model for a vibrating string.

The canonical form for the integral equation is obtained with a continuous function of positive t whose values are matrices

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

with real entries such that the matrix inequality

$$m(a) \leq m(b)$$

holds when a is less than b. It is assumed that $\alpha(t)$ is positive when t is positive, that

$$\lim \alpha(t) = 0$$

as t decreases to zero, and that the integral

$$\int_0^1 \alpha(t) d\gamma(t)$$

is finite.

The matrix

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is applied in the formulation of the integral equation. When a is positive, the integral equation

$$M(a,b,z)I - I = z \int_a^b M(a,t,z) dm(t)$$

admits a unique continuous solution

$$M(a,b,z) = \begin{pmatrix} A(a,b,z) & B(a,b,z) \\ C(a,b,z) & D(a,b,z) \end{pmatrix}$$

as a function of b greater than or equal to a for every complex number z. The entries of the matrix are entire functions of z which are self–conjugate and of Hermite class for every b. The matrix has determinant one. The identity

$$M(a,c,z) = M(a,b,z) M(b,c,z)$$

holds when $a \le b \le c$.

A bar is used to denote the conjugate transpose

$$M^- = \begin{pmatrix} A^- & C^- \\ B^- & D^- \end{pmatrix}$$

of a square matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with complex entries and also for the conjugate transpose

$$c^- = (c_+^-, c_-^-)$$

of a column vector

$$c = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

with complex entries. The space of column vectors with complex entries is a Hilbert space of dimension two with scalar product

$$\langle u, v \rangle = v^- u = v_+^- u_+ + v_-^- u_-.$$

When a and b are positive with a less than or equal to b, a unique Hilbert space $\mathcal{H}(M(a,b))$ exists whose elements are pairs

$$F(z) = \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

of entire functions of z such that a continuous transformation of the space into the Hilbert space of column vectors is defined by taking $F(z)$ into $F(w)$ for every complex number w and such that the adjoint takes a column vector c into the element

$$[M(a,b,z)IM(a,b,w)^- - I]c/[2\pi(z - w^-)]$$

of the space.

An entire function

$$E(c,z) = A(c,z) - iB(c,z)$$

of z which is of Hermite class exists for every positive number c such that the self–conjugate entire functions $A(c,z)$ and $B(c,z)$ satisfy the identity

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))M(a,b,z)$$

when a is less than or equal to b and such that the entire functions

$$E(c,z) \exp[\beta(c)z]$$

of z converge to one uniformly on compact subsets of the complex plane as c decreases to zero.

A space $\mathcal{H}(E(c))$ exists for every positive number c. The space $\mathcal{H}(E(a))$ is contained contractively in the space $\mathcal{H}(E(b))$ when a is less than or equal to b. The inclusion is isometric on the orthogonal complement in the space $\mathcal{H}(E(a))$ of the elements which are linear combinations

$$A(a,z)u + B(a,z)v$$

with complex coefficients u and v. These elements form a space of dimension zero or one since the identity

$$v^- u = u^- v$$

is satisfied.

A positive number b is said to be singular with respect to the function $m(t)$ of t if it belongs to an interval (a,c) such that equality holds in the inequality

$$[\beta(c) - \beta(a)]^2 \leq [\alpha(c) - \alpha(a)][\gamma(c) - \gamma(a)]$$

with $m(b)$ unequal to $m(a)$ and unequal to $m(c)$. A positive number is said to be regular with respect to $m(t)$ if it is not singular with respect to the function of t.

If a and c are positive numbers such that a is less than c and if an element b of the interval (a,c) is regular with respect to $m(t)$, then the space $\mathcal{H}(M(a,b))$ is contained isometrically in the space $\mathcal{H}(M(a,c))$ and multiplication by $M(a,b,z)$ is an isometric transformation of the space $\mathcal{H}(M(b,c))$ onto the orthogonal complement of the space $\mathcal{H}(M(a,b))$ in the space $\mathcal{H}(M(a,c))$.

If a and b are positive numbers such that a is less than b and if a is regular with respect to $m(t)$, then the space $\mathcal{H}(E(a))$ is contained isometrically in the space $\mathcal{H}(E(b))$ and an isometric transformation of the space $\mathcal{H}(M(a,b))$ onto the orthogonal complement of the space $\mathcal{H}(E(a))$ in the space $\mathcal{H}(E(b))$ is defined by taking

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

into

$$\sqrt{2}\ [A(a,z)F_+(z) + B(a,z)F_-(z)].$$

A function $\tau(t)$ of positive t with real values exists such that the function

$$m(t) + Iih(t)$$

of positive t with matrix values is nondecreasing for a function $h(t)$ of t with real values if, and only if, the functions

$$\tau(t) - h(t)$$

and

$$\tau(t) + h(t)$$

of positive t with real values are nondecreasing. The function $\tau(t)$ of t, which is continuous and nondecreasing, is called a greatest nondecreasing function such that

$$m(t) + Ii\tau(t)$$

is nondecreasing. The function is unique within an added constant.

If a and b are positive numbers such that a is less than b, multiplication by

$$\exp(ihz)$$

is a contractive transformation of the space $\mathcal{H}(E(a))$ into the space $\mathcal{H}(E(b))$ for a real number h, if, and only if, the inequalities

$$\tau(a) - \tau(b) \leq h \leq \tau(b) - \tau(a)$$

are satisfied. The transformation is isometric when a is regular with respect to $m(t)$.

An analytic weight function $W(z)$ may exist such that multiplication by

$$\exp(i\tau(c)z)$$

is an isometric transformation of the space $\mathcal{H}(E(c))$ into the weighted Hardy space $\mathcal{F}(W)$ for every positive number c which is regular with respect to $m(t)$. The analytic weight function is unique within a constant factor of absolute value one if the function

$$\alpha(t) + \gamma(t)$$

of positive t is unbounded in the limit of large t. The function

$$W(z) = \lim E(c, z) \exp(i\tau(c)z)$$

can be chosen as a limit uniformly on compact subsets of the upper half–plane.

If multiplication by

$$\exp(i\tau z)$$

is an isometric transformation of a space $\mathcal{H}(E)$ into the weighted Hardy space $\mathcal{F}(W)$ for some real number τ and if the space $\mathcal{H}(E)$ contains an entire function $F(z)$ whenever its product with a nonconstant polynomial belongs to the space, then the space $\mathcal{H}(E)$ is isometrically equal to the space $\mathcal{H}(E(c))$ for some positive number c which is regular with respect to $m(t)$.

A construction of Euler spaces of entire functions is made from Euler weight functions when a hypothesis is satisfied.

Theorem 3. *If for some real number τ a nontrivial entire function $F(z)$ of z exists such that the functions*

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the weighted Hardy space $\mathcal{F}(W)$ of an Euler weight function $W(z)$, then an Euler space of entire functions exists such that multiplication by $\exp(i\tau z)$ is an isometric transformation of the space into the weighted Hardy space and such that the space contains every entire function $F(z)$ of z such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the weighted Hardy space.

Proof of Theorem 3. The set of entire functions $F(z)$ such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the weighted Hardy space is a vector space with scalar product determined by the isometric property of multiplication as a transformation of the space into the weighted Hardy space. The space is shown to be a Hilbert space by showing that a Cauchy sequence of elements $F_n(z)$ of the space converge to an element $F(z)$ of the space.

Since the elements

$$\exp(i\tau z)F_n(z)$$

and

$$\exp(i\tau z)F_n^*(z)$$

of the weighted Hardy space form Cauchy sequences, a function $F(z)$ of z which is analytic separately in the upper half–plane and the lower half–plane exists such that the limit functions

$$\exp(i\tau z)F(z) = \lim \exp(i\tau z)F_n(z)$$

and

$$\exp(i\tau z)F^*(z) = \lim \exp(i\tau z)F_n^*(z)$$

of z belong to the weighted Hardy space. Since

$$|z - z^-|^{\frac{1}{2}} \exp(i\tau z)F(z)/W(z) = \lim |z - z^-|^{\frac{1}{2}} \exp(i\tau z)F_n(z)/W(z)$$

and

$$|z - z^-|^{\frac{1}{2}} \exp(i\tau z) F^*(z)/W(z) = \lim |z - z^-|^{\frac{1}{2}} \exp(i\tau z) F_n^*(z)/W(z)$$

uniformly in the upper half–plane and since the functions

$$\log |F_n(z)/W(z)|$$

and

$$\log |F_n^*(z)/W(z)|$$

of z are subharmonic in the half–plane

$$-1 < iz^- - iz,$$

the convergence of

$$F(z) = \lim F_n(z)$$

is uniform on compact subsets of the complex plane. The limit function $F(z)$ of z is analytic in the complex plane.

A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) is obtained such that multiplication by $\exp(i\tau z)$ is an isometric transformation of the space into the weighted Hardy space. Since the space contains a nonzero element by hypothesis, it is isometrically equal to a space $\mathcal{H}(E)$.

The space is shown to be an Euler space of entire functions by showing that a maximal dissipative transformation is defined in the space for h in the interval $[-1, 1]$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space. The dissipative property of the transformation is a consequence of the dissipative property in the weighted Hardy space.

Maximality is proved by showing that every element of the space is a sum

$$F(z) + F(z + ih)$$

of functions $F(z)$ and $F(z + ih)$ of z which belong to the space.

Since a maximal dissipative transformation exists in the weighted Hardy space, every element of the Hilbert space of entire functions is in the upper half–plane a sum

$$F(z) + F(z + ih)$$

of functions $F(z)$ and $F(z + ih)$ of z such that the functions

$$\exp(i\tau z) F(z)$$

and

$$\exp(i\tau z) F(z + ih)$$

of z belong to the weighted Hardy space. The function $F(z)$ of z admits an analytic continuation to the complex plane. The decomposition applies for all complex z.

The entire function
$$F^*(z) + F^*(z - ih)$$
of z belongs to the Hilbert space of entire functions since the space satisfies the axiom (H3). An entire function $G(z)$ of z exists such that
$$F^*(z) + F^*(z - ih) = G(z) + G(z + ih)$$
and such that the functions
$$\exp(i\tau z)G(z)$$
and
$$\exp(i\tau z)G(z + ih)$$
of z belong to the weighted Hardy space.

Vanishing of the entire function
$$F^*(z) - G(z + ih) = G(z) - F^*(z - ih)$$
of z implies the desired conclusion that the functions $F(z)$ and $F(z + ih)$ of z as well as the functions $G(z)$ and $G(z + ih)$ of z belong to the Hilbert space of entire functions. Vanishing is proved by showing boundedness of the function in the strip
$$-2h < iz^- - iz < 0$$
since the function is periodic of period $2ih$ with modulus which is periodic of period ih.

It can be assumed that the functions
$$\exp(i\tau z)F(z)$$
and
$$\exp(i\tau z)G(z)$$
of z are elements of norm at most one in the weighted Hardy space. The inequalities
$$2\pi|F(z)|^2 \le |\exp(-i\tau z)W(z)|^2/(iz^- - iz)$$
and
$$2\pi|G(z)|^2 \le |\exp(-i\tau z)W(z)|^2/(iz^- - iz)$$
apply when z is in the upper half–plane. Since the inequalities
$$2\pi|F^*(z)|^2 \le |\exp(i\tau z)W^*(z)|^2/(iz - iz^-)$$
and
$$2\pi|G(z + ih)|^2 \le \exp(2\pi h)|\exp(-i\tau z)W(z + ih)|^2/(2h + iz^- - iz)$$

apply when z is in the strip, the inequality

$$\pi|F^*(z) - G(z+ih)|^2 \leq |\exp(i\tau z)W^*(z)|^2/(iz - iz^-)$$
$$+ \exp(2\pi h)|\exp(-i\tau z)W(z+ih)|^2/(2h + iz^- - iz)$$

applies when z is in the strip.

Boundedness of the entire function

$$F^*(z) - G(z+ih)$$

of z in the complex plane follows from the subharmonic property of the logarithm of its modulus. The entire function is a constant which vanishes because of the identity

$$F^*(z) - G(z+ih) = G(z) - F^*(z-ih).$$

This completes the proof of the theorem.

The hypothesis of the theorem are satisfied by an Euler weight function $W(z)$ which satisfies the identity

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\varphi(z)$$

for a function $\phi(z)$ of z which is analytic and has nonnegative real part in the upper half–plane if $\log \phi(z)$ has nonnegative real part in the upper half–plane. The modulus of $W(x+iy)$ is then a nondecreasing function of positive y for every real x.

Since the weight function can be multiplied by a constant, it can be assumed to have value one at the origin. The phase $\psi(x)$ is defined as the continuous function of real x with value zero at the origin such that

$$\exp(i\psi(x))W(x)$$

is positive for all real x. The phase function is a nondecreasing function of real x which is identically zero if it is constant in any interval.

When the phase function vanishes identically, the modulus of $W(x+iy)$ is a constant as a function of positive y for every real x. The weight function is then the restriction of a self–conjugate entire function of Hermite class. For every positive number τ an entire function

$$F(z) = W(z)\sin(\tau z)/z$$

is obtained such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the space $\mathcal{F}(W)$. No nonzero entire function $F(z)$ exists such that the functions $F(z)$ and $F^*(z)$ belong to the space $\mathcal{F}(W)$.

When the phase function does not vanish identically, an entire function $E(z)$ of Hermite class which has no real zeros exists such that $E(x)$ is real for a real number x if, and only if, $\psi(x)$ is an integral is an integral multiple of π, and then

$$\exp(i\psi(x))E(x)$$

is positive. Such an entire function is unique within a factor of a self–conjugate entire function of Hermite class. The factor is chosen so that the function

$$E(z)/W(z)$$

of z has nonnegative real part in the upper half–plane. The entire functions $E(z)$ and $E^*(z)$ are linearly independent. A nontrivial entire function

$$F(z) = [E(z) - E^*(z)]/z$$

is obtained such that the functions $F(z)$ and $F^*(z)$ of z belong to the space $\mathcal{F}(W)$.

The same conclusions are obtained under a weaker hypothesis.

Theorem 4. *If an Euler weight function $W(z)$ satisfies the identity*

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z)$$

for a function $\phi(z)$ of z which is analytic and has nonnegative real part in the upper half–plane such that

$$\sigma(z) + \log \phi(z)$$

has nonnegative real part in the upper half–plane for a function $\sigma(z)$ of z which is analytic and has nonnegative real part in the upper half–plane such that the least upper bound

$$\sup \int_{-\infty}^{+\infty} \mathcal{R}\sigma(x + iy)dx$$

taken over all positive y is finite, then for every positive number τ a nontrivial entire function $F(z)$ exists such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the weighted Hardy space $\mathcal{F}(W)$.

