# THE MEASURE PROBLEM

Louis de Branges*

A problem of Banach is to show that a nonnegative measure defined on all subsets of the continuum vanishes identically if it vanishes on all finite sets. Banach and Kuratowski [1] obtain the desired conclusion when the continuum has the least cardinality of all uncountable sets. Paul Cohen [2] shows that the cardinality of the continuum is not decideable in the Zermelo–Fränkel axiomatization of set theory with axiom of choice. The desired solution of the measure problem is now obtained in the same formulation of set theory with no hypothesis on the cardinality of the continuum: If a nonnegative measure is defined on all subsets of a set, then a countable subset exists whose complement has zero measure. A theorem of Solovay [5] reduces the problem to the case in which the set is the continuum, in which case a theorem of Ulam [6] is applied.

A measure is now defined as a bounded function with complex numbers as values which is defined on the measurable subsets of a given infinite set $\mathcal{S}$. A countable union of measurable sets is measurable. The complement of a measurable set is measurable. The value of the measure on a countable union of disjoint sets is the absolutely convergent sum of the values of the measure on the subsets.

A fundamental example of such a space $\mathcal{S}$ is the set $Z$ of all nonnegative integers. All subsets of $Z$ are measurable. A measure for $Z$ is a bounded function $\mu(E)$ of subsets $E$ of $Z$ such that the sum

$$\mu(E) = \sum \mu(E_n)$$

is absolutely convergent whenever $E$ is a union of pairwise disjoint sets $E_n$. It is sufficient to treat the case in which $E_n$ contains no more than one element. If $\mu'(n)$ is the value of $\mu$ on the set whose only element is $n$, then for every subset $E$ of $Z$

$$\mu(E) = \sum \mu'(n)$$

is an absolutely convergent sum over the elements $n$ of $E$.

A measure is defined for $\mathcal{S}$ by reduction to the case in which $\mathcal{S}$ is $Z$. A mapping $\pi$ of $\mathcal{S}$ onto $Z$ is said to be measurable if for every subset $E$ of $Z$ a measurable subset of $\mathcal{S}$ is defined as the set of elements $s$ such that $\pi(s)$ belongs to $E$.

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A function $\mu$ of measurable subsets of $\mathcal{S}$ with complex numbers as values is mapped by $\pi$ into a function $\mu_\pi$ of subsets of $Z$ whose value on a set $E$ is the value of $\mu$ on the set of elements $s$ of $\mathcal{S}$ such that $\pi(s)$ belongs to $E$.

The function $\mu$ is said to be a measure if the function $\mu_\pi$ is a measure for every measurable mapping $\pi$ of $\mathcal{S}$ onto $Z$ and if the least upper bound

$$\|\mu\|_{\mathcal{M}(\mathcal{S})} = \sup \sum_{n=0}^{\infty} |\mu'_\pi(n)|$$

taken over all measurable mappings $\pi$ of $\mathcal{S}$ onto $Z$ is finite.

The set of all measures on the measurable subsets of $\mathcal{S}$ is a Banach space $\mathcal{M}(\mathcal{S})$ with the given norm. It will be shown that $\mathcal{M}(\mathcal{S})$ is the dual space of a Banach space.

The set $\mathcal{M}(Z)$ of all measures $\mu$ on the subsets of $Z$ is a Banach space in the norm

$$\|\mu\|_{\mathcal{M}(Z)} = \sum_{n=0}^{\infty} |\mu'(n)|.$$

The space is the dual space of the Banach space $\mathcal{C}(Z)$ of all complex–valued functions $f(n)$ of nonnegative integers $n$ which converge to zero

$$\lim f(n) = 0$$

in the limit of large $n$. The norm is the maximum

$$\|f\|_{\mathcal{C}(Z)} = \max |f(n)|$$

taken over all nonnegative integers $n$.

A measure $\mu$ defines a linear functional on $\mathcal{C}(Z)$ which takes a function $f(n)$ of nonnegative integers $n$ into the integral

$$\int f d\mu = \sum_{n=0}^{\infty} f(n)\mu'(n).$$

Continuity of the linear functional results from the inequality

$$|\int f d\mu| \leq \|f\|_{\mathcal{C}(Z)} \|\mu\|_{\mathcal{M}(Z)}.$$

The norm

$$\|\mu\|_{\mathcal{M}(Z)} = \sup |\int f d\mu|$$

of a measure $\mu$ is equal to a least upper bound taken over the set of functions $f(n)$ of $n$ such that

$$\|f\|_{\mathcal{C}(Z)} \leq 1.$$

Every continuous linear functional on $\mathcal{C}(Z)$ is defined by integration with respect to a measure which belongs to $\mathcal{M}(Z)$. The measure is unique.