Proof of Theorem 4. It can be assumed that the symmetry condition

$$\sigma^*(z) = \sigma(-z)$$

is satisfied since otherwise $\sigma(z)$ can be replaced by $\sigma(z)+\sigma^*(-z)$. When h is in the interval $(0,1)$, the integral

$$\log \phi_h(z) = \sin(\pi h) \int_{-\infty}^{+\infty} \frac{\log \phi(z-t)dt}{\cos(2\pi it) + \cos(\pi h)}$$

defines a function $\phi_h(z)$ of z which is analytic and has nonnegative real part in the upper half–plane such that

$$W(z + \tfrac{1}{2}ih) = W(z - \tfrac{1}{2}ih)\phi_h(z).$$

The integral

$$\mathcal{R}\sigma_h(z) = \sin(\pi h) \int_{-\infty}^{+\infty} \frac{\mathcal{R}\sigma(z-t)dt}{\cos(2\pi it) + \cos(\pi h)}$$

and the symmetry condition

$$\sigma_n^*(z) = \sigma_n(-z)$$

define a function $\sigma_h(z)$ which is analytic and has nonnegative real part in the upper half–plane. The function

$$\sigma_h(z) + \log \phi_h(z)$$

of z has nonnegative real part in the upper half–plane since the function

$$\sigma(z) + \log \phi(z)$$

has nonnegative real part in the upper half–plane by hypothesis.

An analytic weight function $U(z)$ which admits an analytic extension without zeros to the half–plane $iz^- - iz > -1$ is defined within a constant factor by the identity

$$\log U(z + \tfrac{1}{2}ih) - \log U(z - \tfrac{1}{2}ih) = \sigma_h(z)$$

for h in the interval $(0,1)$ and by the symmetry

$$U^*(z) = U(-z).$$

The analytic weight function

$$V(z) = U(z)W(z)$$

has an analytic extension without zeros to the half–plane $iz^- - iz > -1$. The modulus of $U(x+iy)$ and the modulus of $V(x+iy)$ are nondecreasing functions of positive y for every real x.

Since

$$\frac{\partial}{\partial y} \log |U(x+iy)| = \pi \int_{-\infty}^{+\infty} \frac{\mathcal{R}\sigma(x+iy-t)dt}{1+\cos(2\pi it)}$$

the integral

$$\int_{-\infty}^{+\infty} \frac{\partial}{\partial y} \log |U(x+iy)| dx = \int_{-\infty}^{+\infty} \mathcal{R}\sigma(x+iy)dx$$

is a bounded function of positive y. The phase $\psi(x)$ is the continuous, nondecreasing, odd function of real x such that

$$\exp(i\psi(x))U(x)$$

is positive for all real x. Since

$$\frac{\partial}{\partial y}\log|U(x+iy)| = \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{d\psi(t)}{(t-x)^2+y^2}$$

when y is positive, the inequality

$$\psi(b)-\psi(a) \le \sup\int_{-\infty}^{+\infty}\mathcal{R}\sigma(x+iy)dx$$

holds when a is less than b with the least upper bound taken over all positive y.

The remaining arbitrary constant in $U(z)$ is chosen so that the integral representation

$$\log U(z) = \frac{1}{2\pi}\int_0^{\infty}\log(1-z^2/t^2)d\psi(t)$$

holds when z is in the upper half–plane with the logarithm of $1-z^2/t^2$ defined continuously in the upper half–plane with nonnegative values when z is on the upper half of the imaginary axis. The inequality

$$|U(z)| \le |U(i|z|)$$

holds when z is in the upper half–plane since

$$|1-z^2/t^2| \le 1+z^-z/t^2.$$

If a positive integer r is chosen so that the inequality

$$\int_{-\infty}^{+\infty}\mathcal{R}\sigma(x+iy)dx \le 2\pi r$$

holds for all positive y, then the function

$$U(z)/(z+i)^r$$

is bounded and analytic in the upper half–plane.

Since the modulus of $V(x+iy)$ is a nondecreasing function of positive y for every real x, there exists for every positive number τ a nontrivial entire function $G(z)$ such that the functions

$$\exp(i\tau z)G(z)$$

and

$$\exp(i\tau z)G^*(z)$$

of z belong to the weighted Hardy space $\mathcal{F}(V)$. Since the entire function

$$G(z) = F(z)P(z)$$

is the product of an entire function $F(z)$ and a polynomial $P(z)$ of degree r, a nontrivial entire function $F(z)$ is obtained such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the weighted Hardy space $\mathcal{F}(W)$.

This completes the proof of the theorem.

The Hilbert spaces of entire functions for the vibrating string of an Euler weight function are Euler spaces of entire functions.

Theorem 5. *A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains a nonzero element is an Euler space of entire functions if it contains an entire function whenever its product with a nonconstant polynomial belongs to the space and if multiplication by* $\exp(i\tau z)$ *is for some real number* τ *an isometric transformation of the space into the weighted Hardy space* $\mathcal{F}(W)$ *of an Euler weight function* $W(z)$.

Proof of Theorem 5. It can be assumed that τ vanishes since the function

$$\exp(-i\tau z)W(z)$$

is an Euler weight function whenever the function $W(z)$ of z is an Euler weight function.

The given Hilbert space of entire functions is isometrically equal to a space $\mathcal{H}(E)$ for an entire function $E(z)$ which has no real zeros since an entire function belongs to the space whenever its product with a nonconstant polynomial belongs to the space.

A dissipative transformation is defined in the space $\mathcal{H}(E)$ when h is in the interval $[0, 1]$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space since the space is contained isometrically in the space $\mathcal{F}(W)$ and since a dissipative transformation is defined in the space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space. It remains to prove the maximal dissipative property of the transformation in the space $\mathcal{H}(E)$.

The ordering theorem for Hilbert spaces of entire functions applies to spaces which satisfy the axioms (H1), (H2), and (H3) and which are contained isometrically in a weighted Hardy space $\mathcal{F}(W)$ when a space contains an entire function whenever its product with a nonconstant polynomial belongs to the space. One space is properly contained in the other when the two spaces are not identical.

A Hilbert space $\mathcal{H}$ of entire functions which satisfies the axioms (H1) and (H2) and which contains a nonzero element need not satisfy the axiom (H3). Multiplication by $\exp(iaz)$ is for some real number a an isometric transformation of the space onto a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3).

A space $\mathcal{H}$ which satisfies the axioms (H1) and (H2) and which is contained isometrically in the space $\mathcal{F}(W)$ is defined as the closure in the space $\mathcal{F}(W)$ of the set of those elements of the space which functions $F(z+ih)$ of z for functions $F(z)$ of z belonging to the space $\mathcal{H}(E)$. An example of a function $F(z+ih)$ of z is obtained for every element of the space $\mathcal{H}(E)$ which is a function $F(z)$ of z such that the function $z^2F(z)$ of z belongs to the space $\mathcal{H}(E)$. The space $\mathcal{H}$ contains an entire function whenever its product with a nonconstant polynomial belongs to the space.

The function

$$E(z)/W(z)$$

of z is of bounded type in the upper half–plane and has the same mean type as the function

$$E(z+ih)/W(z+ih)$$

of z which is of bounded type in the upper half–plane. Since the function

$$W(z+ih)/W(z)$$

of z is of bounded type and has zero mean type in the upper half–plane, the function

$$E(z+ih)/E(z)$$

of z is of bounded type and of zero mean type in the upper half–plane.

If a function $F(z)$ of z is an element of the space $\mathcal{H}(E)$ such that the functions

$$F(z)/W(z)$$

and

$$F^*(z)/W(z)$$

of z have equal mean type in the upper half–plane, and such that the functions

$$G(z) = F(z+ih)$$

and

$$G^*(z) = F^*(z-ih)$$

of z belong to the space $\mathcal{F}(W)$, then the functions

$$G(z)/W(z)$$

and

$$G^*(z)/W(z)$$

of z have equal mean type in the upper half–plane. It follows that the space $\mathcal{H}$ satisfies the axiom (H3).

Equality of the spaces $\mathcal{H}$ and $\mathcal{H}(E)$ is shown when the space $\mathcal{H}$ is contained in the space $\mathcal{H}(E)$ and when the space $\mathcal{H}(E)$ is contained in the space $\mathcal{H}$.

The function $F(z + ih)$ of z belongs to the space $\mathcal{H}(E)$ whenever the function $F(z)$ of z belongs to the space $\mathcal{H}$ and the function $F(z+ih)$ of z belongs to the space $\mathcal{F}(W)$ since the spaces $\mathcal{H}$ and $\mathcal{H}(E)$ satisfy the axiom (H3). If the space $\mathcal{H}(E)$ is contained in the space $\mathcal{H}$, then the space $\mathcal{H}$ is contained in the space $\mathcal{H}(E)$. If the space $\mathcal{H}$ is contained in the space $\mathcal{H}(E)$, then the space $\mathcal{H}(E)$ is contained in the space $\mathcal{H}$.

Since the transformation T which takes $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space $\mathcal{H}(E)$ is subnormal, the domain of the adjoint T^* of T contains the domain of T. A dense subspace of the graph of T^* is determined by elements of the domain of T^* which belong to the domain of T. The dissipative property of T implies the dissipative property of T^*. The maximal dissipative property of T follows since T is the adjoint of T^*.

This completes the proof of the theorem.

Computable examples of Hilbert spaces of entire functions are constructed from the gamma function. The Euler weight function

$$W_h(z) = \Gamma(h - iz)$$

has special properties when h is positive since

$$W_h^*(z) = W_h(-z)$$

and since a self–adjoint transformation is defined in the weighted Hardy space $\mathcal{F}(W_h)$ by taking $F(z)$ into

$$F(z + i)/(h - iz)$$

whenever the functions of z belong to the space.

A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains an entire function whenever its product with a nonconstant polynomial belongs to the space is contained isometrically in the weighted Hardy space if it is contained contractively in the space and if the inclusion is isometric on elements of the space whose product with z belongs to the space.

A defining function $E(z)$ for the space can be chosen with the symmetry

$$E^*(z) = E(-z)$$

since an isometric transformation of the space onto itself is defined by taking $F(z)$ into $F(-z)$.

When $h = \frac{1}{2}$, a self–adjoint transformation in the space is defined by taking $F(z)$ into

$$[F(z+i) - F(-z)]/(\tfrac{1}{2} - iz)$$

whenever the functions of z belong to the space. If

$$E(z) = A(z) - iB(z)$$

for self–conjugate entire functions $A(z)$ and $B(z)$, the identities

$$[A(z+i) - A(-z)]/(\tfrac{1}{2} - iz) = A(z)s - iB(z)r$$

and

$$[B(z+i) - B(-z)]/(\tfrac{1}{2} - iz) = iA(z)p + B(z)s$$

hold for positive numbers p, r, and s such that

$$pr = s^2.$$

The spaces are parametrized by positive numbers t so that the space with parameter b is contained isometrically in the space with parameter a when a is less than b. The defining function $E(t, z)$ of the space with parameter t satisfies the identities with

$$s(t) = 2/t.$$

The function is chosen so that the identities are satisfied with

$$p(t) = \exp(-t)s(t)$$

and

$$r(t) = \exp(t)s(t).$$

The functions $E(t, z)$ depend continuously on t and satisfy the integral equation

$$(A(b, z), B(b, z))I - (A(a, z), B(a, z))I = z \int_a^b (A(t, z), B(t, z))dm(t)$$

when a is less than b with

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

a nonincreasing matrix–valued function of positive t with differentiable entries such that

$$-t\alpha'(t) = p(t)/s(t)$$

and

$$-t\gamma'(t) = r(t)/s(t)$$

and

$$\beta(t) = 0.$$

The analytic weight function

$$W(z) = \Gamma(h - iz)$$

is treated in relation to the contiguous analytic weight functions

$$W_-(z) = \Gamma(h - \tfrac{1}{2} - iz)$$

and

$$W_+(z) = \Gamma(h + \tfrac{1}{2} - iz)$$

when $h - \frac{1}{2}$ is positive. Contiguity applies also to associated Hilbert spaces of entire functions. Assume that a space $\mathcal{H}(E)$ is contained isometrically in the weighted Hardy space $\mathcal{F}(W)$ and that the space contains an entire function whenever its product with a polynomial belongs to the space. The defining function $E(z)$ of the space is chosen to satisfy the symmetry condition

$$E^*(z) = E(-z)$$

since the symmetry condition

$$W^*(z) = W(-z)$$

is satisfied. The function $E(z)$ is uniquely determined by the requirement that $E(z)$ has value one at the origin. Contiguous spaces $\mathcal{H}(E_-)$ contained isometrically in the weighted Hardy space $\mathcal{F}(W_-)$ and $\mathcal{H}(E_+)$ contained isometrically in the weighted Hardy space $\mathcal{F}(W_+)$ are constructed with analogous properties.

Since multiplication by

$$h - \tfrac{1}{2} - iz$$

is an isometric transformation of the space $\mathcal{F}(W_-)$ onto the space $\mathcal{F}(W_+)$, the spaces $\mathcal{H}(E_-)$ and $\mathcal{H}(E_+)$ are chosen so that the multiplication is an isometric transformation of the space $\mathcal{H}(E_-)$ onto the set of elements of the space $\mathcal{H}(E_+)$ which vanish at $\frac{1}{2}i - ih$. The adjoint transformation of the space $\mathcal{H}(E_+)$ into the space $\mathcal{H}(E_-)$ takes a function $F(z)$ of z into the function

$$[F(z) - L_-(z)F(\tfrac{1}{2}i - ih)]/(h - \tfrac{1}{2} - iz)$$

of z with

$$L_-(z) = K_+(\tfrac{1}{2}i - ih, z)/K_+(\tfrac{1}{2}i - ih, \tfrac{1}{2}i - ih)$$

the constant multiple of the reproducing kernel function for function values at $\frac{1}{2}i - ih$ in the space $\mathcal{H}(E_+)$ which has value one at $\frac{1}{2}i - ih$. The equations

$$\lambda A(z) = [A_+(z) - L_-(z)A_+(\tfrac{1}{2}i - ih)]/(h - \tfrac{1}{2} - iz)$$

and

$$\lambda^{-1}B(z) = [B_+(z) - L_-(z)B_+(\tfrac{1}{2}i - ih)]/(h - \tfrac{1}{2} - iz)$$

apply with

$$\lambda = [1 - L_-(0)A_+(\tfrac{1}{2}i - ih)]/(h - \tfrac{1}{2})$$

a nonzero real number.

A contractive transformation of the space $\mathcal{F}(W_-)$ into the space $\mathcal{F}(W)$ and of the space $\mathcal{F}(W)$ into the space $\mathcal{F}(W_+)$ is defined by taking a function $F(z)$ of z into the function $F(z + \frac{1}{2}i)$ of z. The adjoint transformation of the space $\mathcal{F}(W)$ into the space $\mathcal{F}(W_-)$ takes a function $F(z)$ of z into the function

$$F(z + \tfrac{1}{2}i)/(h - \tfrac{1}{2} - iz)$$

of z. The adjoint transformation of the space $\mathcal{F}(W_+)$ into the space $\mathcal{F}(W)$ takes a function $F(z)$ of z into the function

$$F(z + \tfrac{1}{2}i)/(h - iz)$$

of z.

The spaces $\mathcal{H}(E_-)$ and $\mathcal{H}(E_+)$ are constructed so that a contractive transformation of the space $\mathcal{H}(E_-)$ into the space $\mathcal{H}(E)$ and of the space $\mathcal{H}(E)$ into the space $\mathcal{H}(E_+)$ is defined by taking a function $F(z)$ of z into the function $F(z + \frac{1}{2}i)$ of z and so that the adjoint transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{H}(E_-)$ takes a function $F(z)$ of z into the function

$$[F(z + \tfrac{1}{2}i) - L_-(z)F(i - ih)]/(h - \tfrac{1}{2} - iz)$$

of z.

The transformation of the space $\mathcal{H}(E_-)$ into the space $\mathcal{H}(E)$ takes the function

$$[B_-(z)A_-(w)^- - A_-(z)B_-(w)^-]/[\pi(z - w^-)]$$

of z into the function

$$[B_-(z + \tfrac{1}{2}i)A_-(w)^- - A_-(z + \tfrac{1}{2}i)B_-(w)^-]/[\pi(z + \tfrac{1}{2}i - w^-)]$$

of z for every complex number w. The adjoint transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{H}(E_-)$ takes the function

$$[B(z)A(w)^- - A(z)B(w)^-]/[\pi(z - w^-)]$$

of z into the function

$$\frac{B(z + \frac{1}{2}i)A(w)^- - A(z + \frac{1}{2}i)B(w)^-}{\pi(z + \frac{1}{2}\, i - w^-)(h - \frac{1}{2} - iz)} - L_-(z)\,\frac{B(i - ih)A(w)^- - A(i - ih)B(w)^-}{\pi(i - ih - w^-)(h - \frac{1}{2} - iz)}$$

of z for every complex number w.

The identity

$$[B_-(z+\tfrac{1}{2}i)A_-(w)^- - A_-(z+\tfrac{1}{2}i)B_-(w)^-]/[\pi(z+\tfrac{1}{2}i-w^-)]$$
$$= \frac{B(z)A(w+\frac{1}{2}i)^- - A(z)B(w+\frac{1}{2}i)^-}{\pi(z+\frac{1}{2}i-w^-)(h-\frac{1}{2}+iw^-)}$$
$$-L_-(w)^- \frac{B(z)A(i-ih)^- - A(z)B(i-ih)^-}{\pi(z+i-ih)(h-\frac{1}{2}+iw^-)}$$

follows by properties of reproducing kernel functions for all complex numbers z and w.

The identity can be written

$$B_-(z+\tfrac{1}{2}i)A_-(w)^- - A_-(z+\tfrac{1}{2}i)B_-(w)^-$$
$$= B(z)[A(w+\tfrac{1}{2}i)^- - L_-(w)^-A(i-ih)^-]/(h-\tfrac{1}{2}+iw^-)$$
$$-A(z)[B(w+\tfrac{1}{2}i)^- - L_-(w)^-B(i-ih)^-]/(h-\tfrac{1}{2}+iw^-)$$
$$+L_-(w)^-[B(z)A(i-ih)^- - A(z)B(i-ih)^-]/(h-1+iz).$$

Since the functions $A(z)$, $B(z)$, and

$$[B(z)A(i-ih)^- - A(z)B(i-ih)^-]/(h-1+iz)$$

of z are linearly independent, the equations

$$A_-(z+\tfrac{1}{2}i) = A(z)P + B(z)R$$
$$-v_-^- \,[B(z)A(i-ih)^- - A(z)B(i-ih)^-]/(h-1+iz)$$

and

$$B_-(z+\tfrac{1}{2}i) = A(z)Q + B(z)S$$
$$+u_-^- \,[B(z)A(i-ih)^- - A(z)B(i-ih)^-]/(h-1+iz)$$

apply for unique complex numbers u_- and v_- and for a unique matrix

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

with complex entries.

Symmetry implies that u_- and the diagonal entries of the matrix are real and that v_- and the off–diagonal entries of the matrix are imaginary. The equations

$$A_-(z)S + B_-(z)R = [A(z+\tfrac{1}{2}\,i) - L_-(z)A(i-ih)]/(h-\tfrac{1}{2}-iz)$$

and

$$A_-(z)Q + B_-(z)P = [B(z+\tfrac{1}{2}\,i) - L_-(z)B(i-ih)]/(h-\tfrac{1}{2}-iz)$$

are satisfied with

$$L_-(z) = A_-(z)u_- + B_-(z)v_-.$$

Consistency of the equations implies the constraints

$$PS = QR$$

and

$$1 = [PA(ih-i) + RB(ih-i)]\ u_- - [QA(ih-i) + SB(ih-i)]\ v_-.$$

The recurrence relations

$$\begin{aligned}[A_-(z+i) - L_-(z)A_-(\tfrac{3}{2}i - ih)]/(h - \tfrac{1}{2} - iz) \\ + [B_-(z)A_-(\tfrac{3}{2}i - ih)^- - A_-(z)B_-(\tfrac{3}{2}i - ih)]v_-/(\tfrac{3}{2} - h - iz) \\ = 2P[A_-(z)S + B_-(z)R]\end{aligned}$$

and

$$\begin{aligned}[B_-(z+i) - L_-(z)B_-(\tfrac{3}{2}i - ih)]/(h - \tfrac{1}{2} - iz) \\ + [B_-(z)A_-(\tfrac{3}{2}i - ih)^- - A_-(z)B_-(\tfrac{3}{2}i - ih)]u_-/(\tfrac{3}{2} - h - iz) \\ = 2S[A_-(z)Q + B_-(z)P]\end{aligned}$$

are satisfied.

The recurrence relations

$$\begin{aligned}[A(z+i) - L(z)A(i - ih)]/(h - iz) \\ + [B(z)A(i - ih)^- - A(z)B(i - ih)^-]\ v/(1 - h - iz) \\ = 2S[A(z)P + B(z)R]\end{aligned}$$

and

$$\begin{aligned}[B(z+i) - L(z)B(i - ih)]/(h - iz) \\ + [B(z)A(i - ih)^- - A(z)B(i - ih)^-]\ u/(1 - h - iz) \\ = 2P[A(z)Q + B(z)S]\end{aligned}$$

are obtained with

$$L(z) = A(z)u + B(z)v$$

where

$$u = Pu_- - Qv_- + B(ih-i)u_-v_-/(h-1)$$

and

$$v = -Ru_- + Sv_- + A(ih-i)u_-v_-/(h-1).$$

The entire function $E(t,z)$ of z depends on a positive parameter t and satisfies the differential equations

$$\frac{\partial}{\partial t}\ B(t,z) = zA(t,z)\alpha'(t)$$

and

$$-\frac{\partial}{\partial t} A(t,z) = zB(t,z)\gamma'(t)$$

for differentiable functions $\alpha(t)$ and $\gamma(t)$ of positive t. The coefficients $u(t)$ and $v(t)$ and the entries of the matrix

$$\begin{pmatrix} P(t) & Q(t) \\ R(t) & S(t) \end{pmatrix}$$

are differentiable functions of t. The derivatives $\alpha'(t)$ and $\gamma'(t)$ are negative since the space parametrized by b is contained in the space parametrized by a when a is less than b. The parametrization is made so that

$$-t\tau'(t) = 1.$$

The constant of integration in $\tau(t)$ is chosen so that

$$t = \exp(-\tau(t)).$$

Similar constructions are made with subscripts plus and minus. The differential equations

$$-t\alpha'(t) = -iQ(t)/S(t) \text{ and } -t\alpha'_-(t) = -iQ(t)/P(t)$$

and

$$-t\gamma'(t) = iR(t)/P(t) \text{ and } -t\gamma'_-(t) = iR(t)/S(t)$$

and

$$u'(t) = ihv(t)\alpha'(t)$$

and

$$v'(t) = -ihu(t)\gamma'(t)$$

are obtained on differentiation.

The differential equations

$$\begin{aligned} &Q(t)A(t,ih-i)v_-(t) + R(t)B(t,ih-i)u_-(t) \\ &= S'(t)/S(t) + \tfrac{1}{2}\, t^{-1} = -P'(t)/P(t) - \tfrac{1}{2}\, t^{-1} \end{aligned}$$

and

$$\begin{aligned} &P(t)A(t,ih-i)u_-(t) + S(t)B(t,ih-i)v_-(t) \\ &= R'(t)/R(t) + \tfrac{1}{2}\, t^{-1} = -Q'(t)/Q(t) - \tfrac{1}{2}\, t^{-1} \end{aligned}$$

are obtained where

$$t\, P(t)S(t) = 1 = tQ(t)R(t)$$

and

$$\begin{aligned} 1 = &P(t)A(t,ih-i)u_-(t) - S(t)B(t,ih-i)v_-(t) \\ &-Q(t)A(t,ih-i)v_-(t) + R(t)B(t,ih-i)u_-(t). \end{aligned}$$

The equations

$$[R(t)/P(t)]' = L(t,ih-i)\; R(t)/P(t)$$

and

$$[S(t)/Q(t)]' = L(t,ih-i)\; S(t)/Q(t)$$

are satisfied.

2. Fourier Analysis on the Complex Skew–Plane

The Hilbert spaces of entire functions constructed from the gamma function apply to Fourier analysis for the complex skew–plane. The complex skew–plane is a vector space of dimension four over the real numbers which contains the complex plane as a vector subspace of dimension two. The multiplicative structure of the complex plane as a field is generalized as the multiplicative structure of the complex skew–plane as a skew–field. The conjugation of the complex plane is an automorphism which extends as an anti–automorphism of the complex skew–plane.

An element

$$\xi = t + ix + jy + kz$$

of the complex skew–plane has four real coordinates x, y, z, and t. The conjugate element is

$$\xi^- = t - ix - jy - kz.$$

The multiplication table

$$ij = k, jk = i, ki = j$$
$$ji = -k, kj = -i, ki = -j$$

defines a conjugated algebra in which every nonzero element has an inverse. An automorphism of the skew–field is an inner automorphism which is defined by an element with conjugate as inverse. A plane is a maximal commutative subalgebra. Every plane is isomorphic to every other plane under an automorphism of the skew–field. The complex plane is the subalgebra of elements which commute with i.

The topology of the complex skew–plane is derived from the topology of the real line as is the topology of the complex plane. Addition and multiplication are continuous as transformations of the Cartesian product of the space with itself into the space. The topology of the real line is derived from Dedekind cuts. A real number t divides the real line into two open half–lines (t, ∞) and $(-\infty, t)$. The intersection of open half–lines is an open interval (a, b) when it is nonempty and not a half–line. A open subset of the line is a union of open intervals. The topology of the plane is the Cartesian product topology of two Dedekind topologies of two coordinate lines. The topology of the complex skew–plane is the Cartesian product topology of the Dedekind topologies of four coordinate lines.

The canonical measure for the complex skew–plane is derived from the canonical measure for the real line as is the canonical measure for the complex plane. In all cases the canonical measure is defined on the Baire subsets of the space defined as the smallest class of sets containing the open sets and the closed sets and containing countable unions and countable intersections of sets of the class. A measure preserving transformation of the space onto itself is defined by taking ξ into $\xi + \eta$ for every element η of the space. This condition determines the canonical measure within a constant factor.

Multiplication by an element ξ of the space multiplies the canonical measure by $|\xi|$ in the case of the line, by $|\xi|^2$ in the case of the plane, and by $|\xi|^4$ in the case of the skew–plane with $|\xi|$ the nonnegative solution of the equation

$$|\xi|^2 = \xi^- \xi.$$

The modulus $|\xi|$ of ξ defines a metric on the space whose topology is identical with the Dedekind topology. The identity

$$|\xi\eta| = |\xi||\eta|$$

holds for all elements ξ and η of the space.

The canonical measure for the real line is Lebesgue measure, whose normalization is made with respect to the integral elements of the line. The additive group of integral elements inherits a discrete topology. The quotient group is compact. Elements ξ and η of the line are defined as equivalent with respect to the subgroup if their difference $\eta - \xi$ belongs to the subgroup. A fundamental domain for the equivalence relation consists of the elements which are closer to the origin than to any other integral element. Every element of the line is equivalent to an element of the closure of the fundamental domain. Elements of the fundamental domain are equal if they are equivalent. The fundamental domain is the open interval $(-\frac{1}{2}, \frac{1}{2})$. The closure $[-\frac{1}{2}, \frac{1}{2}]$ is compact and has measure one.

The ring of integral elements of the real line has special multiplicative structure. An ideal of the ring of integral elements is defined as an additive subgroup such that for every element γ of the ring $\beta\gamma$ belongs to the subgroup whenever β belongs to the subgroup. A computation of ideals is made by the Euclidean algorithm. If α is an element of the ring and if β is a nonzero element of the ring, then an element γ of the ring exists which satisfies the inequality

$$|\alpha - \beta\gamma| < |\beta|.$$

An ideal of the ring which contains a nonzero element contains a nonzero element β which minimizes the positive integer $\beta^-\beta$. Every element α of the ideal is a product

$$\alpha = \beta\gamma$$

with an element γ of the ring.

A character χ modulo ρ is defined for a positive integer ρ as a function of rational numbers which vanishes at nonintegers and at integers not relatively prime to ρ, which is periodic of period ρ, and which acts as a homomorphism of the multiplicative group of integers modulo ρ relatively prime to ρ into the multiplicative group of complex numbers of absolute value one.

A character modulo ρ is said to be primitive modulo ρ if it does not agree on integers relatively prime to ρ with a character modulo a proper divisor of ρ. The conjugate character of a character χ modulo ρ is the character χ^- modulo ρ with conjugate values:

$$\chi^-(\xi) = \chi(\xi)^-$$

for every rational number ξ. The conjugate character χ^- is primitive modulo ρ when the given character χ is primitive modulo ρ.

The integral elements of the complex plane are according to Gauss the elements whose coordinates are integral elements of the line. An Euclidean algorithm applies to the ring of integral elements of the complex plane. If α is an integral element and if β is a nonzero integral element, then an integral element γ exists which satisfies the inequality

$$|\alpha - \beta\gamma| < |\beta|.$$

An ideal of the ring of integral elements which contains a nonzero element contains an element β which minimizes the positive integer $\beta^-\beta$. Every element α of the ideal is a product

$$\alpha = \beta\gamma$$

with an element γ of the ring.