A Banach space always exists which has $\mathcal{M}(\mathcal{S})$ as dual space. The set $Z$ is a locally compact Hausdorff space in the discrete topology. The smallest compact Hausdorff space which contains $Z$ contains only one element, said to be infinite, which does not belong to $Z$. The closed subsets of the compactification are the subsets which contain the infinite element and the finite subsets which do not contain the infinite element.

The Cartesian product $\mathcal{P}$ of isomorphic images of the compactification of $Z$ is taken over an indexing set whose elements are the measurable mappings $\pi$ of $\mathcal{S}$ onto $Z$. The space $\mathcal{S}$ is mapped into the Cartesian product by taking an element $s$ of $\mathcal{S}$ into the element of $\mathcal{P}$ whose component at $\pi$ is $\pi(s)$. Elements of $\mathcal{S}$ are defined as equivalent if they map into the same element of $\mathcal{P}$. Every measurable subset of $\mathcal{S}$ is a union of equivalence classes. It can without loss of generality be assumed that $\mathcal{S}$ is mapped injectively into $\mathcal{P}$.

The Cartesian product is a compact Hausdorff space in the Cartesian product topology of component compact Hausdorff spaces. If $\pi$ is a measurable mapping of $\mathcal{S}$ onto $Z$, the projection of the Cartesian product into the component space at $\pi$ is also denoted $\pi$. The space $\mathcal{S}$ is identified with a subspace of $\mathcal{P}$ such that the projection into the component space at $\pi$ always agrees with the measurable mapping $\pi$ of $\mathcal{S}$ onto $Z$.

The closure of $\mathcal{S}$ in $\mathcal{P}$ is a compact Hausdorff space in the subspace topology inherited from $\mathcal{P}$. A subset $\mathcal{S}^\wedge$ of the closure of $\mathcal{S}$ is defined as the set of elements $s$ such that the continuous extension of some measurable mapping $\pi$ of $\mathcal{S}$ onto $Z$ has a finite value at $s$. The continuous extension of every measurable mapping of $\mathcal{S}$ onto $Z$ then has a finite value at $s$ since measurable mappings $\pi_+$ and $\pi_-$ of $\mathcal{S}$ onto $Z$ are comparable: a measurable mapping $\pi$ of $\mathcal{S}$ onto $Z$ exists such that every set of constancy for $\pi$ is a union of sets of constancy for $\pi_+$ and a union of sets of constancy for $\pi_-$. The set $\mathcal{S}^\wedge$ is open and contains $\mathcal{S}$. A unique element of the closure of $\mathcal{S}$ exists which does not belong to $\mathcal{S}^\wedge$.

A Banach space $\mathcal{C}(\mathcal{S})$ is defined as the smallest norm closed subset of the space of continuous functions on the closure of $\mathcal{S}^\wedge$ which contains the function

$$f(\pi(s))$$

of $s$ for the continuous extension of every measurable mapping $\pi$ of $\mathcal{S}$ onto $Z$ and every function $f(n)$ of nonnegative integers $n$ which belongs to $\mathcal{C}(Z)$. The norm

$$\|f\|_{\mathcal{C}(\mathcal{S})} = \max |f(s)|$$

of a function $f(s)$ of $s$ in the closure of $\mathcal{S}$ which belongs to $\mathcal{C}(\mathcal{S})$ is a maximum taken over the elements $s$ of $\mathcal{S}^\wedge$.

If $L$ is a continuous linear functional on $\mathcal{C}(\mathcal{S})$, then for every measurable mapping $\pi$ of $\mathcal{S}$ onto $Z$ a continuous linear functional $L_\pi$ on $\mathcal{C}(Z)$ is defined by taking a function $f(s)$ of $s$ which belongs to $\mathcal{C}(Z)$ into the action of $L$ on the composed function $f(\pi(s))$ of $s$ in $\mathcal{S}^\wedge$. A unique measure $\mu_\pi$ which belongs to $\mathcal{M}(Z)$ exists such that the action of $L_\pi$ on a function $f(s)$ of $s$ which belongs to $\mathcal{C}(Z)$ is the sum

$$\sum_{n=0}^{\infty} f(n){\mu'}_{\pi}(n).$$

The inequality

$$\|\mu_\pi\|_{\mathcal{M}(Z)} \leq \|L\|$$

is satisfied.

If $\rho$ is a measurable mapping of $\mathcal{S}$ onto $Z$ such that every set of constancy for $\pi$ is a union of sets of constancy for $\rho$, then

$$\rho = \pi(\sigma)$$

for a mapping $\sigma$ of $Z$ onto $Z$. A unique measure $\mu_\rho$ which belong to $\mathcal{M}(Z)$ exists such that

$$\sum_{n=1}^{\infty} f(\sigma(n))\mu'_\pi(n) = \sum_{n=1}^{\infty} f(n)\mu'_\rho(n)$$

for every function $f(s)$ of $s$ which belongs to $\mathcal{C}(Z)$.