An application of the Euclidean algorithm for the plane is the representation of a positive integer as the sum of four squares

$$a^2 + b^2 = (a + ib)^-(a + ib)$$

for integers a and b. The representation problem reduces to one for prime numbers by the multiplicative structure of the complex plane. The even prime

$$2 = 1 + 1$$

is a sum of two squares. A prime which is congruent to three modulo four is not a sum of two squares since the representation is not possible modulo four. A prime

$$p = a^2 + b^2$$

is a sum of two squares of integers a and b if it is congruent to one modulo four.

The integral elements of the complex skew–plane are according to Hurwitz the elements whose coordinates are either all integral elements of the line or all half–integral elements of the line. An Euclidean algorithm applies to the ring of integral elements of the complex skew–plane. If α is an integral element and if β is a nonzero integral element, then an integral element γ exists which satisfies the inequality

$$|\alpha - \beta\gamma| < |\beta|.$$

An additive subgroup of the ring of integral elements is said to be a right ideal if it contains the product $\beta\gamma$ of every element β with an integral element γ of the ring. A right ideal of the ring which contains a nonzero element contains a nonzero element β which minimizes the positive integer $\beta^-\beta$. Every element α of the ideal is a product

$$\alpha = \beta\gamma$$

with an element γ of the ring.

Twenty–four integral elements of the complex skew–plane are inverses of integral elements of the complex skew–plane. The group has a normal subgroup of eight elements. The quotient group is a cyclic group of three elements.

An application of the Euclidean algorithm for the skew–plane is the representation of a positive integer as a sum of four squares:

$$a^2 + b^2 + c^2 + d^2 = (d + ia + jb + kc)^-(d + ia + jb + kc).$$

The representation problem reduces to one for prime numbers by the multiplicative structure of the complex skew–plane.

The canonical measure for the complex plane is the Cartesian product measure of the Lebesgue measures for two coordinate lines. The canonical measure for the complex skew–plane is the Cartesian product measure of the Lebesgue measures of four coordinate lines.

The Fourier transformation for the real line is an isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into the same space. The Fourier transform of an integrable function $f(\xi)$ of real ξ is the continuous function

$$g(\xi) = \int \exp(2\pi i\xi\eta) f(\eta) d\eta$$

of ξ is defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-2\pi i\xi\eta) g(\eta) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The Fourier transformation for the complex plane is an isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into the same space. The Fourier transformation of an integrable function $f(\xi)$ of ξ is the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^-\eta + \eta^-\xi)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^-\eta + \eta^-\xi)) g(\eta) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The Fourier transformation for the complex skew–plane is an isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure onto the same space. The Fourier transform of an integrable function $f(\xi)$ of ξ is the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^-\eta + \eta^-\xi)) f(\eta) d\eta$$

of ξ which is defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^-\eta + \eta^-\xi)) g(\eta) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The Fourier transformation commutes with the isometric transformations of the Hilbert space onto itself which are defined by taking a function $f(\xi)$ of ξ into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ for every element ω of the complex skew–plane with conjugate as inverse. The Hilbert space decomposes into the orthogonal sum of invariant subspaces for the transformations taking a function $f(\xi)$ of ξ into the function $f(\omega\xi)$ of ξ for every element ω of the complex skew–plane with conjugate as inverse.

A homomorphism of the multiplicative group of nonzero elements of the complex skew–plane onto the multiplicative group of the positive half–line is defined by taking ξ into $\xi^-\xi$. The identity

$$\int |f(\xi^-\xi)|^2 d\xi = 4\pi \int |f(\xi)|^2 \xi d\xi$$

holds with integration on the left with respect to the canonical measure for the complex skew–plane and with integration on the right with respect to Lebesgue measure for every Baire function $f(\xi)$ of ξ in the positive half–line.

A computation of invariant subspaces is made in Hilbert spaces of homogeneous polynomials defined on the complex skew–plane. A homogeneous polynomial of degree ν is a function $f(\xi)$ of

$$\xi = t + ix + jy + kz$$

of ξ in the complex skew–plane which is a linear combination of monomials

$$x^a y^b z^c t^d$$

whose exponents are nonnegative integers with sum ν. The functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the complex skew–plane are homogeneous polynomials of degree ν for every element ω of the complex skew–plane with conjugate as inverse if the function $f(\xi)$ of ξ in the complex skew–plane is a homogeneous polynomial of degree ν. The set of homogeneous polynomials of degree ν is a Hilbert space of dimension

$$(1+\nu)(2+\nu)(3+\nu)/6$$

with scalar product defined so that the monomials form an orthogonal set and so that

$$\frac{a!b!c!d!}{\nu!}$$

is the scalar self–product of the monomial with exponents a, b, c, and d. Isometric transformations of the space onto itself are defined by taking a function $f(\xi)$ of ξ into the functions $f(\omega\xi)$ and $f(\xi\omega)$ for every element ω of the complex skew–plane with conjugate as inverse.

The set of elements of the complex skew–plane with conjugate as inverse is a compact group which is the kernel of the homomorphism of the multiplicative group of the complex skew–plane onto the positive half–line. The canonical measure for the complex skew–plane, as it acts on the multiplicative group, is the Cartesian product measure of an invariant measure on the compact subgroup and the measure on the positive half–line whose value on a Baire set is the integral

$$\int tdt$$

with respect to Lebesgue measure over the set. Measure preserving transformations of the compact group onto itself are defined by taking ξ into $\omega\xi$ and into $\xi\omega$ for every element ω of the group. The integral

$$\int |f(\xi)|^2 d\xi = 2\pi \|f\|^2$$

with respect to the invariant measure computes the scalar self–product of a function $f(\xi)$ of ξ in the complex skew–plane which is a homogeneous polynomial of degree ν.

The Laplacian

$$\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is applied in the decomposition of the Hilbert space of homogeneous polynomials of degree ν into invariant subspaces. The Laplacian annihilates homogeneous polynomials of degree less than two and takes homogeneous polynomials of greater degree ν into homogeneous polynomials of degree $\nu - 2$. The Laplacian commutes with the transformations which take a function $f(\xi)$ of ξ into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ for every element ω of the complex skew–plane with conjugate as inverse. A homogeneous polynomial of degree ν is said to be harmonic if it is annihilated by the Laplacian. Homogeneous polynomials of degree less than two are harmonic. Homogeneous polynomials of degree ν greater than two are harmonic if, and only if, they are orthogonal to products of $\xi^-\xi$ with homogeneous polynomials of degree $\nu - 2$. The dimension of the space of homogeneous harmonic polynomials of degree ν is

$$(1 + \nu)^2.$$

A conjugated ideal of the ring of integral elements of the complex skew–plane is generated by two. The quotient ring contains sixteen elements, of which twelve are invertible. The group of invertible elements of the quotient ring has a normal subgroup of four elements which is noncommutative and whose quotient group is cyclic of three elements.

Hecke operators are transformations which act on functions $f(\xi)$ of ξ in the complex skew–plane which are homogeneous of degree zero. The identity

$$f(\lambda\xi) = f(\xi)$$

holds for every positive number λ.

A Hecke operator $\Delta(n)$ is defined for every positive integer n. The transformation takes a function $f(\xi)$ of ξ in the complex skew–plane into the function $g(\xi)$ of ξ in the complex skew–plane defined by the sum

$$24\ g(\xi) = \sum f(\xi\omega)$$

over the integral elements ω of the complex skew–plane which represent

$$n = \omega^- \omega.$$

The transformation $\Delta(1)$ is a projection onto the set of functions $f(\xi)$ of ξ in the complex skew–plane which satisfy the identity

$$f(\xi) = f(\xi\omega)$$

for every integral element ω of the complex skew–plane with integral inverse.

The identity

$$\Delta(m)\Delta(n) = \sum \Delta(n/k^2)$$

holds for all positive integers m and n with summation over the odd positive integers k which are common divisors of m and n.

Multiplication by

$$(\xi^- \xi)^{-\frac{1}{2}\nu}$$

is an isometric transformation of the Hilbert space of homogeneous harmonic polynomials of degree ν onto a Hilbert space of homogeneous functions of degree zero. The Hecke operator $\Delta(n)$ is a self–adjoint transformation of the space into itself for every positive integer n. The image space is the orthogonal sum of invariant subspaces whose elements are eigenfunctions of Hecke operators.

The eigenvalue of $\Delta(n)$ is a real number $\tau(n)$ for every positive integer n. The identity

$$\tau(m)\tau(n) = \sum \tau(mn/k^2)$$

holds for all positive integers m and n with summation over the odd positive integers k which are common divisors of m and n.

The complementary space to the complex plane in the complex skew–plane is the set of elements η of the complex skew–plane which satisfy the identity

$$\xi\eta = \eta\xi^-$$

for every element ξ of the complex plane. An element η of the complex skew–plane is skew–conjugate:

$$\eta^- = -\eta.$$

Multiplication on left or right by η is an injective transformation of the complex plane onto the complementary space for every nonzero element η of the complementary space. The transformation is continuous when the complementary space is given the topology inherited from the complex skew–plane. The transformation takes the canonical measure for the complex plane into $\eta^-\eta$ times the measure defined as the canonical measure for the complementary space.

An element of the complex skew–plane is the unique sum $\alpha + \beta$ of an element α of the complex plane and an element β of the complementary space. The topology of the complex skew–plane is the Cartesian product topology of the topology of the complex plane and the topology of the complementary space. The canonical measure for the complex skew–plane is the Cartesian product measure of the canonical measure for the complex plane and the canonical measure for the complementary space.

The Radon transformation for the complex skew–plane is a transformation with closed graph whose domain and range are contained in the Hilbert space of functions which are square integrable with respect to the canonical measure for the complex skew–plane and whose graph contains the pair $(f(\omega\xi), g(\omega\xi))$ of functions of ξ for every element ω of the complex skew–plane with conjugate as inverse whenever it contains the pair $(f(\xi), g(\xi))$ of functions of ξ. The transformation is defined as an integral on elements of its domain which are integrable with respect to the canonical measure.

The Radon transform of an integrable function $f(\xi)$ of ξ in the complex skew–plane is a function $g(\xi)$ of ξ in the complex skew–plane which satisfies the identity

$$g(\omega\xi) = \int f(\omega\xi + \omega\eta) d\eta$$

for every element ω of the complex skew–plane with conjugate as inverse when ξ is in the complex plane with integration with respect to the canonical measure for the complementary space to the complex plane in the complex skew–plane. The inequality

$$\int |g(\omega\xi)| d\xi \leq \int |f(\xi)| d\xi$$

holds for every element ω of the complex skew–plane with conjugate as inverse with integration on the left with respect to the canonical measure for the complex plane and integration on the right with respect to the canonical measure for the complex skew–plane.

The Radon transformation for the complex skew–plane factors the Fourier transformation for the complex skew–plane as a composition with the Fourier transformation for the complex plane. If the Radon transformation takes a function $f(\xi)$ of ξ into a function $g(\xi)$ of ξ and if the function $f(\xi)$ of ξ is integrable with respect to the canonical measure for the complex skew–plane, then the function $g(\omega\xi)$ of ξ is integrable with respect to the canonical measure for the complex plane for every element ω of the complex skew–plane with conjugate as inverse. The Fourier transform of the function $g(\omega\xi)$ of ξ in the complex plane is the restriction to the complex plane of the Fourier transform of the function $f(\omega\xi)$ of ξ in the complex skew–plane.

The Radon transformation for the complex skew–plane is a subnormal operator whose spectral analysis is made by Laplace transformations. A Laplace transformation of harmonic ϕ is defined for every homogeneous harmonic polynomial $\phi(\xi)$ of degree ν such that the normalization

$$\int |\phi(\xi)|^2 d\xi = \int (\xi^- \xi)^\nu d\xi$$

applies with integration with respect to the canonical measure for the complex skew–plane over the unit disk $\xi^- \xi < 1$. The function

$$\phi(\xi) \exp(\pi i z \xi^- \xi)$$

of ξ in the complex skew–plane is an eigenfunction of the Radon transformation for the eigenvalue

$$i/z$$

when z is in the upper half–plane.

The domain of the Laplace transformation of harmonic ϕ is the Hilbert space of functions $f(\xi)$ of ξ in the complex skew–plane which are square integrable with respect to the canonical measure and which satisfy the identity

$$\phi(\xi) f(\omega \xi) = \phi(\omega \xi) f(\xi)$$

for every element ω of the complex skew–plane with conjugate as inverse.

The canonical measure for the upper half–plane is defined as the restriction to Baire subsets of the upper half–plane of the canonical measure for the complex plane.

A function

$$f(\xi) = \phi(\xi)(\xi^- \xi)^{-\frac{1}{2}\nu} h(\tfrac{1}{2}\xi^- \xi)$$

of ξ in the complex skew–plane which belongs to the domain of the Laplace transformation of harmonic ϕ is parametrized by a function $h(\xi)$ of ξ in the upper half–plane which is square integrable with respect to the canonical measure for the upper half–plane and which admits an extension to the complex plane satisfying the identity

$$h(\omega \xi) = h(\xi)$$

for every element ω of the complex plane with conjugate as inverse. The identity

$$\int |f(\xi)|^2 d\xi = 16 \int |h(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the complex skew–plane and integration on the right with respect to the canonical measure for the upper half–plane.

The Laplace transform of harmonic ϕ of the function $f(\xi)$ of ξ in the complex skew–plane is defined as the analytic function

$$g(z) = \pi \int_0^\infty t^{\frac{1}{2}\nu} h(t) \exp(2\pi i t z) t dt$$

of z in the upper half–plane. The identity

$$\int_0^\infty \int_{-\infty}^{+\infty} |g(x+iy)|^2 y^\nu dx dy = \tfrac{1}{4}(4\pi)^{-\nu} \Gamma(1+\nu) \int |h(\xi)|^2 d\xi$$

holds with integration on the right with respect to the canonical measure for the upper half–plane. An analytic function $g(z)$ of z in the upper half–plane is a Laplace transform of harmonic ϕ if the integral on the left converges.

The domain of the Laplace transformation of harmonic ϕ is an invariant subspace for the Radon transformation and its adjoint. The Radon transformation acts as a maximal dissipative transformation on the space. The adjoint takes a function $f(\xi)$ of ξ into a function $g(\xi)$ of ξ when the identity

$$\int \phi(\xi)^- g(\xi) \exp(\pi i z \xi^- \xi) d\xi = (i/z) \int \phi(\xi)^- f(\xi) \exp(\pi i z \xi^- \xi) d\xi$$

holds when z is in the upper half–plane with integration with respect to the canonical measure for the complex skew–plane.

The Fourier transform for the complex skew–plane of the function

$$\phi(\xi) \exp(\pi i z \xi^- \xi)$$

of ξ in the complex skew–plane is the function

$$i^\nu (i/z)^{2+\nu} \phi(\xi) \exp(-\pi i z^{-1} \xi^- \xi)$$

of ξ in the complex skew–plane when z is in the upper half–plane. Since the Fourier transformation commutes with the transformations which take a function $f(\xi)$ of ξ into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ for every element ω of the complex skew–plane with conjugate as inverse, it is sufficient to make the verification when

$$\phi(t + ix + jy + kz) = (t + ix)^\nu.$$

The Radon transformation for the complex skew–plane reduces the verification to showing that the Fourier transform for the complex plane of the function

$$\xi^\nu \exp(\pi i z \xi^- \xi)$$

of ξ in the complex plane is the function

$$i^\nu (i/z)^{1+\nu} \xi^\nu \exp(-\pi i z^{-1} \xi^- \xi)$$

of ξ in the complex plane. It is sufficient by analytic continuation to make the verification when z lies on the imaginary axis. It remains by a change of variable to show that the Fourier transform of the function

$$\xi^\nu \exp(-\pi \xi^- \xi)$$

of ξ in the complex plane is the function

$$i^\nu \xi^{-\nu} \exp(-\pi \xi^- \xi)$$

of ξ in the complex plane.

The desired identity

$$i^\nu \xi^\nu \exp(-\pi\xi^-\xi) = \sum_{k=0}^{\infty} \xi^\nu \int \frac{(\pi i\xi^-\xi)^k(\pi i\eta^-\eta)^{\nu+k}}{k!(\nu+k)!} \exp(-\pi\eta^-\eta)d\eta$$

follows since

$$\exp(\pi i(\xi^-\eta + \eta^-\xi)) = \sum_{n=0}^{\infty} \frac{(\pi i\xi^-\eta + \pi i\eta^-\xi)^n}{n!}$$

and since

$$(\pi i\xi^-\eta + \pi i\eta^-\xi)^n = \sum_{k=0}^{\infty} \frac{(\pi i\eta^-\xi)^{n-2k}(\pi i\xi^-\xi)^k(\pi i\eta^-\eta)^k}{k!(n-k)!}$$

where

$$\int (\pi i\eta^-\eta)^{\nu+k} \exp(-\pi\eta^-\eta)d\eta = i^{\nu+k}(\nu+k)!$$

and

$$i^\nu \exp(-\pi\xi^-\xi) = \sum_{k=0}^{\infty} i^{\nu+k} \frac{(\pi i\xi^-\xi)^k}{k!}$$

Integrations are with respect to the canonical measure for the complex plane. Interchanges of summation and integration are justified by absolute convergence.

If a function $f(\xi)$ of ξ in the complex skew–plane is square integrable with respect to the canonical measure and satisfies the identity

$$\phi(\xi)f(\omega\xi) = \phi(\omega\xi)f(\xi)$$

for every element ω of the complex skew–plane with conjugate as inverse, then its Fourier transform is a function $g(\xi)$ of ξ in the complex skew–plane which is square integrable with respect to the canonical measure and which satisfies the identity

$$\phi(\xi)g(\omega\xi) = \phi(\omega\xi)g(\xi)$$

for every element ω of the complex skew–plane with conjugate as inverse. The Laplace transforms of harmonic ϕ are functions $F(z)$ and $G(z)$ of z in the upper half–plane which satisfy the identity

$$G(z) = i^\nu (i/z)^{2+\nu} F(-1/z).$$

A construction of Euler weight functions is obtained on applying the Mellin transformation. The Mellin transformation reformulates the Fourier transformation for the real line on the multiplicative group of the positive half–line. Analytic weight functions constructed from the gamma function appear when the Mellin transformation is adapted to the domain of the Laplace transformation of harmonic ϕ.

The Mellin transform of harmonic ϕ of the function $f(\xi)$ of ξ in the complex skew–plane is an analytic function $F(z)$ of z in the upper half–plane which is defined when the function of ξ vanishes in the disk $\xi^-\xi < a$ for some positive number a. The function is defined by the integral

$$\pi F(z) = \int_0^\infty g(it) t^{\frac{1}{2}\nu - iz} dt.$$

Since

$$g(iy) = \pi \int_0^\infty t^{\frac{1}{2}\nu} h(t) \exp(-2\pi t y) t dt$$

when y is positive, the identity

$$F(z)/W(z) = \int_0^\infty h(it) t^{iz} dt$$

holds with

$$W(z) = (2\pi)^{-\frac{1}{2}\nu - 1 + iz} \Gamma(\tfrac{1}{2}\nu + 1 - iz).$$

The identity

$$\int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 dx = 2\pi \int_0^\infty |h(it)|^2 t^{-2y} t dt$$

holds when y is positive.

The analytic function

$$2^{-iz} a^{iz} F(z)$$

of z in the upper half–plane belongs to the weighted Hardy space $\mathcal{F}(W)$ since the function $f(\xi)$ of ξ in the complex skew–plane vanishes when $\xi^-\xi < a$. An analytic function $F(z)$ of z in the upper half–plane such that the function

$$2^{-iz} a^{iz} F(z)$$

of z belongs to the space $\mathcal{F}(W)$ is the Mellin transform of a function $f(\xi)$ of ξ in the complex skew–plane which belongs to the domain of the Laplace transformation of harmonic ϕ and vanishes in the disk $\xi^-\xi < a$.

The Radon transformation for the complex skew–plane takes elements of the domain of the Laplace transformation of harmonic ϕ which vanish in the disk $\xi^-\xi < a$ into elements of the space which vanish in the disk. The Radon transformation acts as a maximal dissipative transformation on the subspace. The adjoint is a maximal dissipative transformation which is unitarily equivalent to multiplication by

$$i/z$$

in the Hilbert space which is the image of the subspace under the Laplace transformation of harmonic ϕ. When $\xi^-\xi < 1$ is the unit disk, the transformation is unitarily equivalent to the transformation which takes $F(z)$ into $F(z-i)$ whenever the functions of z belong to the space $\mathcal{F}(W)$.

An isometric transformation of the space of square integrable functions with respect to the canonical measure for the complex skew–plane into itself is defined by taking a function $f(\xi)$ of ξ in the complex skew–plane into the function

$$\exp(\pi i(\beta^{-}\xi + \xi^{-}\beta)) f(\xi + \alpha)$$

of ξ in the complex skew–plane for every pair (α, β) of elements of the complex skew–plane.

A matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with entries in the complex skew–plane is said to be symplectic if it has the matrix

$$\begin{pmatrix} D^{-} & -B^{-} \\ -C^{-} & A^{-} \end{pmatrix}$$

as inverse.

An isometric transformation of the space of square integrable functions with respect to the canonical measure for the complex skew–plane into itself is said to be symplectic with respect to a symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

if the transformation takes the function

$$\exp(\tfrac{1}{2}\pi i(\alpha^{-}\beta + \beta^{-}\alpha)) \exp(\pi i(\beta^{-}\xi + \xi^{-}\beta)) f(\xi + \alpha)$$

of ξ in the complex skew–plane into the function

$$\exp(\tfrac{1}{2}\pi i(\gamma^{-}\delta + \delta^{-}\gamma)) \exp(\pi i(\delta^{-}\xi + \xi^{-}\delta)) g(\xi + \gamma)$$

of ξ in the complex skew–plane whenever it takes a function $f(\xi)$ of ξ in the complex skew–plane into a function $g(\xi)$ of ξ in the complex skew–plane if (α, β) and (γ, δ) are pairs of elements of the complex skew–plane such that

$$(\gamma, \delta) = (\alpha, \beta) \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The composition of a symplectic transformation with respect to a symplectic matrix

$$\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$$

and a symplectic transformation with respect to a symplectic matrix

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

is a symplectic transformation with respect to the product symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}.$$

The inverse of a symplectic transformation with respect to a symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a symplectic transformation with respect to the inverse symplectic matrix

$$\begin{pmatrix} D^- & -B^- \\ -C^- & A^- \end{pmatrix}.$$

The Fourier transformation for the complex skew–plane is a symplectic transformation with respect to the symplectic matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A symplectic transformation with respect to the symplectic matrix

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

is defined for every real number λ by taking a function $f(\xi)$ of ξ in the complex skew–plane into the function

$$\exp(-\pi i \lambda \xi^- \xi) f(\xi)$$

of ξ in the complex skew–plane.

A symplectic transformation with respect to the symplectic matrix

$$\begin{pmatrix} \omega^{-1} & 0 \\ 0 & \omega^- \end{pmatrix}$$

is defined for every nonzero element ω of the complex skew–plane by taking a function $f(\xi)$ of ξ in the complex skew–plane into the function

$$\omega^- \omega f(\xi \omega)$$

of ξ in the complex skew–plane.

A symplectic transformation exists with respect to every symplectic matrix and is unique within a constant factor of absolute value one. The span of the domain and range of the Fourier transformation of harmonic ϕ is an invariant subspace of every symplectic matrix.

If

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a symplectic matrix with entries in the complex plane, the restriction of a symplectic transformation with respect to the inverse symplectic matrix

$$\begin{pmatrix} D^- & -B^- \\ -C^- & A^- \end{pmatrix}$$

is defined by taking a function $f(\xi)$ of ξ in the complex skew–plane into a function $g(\xi)$ when the identity

$$G(z) = \frac{1}{(Cz+D)^{2+\nu}} F\left(\frac{Az+B}{Cz+D}\right)$$

holds for the analytic function $F(z)$ of z in the upper half–plane which is the Laplace transform of harmonic ϕ of the function $f(\xi)$ of ξ in the complex skew–plane and the analytic function $G(z)$ of z in the upper half–plane which is the Laplace transform of harmonic ϕ of the function $g(\xi)$ of ξ in the complex skew–plane.

3. Fourier Analysis on an r–adic Skew–Plane

The integral elements of the complex plane according to Gauss are the elements

$$t + ix$$

whose coordinates are integers. The Gauss plane is the set of elements of the complex plane whose coordinates are rational numbers. The Gauss skew–plane is the set of elements

$$t + ix + jy + kz$$

of the complex skew–plane whose coordinates are rational numbers. The integral elements of the Gauss skew–plane according to Hurwitz are the elements whose coordinates are all integers or all halves of odd integers. The r–adic skew–plane is defined for a positive integer r other than one as the completion of the Gauss skew–plane in the r–adic topology of the Gauss skew–plane. An adic skew–plane is defined by removing the constraints supplied by the positive integer r.

The topology is determined on the ring of integral elements of the Gauss skew–plane. A nonzero element ρ of the ring generates a left ideal whose quotient space contains $(\rho^-\rho)^2$ elements. A quotient ring is obtained when ρ is a positive integer since the left ideal is also a right–ideal. The quotient ring inherits a conjugation since the ideal contains the conjugate ξ^- of every element ξ. Representatives in equivalences classes modulo ρ can be chosen when r is odd as elements of the Gauss skew–plane whose coordinates are integers. An element represents the origin of the quotient ring if, and only if, its coordinates are divisible by ρ. The quotient ring is isomorphic to the ring of quaternions

$$t + ix + jy + kz$$

whose coordinates are integers modulo ρ.

The discrete topology is the unique Hausdorff topology of the quotient ring. Addition and multiplication are continuous as transformations of the Cartesian product of the quotient ring with itself into the quotient ring. Conjugation is continuous as a transformation of the quotient ring into itself.

The r–adic topology of the ring of integral elements of the Gauss skew–plane is defined as the least topology with respect to which the homomorphism onto the quotient ring modulo ρ is continuous for every positive integer ρ whose prime divisors are divisors of r. The ring of integral elements is a compact Hausdorff space in the r–adic topology. Addition and multiplication are continuous as transformations of the Cartesian product of the ring with itself into the ring. Conjugation is continuous as a transformation of the ring into itself.

Multiplication by a positive integer ρ whose prime divisors are divisors of r is a homomorphism of the ring of integral elements onto a closed ideal which is given its subspace topology. The r–adic topology of the Gauss skew–plane is defined so that multiplication by ρ is a homeomorphism of the space onto itself for every positive integer ρ whose prime divisors are divisors of r and so that the ring of integral elements is a closed subspace with the given topology as subspace topology.

Addition is continuous as a transformation of the Cartesian product of the Gauss skew–plane with itself into the Gauss skew–plane when the Gauss skew–plane is given the r–adic topology. Uniformity of topology is determined by translations of the additive group. The r–adic skew–plane is defined as the completion in the uniform topology. Addition extends continuously as a transformation of the Cartesian product of the r–adic skew–plane with itself into the r–adic skew–plane. Conjugation extends continuously as a transformation of the r–adic skew–plane onto itself.

An element of the r–adic skew–plane is defined as integral if it belongs to the closure of the ring of integral elements of the Gauss skew–plane. Multiplication extends continuously as a transformation of the Cartesian product of the set of integral elements with itself into the set. Multiplication by an element of the Gauss skew–plane extends continuously as a transformation of the r–adic skew–plane into itself. An integral element of the r–adic skew–plane is said to be a unit if it is the inverse of an integral element of the r–adic skew–plane. An invertible element of the r–adic skew–plane is the product of a unit and an element of the Gauss skew–plane. The set of integral elements of the r–adic skew–plane is a compact ring which has the r–adic skew–plane as ring of quotients. Conjugation is an anti–automorphism of the r–adic skew–plane which takes integral elements into integral elements.

The canonical measure for the r–adic skew–plane is the unique nonnegative measure on its Baire subsets such that a measure preserving transformation is defined by taking ξ into $\xi + \eta$ for every element η and such that the ring of integral elements has measure one when r is odd and two when r is even. If ξ is an element of the r–adic skew–plane, define the r–adic modulus of ξ as the unique positive number $\lambda_r(\xi)$ such that $\lambda_r(\xi)^2$ is rational and such that

$$\lambda_r(\xi)^2 \xi^- \xi$$

is a unit if ξ is invertible and as zero otherwise. Multiplication by ξ multiplies the canonical measure by a factor of

$$\lambda_r(\xi)^4 = \lambda_r(\xi^- \xi)^2.$$

Conjugation is a measure preserving transformation.

The r–adic line is a commutative subring of the r–adic skew–plane whose elements are the self–conjugate elements of the r–adic skew–plane. The r–adic skew–plane is the algebra of quaternions

$$t + ix + jy + kz$$

with coordinates in the r–adic line. The r–adic line is given the subspace topology of the r–adic skew–plane. The r–adic topology of the r–adic skew–plane is the Cartesian product topology of the topologies of four r–adic lines.

The canonical measure for the r–adic line is the unique nonnegative measure on its Baire subsets such that a measure preserving transformation is defined by taking ξ into $\xi + \eta$ for every element η and such that the set of integral elements has measure one. The canonical measure for the r–adic skew–plane is the Cartesian product measure of the canonical measures for four r–adic lines. Multiplication by an element ξ of the r–adic line multiplies the canonical measure by a factor of $\lambda_r(\xi)$.

The r–adic line is the field p–adic numbers when r is a prime p. The r–adic skew–plane is a skew–field when r is a prime p. The r–adic skew–plane is the Cartesian product of the p–adic skew–planes taken over the prime divisors p of r.

A continuous homomorphism of the group of invertible elements of the r–adic skew–plane onto the group of invertible elements of the r–adic line is defined by taking ξ into $\xi^- \xi$. The homomorphism takes the canonical measure for the r–adic skew–plane into a nonnegative measure on the Baire subsets of invertible elements of the r–adic line whose value on a set is the integral

$$\prod (1 + p^{-1}) \int \lambda_r(\xi) d\xi$$

over the set with respect to the canonical measure for the r–adic line when r is odd and twice the same integral when r is even with the product taken over the prime divisors p of r.

The function $\exp(2\pi i \xi)$ of rational numbers ξ admits a unique continuous extension as a function of ξ in the r–adic line. The function acts as a homomorphism of the additive group of the r–adic line into the multiplicative group of complex numbers of absolute value one.

The Fourier transformation for the r–adic line is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the r–adic line into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(2\pi i \xi \eta) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure for the r–adic line. Fourier inversion states that the integral

$$f(\xi) = \int \exp(-2\pi i \xi \eta) g(\eta) d\eta$$

with respect to the canonical measure for the r–adic line represents the function $f(\xi)$ of ξ if the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The Fourier transformation for the r–adic skew–plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the r–adic skew–plane into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i (\xi^- \eta + \eta^- \xi)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure for the r–adic skew–plane. Fourier inversion states that the integral

$$f(\xi) = \int \exp(-\pi i (\xi^- \eta + \eta^- \xi)) g(\eta) d\eta$$

with respect to the canonical measure for the r–adic skew–plane represents the function $f(\xi)$ of ξ if the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

An r–adic plane is a maximal commutative subring of the r–adic skew–plane which is isomorphic to a Cartesian product of p–adic taken over the prime divisors p of r. A p–adic plane is determined by an integral element ι_p of the complex skew–plane which represents

$$p = \iota_p^- \iota_p.$$

The elements of the p–adic plane are the elements of the p–adic skew–plane which commute with ι_p. An element

$$\xi = \alpha + \iota_p \beta$$

of the p–adic plane has coordinates α and β in the p–adic line. The conjugate

$$\xi^- = \alpha + \iota_p^- \beta$$

of an element ξ of the p–adic plane is an element of the p–adic plane since

$$\iota_p + \iota_p^-$$

is an integer. An element of the p–adic plane is integral if, and only if, its coordinates are integral elements of the p–adic line.

An ideal of the ring of integral elements of the p–adic plane which contains a nonzero element contains ι_p^n for some nonnegative integer n. If n is the least nonnegative integer such that ι_p^n belongs to the ideal, then multiplication by ι_p^n takes the ring of integral elements of the p–adic plane onto the ideal. The quotient ring modulo the ideal contains p^n elements.

The construction of the p–adic plane requires an integral element ι_p of the complex skew–plane which represents

$$p = \iota_p^- \iota_p.$$

Such an element is

$$\iota_p = 1 + i$$

when p is the even prime. The integral element ι_p is otherwise obtained by application of the Euclidean algorithm.

When p is an odd prime, the elements of the quotient ring of the ring of integral elements of the complex skew–plane modulo the ideal generated by p are represented

$$\xi = t + ix + jy + kz$$

with coordinates in the integers modulo p. The image of ι_p in the quotient ring is a nonzero element ξ such that

$$\xi^- \xi = 0.$$

Such an element can be chosen with

$$z = 0 \text{ and } t = 1.$$

Since there are $(p+1)/2$ squares of integers modulo p, the coordinates x and y can be chosen as integers modulo p such that

$$1 + x^2 = -y^2.$$

A left ideal of the ring of integral elements of the p–adic skew–plane is defined as the set of elements whose image in the quotient ring belongs to the left ideal generated by an element ξ. Since the ideal contains a nonzero element, it is generated by a nonzero element ι_p which minimizes the positive integer $\iota_p^- \iota_p$. The element is unique within a left factor of an integral unit of the complex skew–plane.

The ring of integral elements of the r–adic plane is a compact Hausdorff space in the subspace topology inherited from the r–adic skew–plane. Addition and multiplication are continuous as transformations of the Cartesian product of the ring with itself into the ring. In the subspace topology inherited from the r–adic skew–plane the r–adic plane is a Hausdorff space which contains the ring of integral elements as an open and closed subset containing the origin. A continuous transformation of the r–adic plane into itself is defined by taking ξ into $\xi + \eta$ for every element η of the r–adic plane.

The canonical measure for the r–adic plane is the unique nonnegative measure on its Baire subsets such that a measure preserving transformation of the space into itself is defined by taking ξ into $\xi + \eta$ for every element η of the space and such that the ring of integral elements has measure one. Multiplication by an element ξ of the r–adic plane multiplies the canonical measure by a factor of $\lambda_r(\xi)^2 = \lambda_r(\xi^- \xi)$.

The conjugation of the r–adic skew–plane acts as a continuous isomorphism of the r–adic plane onto itself. The set of self–conjugate elements of the r–adic plane is the r–adic line. If ι_r is the element of the r–adic plane whose p–adic component is ι_p for every prime divisor p of r, then an element

$$\xi = \alpha + \iota_r \beta$$

of the r–adic plane has coordinates α and β in the r–adic line. The topology of the r–adic plane is the Cartesian product topology of the r–adic topologies of two r–adic lines. The canonical measure for the r–adic plane is the Cartesian product measure of the canonical measures of two r–adic lines.

The complementary space to the r–adic plane in the r–adic skew–plane is the set of elements η of the r–adic skew–plane which satisfy the identity

$$\xi \eta = \eta \xi^-$$

for every element ξ of the r–adic plane. An element η of the complementary space to the r–adic plane in the r–adic skew–plane is skew–conjugate:

$$\eta^- = -\eta.$$

Multiplication on left or right by an invertible element of the complementary space is an injective transformation of the r–adic plane onto the complementary space. The transformation is continuous when the complementary space is given the topology inherited from the r–adic skew–plane. The canonical measure for the complementary space is defined as the image of the canonical measure for the r–adic plane under multiplication by a unit of the complementary space.

An element of the r–adic skew–plane is the unique sum $\alpha + \beta$ of an element α of the r–adic plane and an element β of the complementary space. The topology of the r–adic skew–plane is the Cartesian product topology of the topology of the r–adic plane and the topology of the complementary space. The canonical measure for the r–adic skew–plane is the Cartesian product measure of the canonical measure for the r–adic plane and the canonical measure for the complementary space.

The Fourier transformation for the r–adic plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the r–adic plane into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^- \eta + \eta^- \xi)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^- \eta + \eta^- \xi)) g(\eta) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous. The Fourier transformation for the r–adic skew–plane commutes with the transformation which take a function $f(\xi)$ of ξ into the functions $f(\omega\xi)$ of ξ for every element ω of the r–adic skew–plane with conjugate as inverse.

The Radon transformation for the r–adic skew–plane is a transformation with domain and range in the Hilbert space of square integrable functions with respect to the canonical measure which has a closed graph and which commutes with the transformation which takes a function $f(\xi)$ of ξ into the function $f(\omega\xi)$ of ξ for every element ω of the r–adic skew–plane with conjugate as inverse: The graph contains the pair $(f(\omega\xi), g(\omega\xi))$ of functions of ξ whenever it contains the pair $(f(\xi), g(\xi))$ of functions of ξ. The transformation is defined as an integral on those elements of its domain which are integrable with respect to the canonical measure.

The Radon transform of an integrable function $f(\xi)$ of ξ in the r–adic skew–plane is a function $g(\xi)$ of ξ in the r–adic skew–plane defined by the integral

$$g(\omega\xi) = \int f(\omega\xi + \omega\eta) d\eta$$

with respect to the canonical measure for the complementary space to the r–adic plane in the r–adic skew–plane when ξ is in the r–adic plane for every element ω of the r–adic skew–plane with conjugate as inverse. The inequality

$$\int |g(\omega\xi)| d\xi \leq \int |f(\xi)| d\xi$$

holds for every element ω of the r–adic skew–plane with conjugate as inverse with integration on the left with respect to the canonical measure for the r–adic plane and with integration on the right with respect to the canonical measure for the r–adic skew–plane.

The Radon transformation for the r–adic skew–plane factors the Fourier transformation for the r–adic skew–plane as a composition with the Fourier transformation for the r–adic plane. If the Radon transformation takes a function $f(\xi)$ of ξ into a function $g(\xi)$ of ξ and if the function $f(\xi)$ of ξ is integrable with respect to the canonical measure for the r–adic skew–plane, then the function $g(\omega\xi)$ of ξ is integrable with respect to the canonical measure for the r–adic plane for every element ω of the r–adic skew–plane with conjugate as inverse. The restriction to the r–adic plane of the Fourier transform of the function $f(\omega\xi)$ of ξ in the r–adic skew–plane is the Fourier transform of the function $g(\omega\xi)$ of ξ in the r–adic plane.

The Radon transformation for the r–adic skew–plane commutes with the transformation which takes a function $f(\xi)$ of ξ in the r–adic skew–plane into the function $f(\omega\xi)$ of ξ in the r–adic skew–plane for every unit ω of the r–adic skew–plane. The Hilbert space of functions $f(\xi)$ of ξ in the r–adic skew–plane which are square integrable with respect to the canonical measure and which satisfy the identity

$$f(\omega\xi) = f(\xi)$$

for every element ω of the r–adic skew–plane with conjugate as inverse is the orthogonal sum of invariant subspaces defined by primitive characters χ modulo ρ for positive integers ρ whose prime divisors are divisors of r.

A primitive character χ modulo ρ admits an extension as a continuous function of ξ in the r–adic line which vanishes when ξ is not integral and which has equal values at integral elements which are congruent modulo ρ. The identity

$$\chi(\xi\eta) = \chi(\xi)\chi(\eta)$$

holds for all integral elements ξ and η of the r–adic line.

The invariant subspace for the Radon transformation defined by a primitive character χ modulo ρ is the set of functions $f(\xi)$ of ξ in the r–adic skew–plane which are square integrable with respect to the canonical measure for the r–adic skew–plane and which satisfy the identity

$$f(\omega\xi) = \chi(\omega^-\omega)f(\xi)$$

for every unit ω of the r–adic skew–plane.

An element of the domain of the Radon transformation for the r–adic skew–planes is the orthogonal sum of components belonging to subspaces defined by characters. Every component belongs to the domain of the Radon transformation and is mapped by the Radon transformation into the same subspace. The graph of the Radon transformation for the r–adic skew–plane is decomposed into components defined by characters. The decomposition applies to functions $f(\xi)$ of ξ in the r–adic skew–plane which satisfy the identity

$$f(\omega\xi) = f(\xi)$$

for every element ω of the complex skew–plane with conjugate as inverse. The Radon transformation is not treated for other square integrable functions with respect to the canonical measure for the complex skew–plane.

An r–adic half–plane is defined as a maximal commutative subring of the r–adic skew–plane which is isomorphic to a Cartesian product of p–adic half–planes taken over the prime divisors p of r. A p–adic half–plane is defined as a maximal commutative subring of the p–adic skew–plane which is not a p–adic plane. An element of the p–adic half–plane is the product of an element of the p–adic line and a unit of the p–adic half–plane. An element of the p–adic line with conjugate as inverse is either the unit or minus the unit of the p–adic line.

When p is the even prime or is an odd prime which is not congruent to one modulo three, a p–adic half–plane is defined as the set of elements of the p–adic skew–plane which commute with one of the units

$$\pm\tfrac{1}{2} \pm \tfrac{1}{2}i \pm \tfrac{1}{2}j \pm \tfrac{1}{2}k.$$

When an odd prime p is not congruent to one modulo four, a p–adic half–plane is defined as the set of elements of the p–adic skew–plane which commute with one of the units

$$i, j, k.$$

In the subspace topology inherited from the r–adic skew–plane the r–adic half–plane is a Hausdorff space containing the ring of integral elements as an open and compact subset containing the origin. Addition is continuous as a transformation of the Cartesian product of the r–adic half–plane with itself into the r–adic half–plane. Multiplication by an element of the r–adic half–plane is a continuous transformation of the r–adic half–plane into itself.

The canonical measure for the r–adic half–plane is the unique nonnegative measure on its Baire subsets such that a measure preserving transformation is defined by taking ξ into $\xi + \eta$ for every element η of the space and such that the ring of integral elements has measure one. Multiplication by an element ξ of the r–adic half–plane multiplies the canonical measure by a factor of $\lambda_r(\xi^-\xi)$. Define $\lambda_r(\xi)$ as the nonnegative solution of the equation

$$\lambda_r(\xi)^2 = \lambda_r(\xi^-\xi)$$

when ξ is in the r–adic half–plane.

The conjugation of the r–adic skew–plane acts as a continuous isomorphism of the r–adic half–plane onto itself. The r–adic line is the set of self–conjugate elements of the r–adic half–plane. If an integral element κ of the complex skew–plane belongs to the r–adic half–plane and has no self–conjugate p–adic component for a prime divisor p of $\kappa^-\kappa$ then an element

$$\alpha + \kappa\beta$$

of the r–adic half–plane is decomposed with coordinates α and β in the r–adic line.

The topology of the r–adic half–plane is the Cartesian product topology of the topologies of two r–adic lines. The canonical measure for the r–adic half–plane is a constant multiple of the Cartesian product of the canonical measures of two r–adic lines.

The Fourier transformation for the r–adic half–plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the r–adic half–plane into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^-\eta + \eta^-\xi))f(\eta)d\eta$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^-\eta + \eta^-\xi))g(\eta)d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The Fourier transformation for the r–adic half–plane commutes with the transformation which takes a function $f(\xi)$ of ξ into the function $f(\omega\xi)$ of ξ for every element ω of the r–adic half–plane with conjugate as inverse.

A primitive character χ modulo ρ for a positive integer ρ whose prime divisors are divisors of r is a continuous function $\chi(\xi)$ of ξ in the r–adic line having an extension as a function $\chi(\xi)$ of ξ in the r–adic half–plane which vanishes at nonintegral elements, which has equal values at elements congruent modulo ρ, and which satisfies the identity

$$\chi(\xi\eta) = \chi(\xi)\chi(\eta)$$

for all integral elements ξ and η of the r–adic half–plane. An extension is chosen for the parametrization of functions $f(\xi)$ of ξ in the r–adic skew–plane which satisfy the identity

$$f(\omega\xi) = \chi(\omega^-\omega)f(\xi)$$

for every unit ω of the r–adic skew–plane.

A function

$$f(\xi) = h(\tfrac{1}{2}\xi^-\xi)$$

of ξ in the r–adic skew–plane which is square integrable with respect to the canonical measure for the r–adic skew–plane and which satisfies the identity

$$f(\omega\xi) = \chi(\omega^-\omega)f(\xi)$$

for every unit ω of the r–adic skew–plane is parametrized by a square integrable function $h(\xi)$ of ξ in the r–adic half–plane which satisfies the identity

$$h(\omega\xi) = \chi(\omega)h(\xi)$$

for every unit ω of the r–adic half–plane. The identity

$$\int |f(\xi)|^2 d\xi = \tfrac{1}{2} \int |h(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the r–adic skew–plane and with integration on the right with respect to the canonical measure for the r–adic half–plane.

The Laplace transform of character χ of the function $f(\xi)$ of ξ in the r–adic skew–plane is defined as the function $g(\xi)$ of ξ in the r–adic half–plane which is the Fourier transform of the function $h(\xi)$ of ξ in the r–adic half–plane. The identity

$$\int |f(\xi)|^2 d\xi = \tfrac{1}{2} \int |g(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the r–adic skew–plane and with integration on the right with respect to the canonical measure for the r–adic half–plane.

If χ is an extension to the r–adic half–plane of a primitive character modulo ρ for a positive integer ρ whose prime divisors are divisors of r, a number $\epsilon(\chi)$ of absolute value one exists such that the function

$$\epsilon(\chi)\rho^{-1}\chi^{-}(\rho\xi) = \int \exp(\pi i(\xi^{-}\eta + \eta^{-}\xi))\chi(\eta)d\eta$$

of ξ in the r–adic half–plane is the Fourier transform of the function $\chi(\xi)$ of ξ in the r–adic half–plane. When χ is the unique character modulo one,

$$\epsilon(\chi) = 1.$$

The identity

$$\min(n, \lambda_r(\eta)^{-1}) = \int \exp(2\pi i\xi^{-}\xi\eta)d\xi$$

holds for an invertible element η of the r–adic line with integration with respect to the canonical measure for the r–adic plane over the set of element ξ such that $n\xi^{-}\xi$ is integral for a positive integer n whose prime divisors are divisors of r.

If n is a positive integer whose prime divisors are divisors of r, a continuous transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the r–adic half–plane into itself is defined by taking a function $f(\xi)$ of ξ into the function

$$\min(n, \lambda_r(\xi)^{-1})f(\xi)$$

of ξ. The transformation is self–adjoint and nonnegative. It commutes with the transformation which takes a function $f(\xi)$ of ξ in the r–adic half–plane into the function $f(\omega\xi)$ of ξ in the r–adic half–plane for every unit ω of the r–adic half–plane.

Assume that a function

$$f(\xi) = h(\tfrac{1}{2}\xi^{-}\xi)$$

of ξ in the r–adic skew–plane is square integrable with respect to the canonical measure for the r–adic skew–plane and is parametrized by a function $h(\xi)$ of ξ in the r–adic half–plane which is square integrable with respect to the canonical measure for the r–adic half–plane and has the function $g(\xi)$ of ξ in the r–adic half–plane as Fourier transform.

For every positive integer n whose prime divisors are divisors of r the function

$$g_n(\xi) = \min(n, \lambda_r(\xi)^{-1})g(\xi)$$

of ξ in the r–adic half–plane is square integrable with respect to the canonical measure for the r–adic half–plane and is the Fourier transform of a function $h_n(\xi)$ of ξ in the r–adic half–plane which is square integrable with respect to the canonical measure for the r–adic half–plane and parametrizes the function

$$f_n(\xi) = h_n(\tfrac{1}{2}\xi^{-}\xi)$$

of ξ in the r–adic skew–plane which is square integrable with respect to the canonical measure for the r–adic skew–plane. The identity

$$f_n(\xi) = \int f(\xi + \eta) d\eta$$

holds for ξ in the r–adic plane with integration with respect to the canonical measure for the complementary space to the r–adic plane in the r–adic skew–plane over the set of elements η such that $n\eta^- \eta$ is integral.

If the function

$$g_\infty(\xi) = \lambda_r(\xi)^{-1} g(\xi)$$

of ξ in the r–adic half–plane is square integrable with respect to the canonical measure for the r–adic half–plane, then the function is the Fourier transform of a function $h_\infty(\xi)$ of ξ in the r–adic half–plane which is square integrable with respect to the canonical measure for the r–adic half–plane and which parametrizes the function

$$f_\infty(\xi) = h_\infty(\tfrac{1}{2}\xi^- \xi)$$

of ξ in the r–adic skew–plane which is square integrable with respect to the canonical measure for the r–adic skew–plane.

The construction applies to a function $f(\xi)$ of ξ in the r–adic skew–plane which is square integrable with respect to the canonical measure for the r–adic skew–plane and satisfies the identity

$$f(\omega\xi) = f(\xi)$$

for every element ω of the r–adic skew–plane with conjugate as inverse if for some positive integer n whose prime divisors are divisors of r the function vanishes when $n\xi^- \xi$ is nonintegral. It can be assumed that for some primitive character χ modulo ρ the identity

$$f(\omega\xi) = \chi(\omega^- \omega) f(\xi)$$

holds for every unit ω of the r–adic skew–plane. The parameter function $h(\xi)$ of ξ in the r–adic half–plane is chosen so that the identity

$$h(\omega\xi) = \chi(\omega) h(\xi)$$

holds for every unit ω of the r–adic half–plane. The Radon transformation for the r–adic skew–plane takes the function $f(\xi)$ of ξ in the r–adic skew–plane into the function $f_\infty(\xi)$ of ξ in the r–adic skew–plane.

If no positive integer n whose prime divisors are divisors of r exists such that $f(\xi)$ vanishes when $n\xi^- \xi$ is nonintegral, the function can be approximated by functions for which such a positive integer exists. If n is given, the approximating function is defined to agree with $f(\xi)$ when $n\xi^- \xi$ is integral and to vanish otherwise. The Radon transformation takes the function $f(\xi)$ of ξ into the function $f_\infty(\xi)$ of ξ whenever the function $f_\infty(\xi)$ of ξ is defined. The function $f(\xi)$ of ξ does not belong to the domain of the Radon transformation when the function $f_\infty(\xi)$ is not defined.

The Radon transformation is a self–adjoint and nonnegative transformation in the Hilbert space of functions $f(\xi)$ of ξ in the r–adic skew–plane which are square integrable with respect to the canonical measure for the r–adic skew–plane and which satisfy the identity

$$f(\omega\xi) = f(\xi)$$

for every element ω of the r–adic skew–plane with conjugate as inverse. The Hilbert space is the orthogonal sum of invariant subspaces whose elements are defined as eigenfunctions of the Radon transformation for a given eigenvalue. The eigenvalues are positive rational numbers which are ratios of positive integers whose prime divisors are divisors of r.

An isometric transformation of the subspace of eigenfunctions for the eigenvalue λ onto the subspace of eigenfunctions for the eigenvalue $n\lambda$ is defined when n is a positive integer whose prime divisors are divisors of r and

$$n = \omega^-\omega$$

for an integral element ω of the complex skew–plane. The transformation takes a function $f(\xi)$ of ξ in the r–adic skew–plane into the function $f(\omega\xi)$ of ξ in the r–adic skew–plane.

An isometric transformation of the space of square integrable functions with respect to the canonical measure for the r–adic skew–plane into itself is defined by taking a function $f(\xi)$ of ξ in the r–adic skew–plane into the function

$$\exp(\pi i(\beta^-\xi + \xi^-\beta))f(\xi + \alpha)$$

of ξ in the r–adic skew–plane for every pair (α, β) of elements of the r–adic skew–plane.

A matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with entries in the r–adic skew–plane is said to be symplectic if it has the matrix

$$\begin{pmatrix} D^- & -B^- \\ -C^- & A^- \end{pmatrix}$$

as inverse.

An isometric transformation of the space of square integrable functions with respect to the canonical measure for the r–adic skew–plane into itself is said to be symplectic with respect to a symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

if the transformation takes the function

$$\exp(\tfrac{1}{2}\pi i(\alpha^-\beta + \beta^-\alpha)) \exp(\pi i(\beta^-\xi + \xi^-\beta))f(\xi + \alpha)$$

of ξ in the r–adic skew–plane into the function

$$\exp(\tfrac{1}{2}\pi i(\gamma^-\delta + \delta^-\gamma)) \exp(\pi i(\delta^-\xi + \xi^-\delta)) g(\xi + \gamma)$$

of ξ in the r–adic skew–plane whenever it takes a function $f(\xi)$ of ξ in the r–adic skew–plane into a function $g(\xi)$ of ξ in the r–adic skew–plane if (α, β) and (γ, δ) are pairs of elements of the r–adic skew–plane such that

$$(\gamma, \delta) = (\alpha, \beta) \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The composition of a symplectic transformation with respect to a symplectic matrix

$$\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$$

and a symplectic transformation with respect to a symplectic matrix

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

is a symplectic transformation with respect to the symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}.$$

The inverse of a symplectic transformation with respect to a symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a symplectic transformation with respect to the inverse symplectic matrix

$$\begin{pmatrix} D^- & -B^- \\ -C^- & A^- \end{pmatrix}.$$

The Fourier transformation for the r–adic skew–plane is a symplectic transformation with respect to the symplectic matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A symplectic transformation with respect to the symplectic matrix

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

is defined for every element λ of the r–adic line by taking a function $f(\xi)$ of ξ in the r–adic skew–plane into the function

$$\exp(-\pi i\lambda\xi^-\xi)f(\xi)$$

of ξ in the r–adic skew–plane.

A symplectic transformation with respect to the symplectic matrix

$$\begin{pmatrix} \omega^{-1} & 0 \\ 0 & \omega^- \end{pmatrix}$$

is defined for every invertible element ω of the r–adic skew–plane by taking a function $f(\xi)$ of ξ in the r–adic skew–plane into the function

$$\lambda_r(\omega)^2 f(\xi\omega)$$

of ξ in the r–adic skew–plane.

A symplectic transformation exists for every symplectic matrix and is unique within a constant factor of absolute value one.

4. Fourier Analysis on an r–Adelic Skew–Plane

The r–adelic skew–plane is the Cartesian product of the complex skew–plane and the r–adic skew–plane. An element ξ of the r–adelic skew–plane has a component ξ_+ in the complex skew–plane and a component ξ_- in the r–adic skew–plane. The r–adelic skew–plane is a conjugated ring with coordinate addition and multiplication.

The sum $\xi + \eta$ of elements ξ and η of the r–adelic skew–plane is the element of the r–adelic skew–plane whose component in the complex skew–plane is the sum

$$\xi_+ + \eta_+$$

of components in the complex skew–plane and whose component in the r–adic skew–plane is the sum

$$\xi_- + \eta_-$$

of components in the r–adic skew–plane.

The product $\xi\eta$ of elements ξ and η of the r–adelic skew–plane is the element of the r–adelic skew–plane whose component in the complex skew–plane is the product

$$\xi_+\eta_+$$

of components in the complex skew–plane and whose component in the r–adic skew–plane is the product

$$\xi_-\eta_-$$

of components in the r–adic skew–plane.

The conjugate of an element ξ of the r–adelic skew–plane is the element ξ^- of the r–adelic skew–plane whose component in the complex skew–plane is the conjugate

$$\xi_+^-$$

of the component in the complex skew–plane and whose component in the r–adic skew–plane is the conjugate

$$\xi_-^-$$

of the component in the r–adic skew–plane.

The r–adelic skew–plane is a locally compact Hausdorff space in the Cartesian product topology of the topology of the complex skew–plane and the topology of the r–adic skew–plane. Addition is continuous as a transformation of the Cartesian product of the r–adelic skew–plane with itself into the r–adelic skew–plane. Multiplication by an element of the r–adelic skew–plane is a continuous transformation of the r–adelic skew–plane into itself. Conjugation is a continuous transformation of the r–adelic skew–plane into itself.

The canonical measure for the r–adelic skew–plane is the Cartesian product measure of the canonical measure for the complex skew–plane and the canonical measure for the r–adic skew–plane. The measure is defined on Baire subsets of the r–adelic skew–plane. A measure preserving transformation of the r–adelic skew–plane into itself is defined by taking ξ into $\xi + \eta$ for every element η of the r–adelic skew–plane. Measure preserving transformations of the r–adelic skew–plane into itself are defined by taking ξ into $\omega\xi$ and into $\xi\omega$ for every element ω of the r–adelic skew–plane whose component ω_+ in the complex skew–plane has conjugate as inverse and whose component ω_- in the r–adic skew–plane is a unit. If ω is an invertible element of the r–adelic skew–plane, multiplication on left or right by ω multiplies the canonical measure by a factor of

$$(\omega_+^-\omega_+)^2\lambda_r(\omega_-^-\omega_-)^2.$$

The set of noninvertible elements of the r–adelic skew–plane has measure zero.

The Fourier transformation for the r–adelic skew–plane is the isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the r–adelic skew–plane into itself which takes an integrable function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane into the continuous function

$$g(\xi_+, \xi_-) = \int \exp(\pi i(\xi_+^-\eta_+ + \eta_+^-\xi_+)) \exp(\pi i(\xi_-^-\eta_- + \eta_-^-\xi_-)) f(\eta_+, \eta_-) d\eta$$

of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi_+, \xi_-) = \int \exp(-\pi i(\xi_+^-\eta_+ + \eta_+^-\xi_+)) \exp(-\pi i(\xi_-^-\eta_- + \eta_-^-\xi_-)) g(\eta_+, \eta_-) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane is integrable and the function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane is continuous.

The r–adelic plane is defined as the set of elements of the r–adelic skew–plane whose component in the complex skew–plane belongs to the complex plane and whose component in the r–adic skew–plane belongs to the r–adic plane. The r–adelic plane is a maximal commutative subring of the r–adelic skew–plane which is isomorphic to the Cartesian product of the complex plane and the r–adic plane. The conjugation of the r–adelic skew–plane acts as an isomorphism of the r–adelic plane onto itself.

The r–adelic plane is a locally compact Hausdorff space in the topology inherited from the r–adelic skew–plane. The topology of the r–adelic plane is identical with the Cartesian product topology of the topology of the complex plane and the topology of the r–adic plane. Addition is continuous as a transformation of the Cartesian product of the r–adelic plane with itself into the r–adelic plane. Multiplication by an element of the r–adelic plane is a continuous transformation of the r–adelic plane into itself. Conjugation is continuous as a transformation of the r–adelic plane into itself.

The canonical measure for the r–adelic plane is the Cartesian product measure of the canonical measure for the complex plane and the canonical measure for the r–adic plane. The measure is defined on Baire subsets of the r–adelic plane. A measure preserving transformation of the r–adelic plane into itself is defined by taking ξ into $\xi + \eta$ for every element η of the r–adelic plane. A measure preserving transformation of the r–adelic plane into itself is defined by taking ξ into $\omega\xi$ for every element ω of the r–adelic plane whose component ω_+ in the complex plane has conjugate as inverse and whose component ω_- in the r–adic plane is a unit. Multiplication by an invertible element ω of the r–adelic plane multiplies the canonical measure by a factor of

$$\omega_+^- \omega_+ \lambda_r(\omega_-^- \omega_-).$$

The set of noninvertible elements has measure zero.

The Fourier transformation for the r–adelic plane is the isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the r–adelic plane into itself which takes an integrable function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic plane into the continuous function

$$g(\xi_+, \xi_-) = \int \exp(\pi i(\xi_+^- \eta_+ + \eta_+^- \xi_+)) \exp(\pi i(\xi_-^- \eta_- + \eta_-^- \xi_-)) f(\eta_+, \eta_-) d\eta$$

of $\xi = (\xi_+, \xi_-)$ in the r–adelic plane defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi_+, \xi_-) = \int \exp(-\pi i(\xi_+^- \eta_+ + \eta_+^- \xi_+)) \exp(-\pi i(\xi_-^- \eta_- + \eta_-^- \xi_-)) g(\eta_+, \eta_-) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic plane is integrable and the function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic plane is continuous.

The complementary space to the r–adelic plane in the r–adelic skew–plane is the set of elements $\xi = (\xi_+, \xi_-)$ of the r–adelic skew–plane whose component ξ_+ in the complex skew–plane belongs to the complementary space to the complex plane in the complex skew–plane and whose component ξ_- in the r–adic skew–plane belongs to the complementary space to the r–adic plane in the r–adic skew–plane.

The identity

$$\xi\eta = \eta\xi^-$$

holds when ξ is in the r–adelic plane and η is in the complementary space to the r–adelic plane in the r–adelic skew–plane. An element η of the complementary space to the r–adelic plane in the r–adelic skew–plane is skew–conjugate:

$$\eta^- = -\eta.$$

An element $\xi + \eta$ of the r–adelic skew–plane is the unique sum of an element ξ of the r–adelic plane and an element η of the complementary space to the r–adelic plane in the r–adelic skew–plane.

The canonical measure for the complementary space to the r–adelic plane in the r–adelic skew–plane is the Cartesian product measure of the canonical measure for the complementary space to the complex plane in the complex skew–plane and the canonical measure for the complementary space to the r–adic plane in the r–adic skew–plane.

The canonical measure for the r–adelic skew–plane is the Cartesian product measure of the canonical measure for the r–adelic plane and the canonical measure for the complementary space to the r–adelic plane in the r–adelic skew–plane.

The Radon transformation for the r–adelic skew–plane is a transformation with domain and range in the Hilbert space of square integrable functions with respect to the canonical measure which has a closed graph and which commutes with the transformation which takes the function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane into the function $f(\omega_+\xi_+, \omega_-\xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane for every element $\omega = (\omega_+, \omega_-)$ of r–adelic skew–plane whose component ω_+ in the complex skew–plane has conjugate as inverse and whose component ω_- in the r–adic skew–plane is a unit. The transformation is defined as an integral on elements of its domain which are integrable with respect to the canonical measure.

The Radon transform of an integrable function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane is a function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane defined by the integral

$$g(\omega_+\xi_+, \omega_-\xi_-) = \int f(\omega_+\xi_+ + \omega_+\eta_+, \omega_-\xi_- + \omega_-\eta_-)d\eta$$

with respect to the canonical measure for the complementary space to the r–adelic plane in the r–adelic skew–plane when $\xi = (\xi_+, \xi_-)$ is in the r–adelic plane for every element $\omega = (\omega_+, \omega_-)$ of the r–adelic skew–plane with conjugate as inverse. The inequality

$$\int |f(\omega_+\xi_+, \omega_-\xi_-)d\xi \leq \int |f(\xi_+, \xi_-)|d\xi$$

holds for every element $\omega = (\omega_+, \omega_-)$ of the r–adelic skew–plane whose conjugate is its inverse with integration on the left with respect to the canonical measure for the r–adelic plane and integration on the right with respect to the canonical measure for the r–adelic skew–plane.

The Radon transformation for the r–adelic skew–plane factors the Fourier transformation for the r–adelic skew–plane as a composition with the Fourier transformation for the r–adelic plane. If the Radon transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane, which is integrable with respect to the canonical measure for the r–adelic skew–plane, into a function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane, then the function $g(\omega_+\xi_+, \omega_-\xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic plane is integrable with respect to the canonical measure for the r–adelic plane for every element $\omega = (\omega_+, \omega_-)$ of the r–adelic skew–plane with conjugate as inverse. The restriction to the r–adelic plane of the Fourier transform of the function $f(\omega_+\xi_+, \omega_-\xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane is the Fourier transform of the function $g(\omega_+\xi_+, \omega_-\xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic plane.

The spectral analysis of the Radon transformation for the r–adelic skew–plane applies a homogeneous harmonic polynomial ϕ of degree ν satisfying the normalization

$$\int |\phi(\xi)|^2 d\xi = \int (\xi^-\xi)^\nu d\xi$$

with integration with respect to the canonical measure for the complex skew–plane over the unit disk $\xi^-\xi < 1$.

A closed subspace of the Hilbert space of square summable functions with respect to the canonical measure for the r–adelic skew–plane is applied which is an invariant subspace for the Radon transformation and its adjoint. The space contains the functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane which satisfy the identity

$$\phi(\xi_+) f(\omega_+\xi_+, \omega_-\xi_-) = \varphi(\omega_+\xi_+) f(\xi_+, \xi_-)$$

for every element $\omega = (\omega_+, \omega_-)$ of the r–adelic skew–plane with conjugate as inverse.

The restriction of the Radon transformation to the subspace is a maximal dissipative transformation. The subspace is the orthogonal sum of invariant subspaces defined by positive rational numbers λ which are ratios of positive integers whose prime divisors are divisors of r.

The elements of the invariant subspace defined by λ are the functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane which are eigenfunctions of the Radon transformation for the r–adic skew–plane for the eigenvalue λ when treated as functions of ξ_- in the r–adic skew–plane for every element ξ_+ of the complex skew–plane. The action of the Radon transformation for the r–adelic skew–plane on the function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane is then equal to λ times the action of the Radon transformation for the complex skew–plane on the function of ξ_+ in the complex skew–plane for every element ξ_- of the r–adic skew–plane.

The Fourier transformation for the r–adelic skew–plane acts as an isometric transformation of the invariant subspace for the eigenvalue λ onto the invariant subspace for the eigenvalue λ^{-1}, which is its own inverse.

If ω is an integral element of the complex skew–plane such that $\omega^- \omega$ is a positive integer whose prime divisors are divisors of r, then an isometric transformation of the invariant subspace for the eigenvalue λ onto the invariant subspace for the eigenvalue

$$\omega^- \omega \lambda$$

is defined by taking a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane into the function

$$f(\xi_+ \omega_+, \xi_- \omega_-)$$

of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane.

A Laplace transformation of harmonic ϕ and character χ for the r–adelic skew–plane is defined when a harmonic polynomial ϕ of degree ν is an eigenfunction of the Hecke operator $\Delta(n)$ for an eigenvalue $\tau(n)$ for every positive integer n whose prime divisors are divisors of r but not of ρ and when χ is a primitive character modulo ρ for a positive integer ρ whose prime divisors are divisors of r. The domain of the transformation is the set of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane which are square integrable with respect to the canonical measure for the r–adelic skew–plane and which satisfy the identity

$$\phi(\xi_+) f(\omega_+ \xi_+, \omega_- \xi_-) = \varphi(\omega_+ \xi_+) \chi(\omega_-^- \omega_-) f(\xi_+, \xi_-)$$

for every element ω_+ of the complex skew–plane with conjugate as inverse and for every unit ω_- of the r–adic skew–plane.

The r–adelic half–plane is defined as the Cartesian product of the upper half–plane and the r–adic half–plane. An element $\xi = (\xi_+, \xi_-)$ in the r–adelic half–plane has a component ξ_+ in the upper half–plane and a component ξ_- in the r–adic half–plane. The topology of the r–adelic half–plane is the Cartesian product topology of the topology of the upper half–plane and the topology of the r–adic half–plane. The canonical measure for the r–adelic half–plane is the Cartesian product measure of the canonical measure for the upper half–plane and the canonical measure for the r–adic half–plane.

A function

$$f(\xi_+, \xi_-) = \varphi(\xi_+)(\xi_+^- \xi_+)^{-\frac{1}{2}\nu} h(\tfrac{1}{2}\xi_+^- \xi_+, \tfrac{1}{2}\xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane which belongs to the domain of the Laplace transformation of harmonic ϕ and character χ is parametrized by a function $h(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic half–plane which satisfies the identity

$$h(\omega_+ \xi_+, \omega_- \xi_-) = \chi(\omega_-) h(\xi_+, \xi_-)$$

for every element ω_+ of the complex plane with conjugate as inverse and every unit ω_- of the r–adic half–plane when ξ_+ and $\omega_+ \xi_+$ belong to the upper half–plane. The identity

$$\int |f(\xi_+, \xi_-)|^2 d\xi = 8 \int |h(\xi_+, \xi_-)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the r–adelic skew–plane and integration on the right with respect to the canonical measure for the r–adelic half–plane.

The Laplace transform of harmonic ϕ and character χ of the function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane is the function

$$g(z, \xi) = \int |\eta_+|^{\frac{1}{2}\nu} h(\eta_+, \eta_-) \exp(2\pi i z |\eta_+|) \exp(\pi i (\xi^- \eta_- + \eta_-^- \xi)) d\eta$$

of (z, ξ) in the r–adelic half–plane which is defined by integration with respect to the canonical measure for the r–adelic half–plane when the integral is absolutely convergent. The identity

$$32 \int \int_0^\infty \int_{-\infty}^{+\infty} |g(x+iy, \xi)|^2 y^\nu dx dy d\xi = (4\pi)^{-\nu} \Gamma(1+\nu) \int |f(\xi_+, \xi_-)|^2 d\xi$$

holds with outer integration on the left with respect to the canonical measure for the r–adic half–plane and with integration on the right with respect to the canonical measure for the r–adelic skew–plane.

A function $g(z, \xi)$ of (z, ξ) in the r–adelic half–plane is a Laplace transform of harmonic ϕ and character χ if, and only if, it is an analytic function of z in the upper half–plane for every element ξ of the r–adic half–plane, the identity

$$g(z, \omega\xi) = \chi^-(\omega) g(z, \xi)$$

holds for every unit ω of the r–adic half–plane, and the outer integral

$$\int \int_0^\infty \int_{-\infty}^{+\infty} |g(x+iy, \xi)|^2 y^\nu dx dy d\xi$$

with respect to the canonical measure for the r–adic half–plane converges.

5. The Riemann Hypothesis

The Riemann hypothesis for Jacobian zeta functions is a construction of Euler weight functions which denies zeros in a half–plane larger than the half–plane of convergence of the Euler product when an infinite number of primes is permitted. The construction of Euler weight functions is made for a finite number of primes since passage to an infinite limit is then immediate.

The Euler product for the adelic zeta function

$$Z(\chi, s) = \sum \tau(n) \chi(n) n^{-s}$$

of harmonic ϕ and character χ is a consequence of the identity

$$\tau(m)\tau(n) = \sum \tau(mn/k^2)$$

which holds for all generating positive integers m and n which are relatively prime to ρ with summation over the common odd divisors k of m and n. The Euler product

$$Z(\chi, s)^{-1} = \prod (1 - \tau(p)\chi(p)p^{-s} + \chi(p)^2 p^{-2s})$$

when ρ is even and

$$\zeta(s)^{-1} = (1 - \tau(2)2^{1-s}) \prod (1 - \tau(p)\chi(p)p^{-s} + \chi(p)^2 p^{-2s})$$

when ρ is odd and is taken over the odd generating primes p which are not divisors of ρ.

The adelic zeta function

$$\zeta(s) = \sum \tau(n)\chi(n)n^{-s}$$

of harmonic ϕ and character χ satisfies a functional identity. The functional identity for the zeta function is obtained from the functional identity for the theta function. The zeta function admits an analytic extension to the complex plane when ν is positive or ρ is not one. When ν is zero and ρ is one, the zeta function admits an analytic extension to the complex plane except for a simple pole at two.

Computable examples of zeta functions are obtained when ν is zero since the homogeneous harmonic polynomial ϕ is a constant. The zeta function

$$\sum \tau(n)n^{-s}$$

associated with the character modulo one has coefficient $\tau(n)$ equal to the sum of the odd positive divisors of n.

Dirichlet zeta functions appear when another character χ is admitted. The Dirichlet zeta function

$$\zeta_\chi(s) = \sum \chi(n)n^{-s}$$

defined by a primitive character χ modulo ρ is a sum over all positive integers n. The Euler product

$$\zeta_\chi(s)^{-1} = \prod (1 - \chi(p)p^{-s})$$

is taken over the primes p. Sum and product define the Dirichlet zeta function in the half–plane

$$\mathcal{R}s > 1.$$

The Dirichlet zeta function admits an analytic extension to the complex plane when ρ is not one. The functional identity states that the analytic extension of the function

$$(\rho/\pi)^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta_\chi(s)$$

of s and the function obtained on replacing s by $1 - s$ and χ by χ^- are linearly dependent when χ is a primitive even character modulo p. The analytic extension of the function

$$(\rho/\pi)^{\frac{1}{2}s+\frac{1}{2}}\Gamma(\tfrac{1}{2}s + \tfrac{1}{2})\zeta_\chi(s)$$

of s and the function obtained on replacing s by $1 - s$ and χ by χ^- are linearly dependent when χ is a primitive odd character modulo p.

The Euler zeta function is the Dirichlet zeta function when ρ is one. The Euler zeta function admits an analytic extension to the complex plane with the exception of a simple pole at one. The Euler functional identity states that the analytic extension of the function

$$\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta_\chi(s)$$

of s and the function obtained on replacing s by $1 - s$ are equal. The conjugate character χ^- is identical with χ since χ is identically one on the integers.

The Euler duplication formula for the gamma function

$$2^s\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s + \tfrac{1}{2}) = 2\Gamma(\tfrac{1}{2})\ \Gamma(s)$$

is applied in relating the functional identities for Dirichlet zeta functions to the functional identities for Hecke zeta functions of order zero.

The identity

$$Z(\chi, s) = (1 - \chi(2)2^{1-s})\zeta_\chi(s)\zeta_\chi(s - 1)$$

expresses a zeta function of order zero associated with a primitive character χ modulo ρ in terms of the Dirichlet zeta function associated with the character. The Dirichlet zeta function is the Euler zeta function when ρ is one.

A Dirichlet zeta function has no zeros in the half–plane

$$\mathcal{R}s > \tfrac{1}{2}.$$

The Euler zeta function has no zeros in the half–plane.

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