A measure $\mu$ which belongs to $\mathcal{M}(\mathcal{S})$ is defined so that for every measurable mapping $\pi$ of $\mathcal{S}$ onto $Z$ and for every nonnegative integer $n$, the measure of the set of elements $s$ of $\mathcal{S}$ such that $\pi(s) = n$ is equal to $\mu'_\pi(n)$. The inequality

$$\|\mu\|_{\mathcal{M}(\mathcal{S})} \leq \|L\|$$

is satisfied. Equality holds in the inequality since the identity

$$\int f(\pi(s))d\mu = \sum_{n=0}^{\infty} f(n)\mu'_\pi(n)$$

holds for every function $f(s)$ of $s$ which belongs to $\mathcal{C}(Z)$. The action of the linear functional $L$ on every functions $f(s)$ of $s$ which belongs to $\mathcal{C}(\mathcal{S})$ is equal to the integral

$$\int f(s)d\mu.$$

This completes the verification that the Banach space $\mathcal{M}(\mathcal{S})$ is the dual Banach space to $\mathcal{C}(\mathcal{S})$. Note that $\mathcal{C}(\mathcal{S})$ is an algebra and since for every measurable mapping $\pi_+$ of $\mathcal{S}$ onto $Z$ and every measurable mapping $\pi_-$ of $\mathcal{S}$ onto $Z$ a measurable mapping $\pi$ of $\mathcal{S}$ onto $Z$ exists such that every set of constancy for $\pi$ is a union of sets of constency for $\pi_+$ and a union of sets of constancy for $\pi_-$.

Every homomorphism of $\mathcal{C}(Z)$ onto the algebra of complex numbers is defined by a nonnegative integer $n$ to take a function $f(s)$ of $s$ into its value $f(n)$ at $n$. Every homomorphism of $\mathcal{C}(\mathcal{S})$ onto the algebra of complex numbers is defined by an element $\eta$ of $\mathcal{S}^\wedge$ to take a function $f(s)$ of $s$ into its value $f(\eta)$ at $\eta$.

By the Stone–Weierstrass theorem the algebra $\mathcal{C}(\mathcal{S})$ contains every continuous functions $f(s)$ of $s$ in the closure of $\mathcal{S}$ which vanishes outside of $\mathcal{S}^\wedge$. The Banach space $\mathcal{M}(\mathcal{S})$ has a weak topology as dual space to the Banach space $\mathcal{C}(\mathcal{S})$. The topology is the weakest topology with respect to which the integral

$$\int f d\mu$$

is a continuous function of measures $\mu$ for every function $f(s)$ of $s$ which belongs to $\mathcal{C}(\mathcal{S})$. A subset $E$ of $\mathcal{M}(\mathcal{S})$ is said to be bounded if the function of $\mu$ in $E$ is bounded for every element of $\mathcal{C}(\mathcal{S})$. A closed and bounded subset of $\mathcal{M}(\mathcal{S})$ is compact. A subset of $\mathcal{M}(\mathcal{S}^\wedge)$ is bounded if, and only if, the norms of elements of the set are bounded.

The Krein–Milman theorem [4] states that every closed and bounded convex subset of the dual Banach space of a Banach space is the closed convex span of its extreme elements in the weak topology induced by duality with the Banach space.

An extreme element of a convex set is an element which is not a proper convex combination of elements of the set. A convex combination is proper if it is distinct from the elements of which it is a convex combination.

The closed convex span of a set is the smallest closed convex set which contains the given set.

The proof of the Krein–Milman theorem is an application of the Hahn–Banach theorem.

The proof [3] of the Stone–Weierstrass theorem from the Krein–Milman theorem applies a computation of extreme elements which is reformulated in the present context.

Assume that a subalgebra $\mathcal{A}$ of $\mathcal{C}(\mathcal{S})$ contains the conjugate function $f(s)^-$ and the product function $h(s)f(s)$ of $s$ in $\mathcal{S}^\wedge$ for every continuous function $h(s)$ of $s$ in the closure of $\mathcal{S}$ whenever it contains a function $f(s)$ of $s$.

The convex set of all measures $\mu$ with real values, of norm at most one, which annihilate

$$\int f d\mu = 0$$

every function $f(s)$ of $s$ which belongs to $\mathcal{A}$ is compact and is the closed convex span of its extreme elements in the weak topology induced by duality with $\mathcal{C}(\mathcal{S})$.

If $h(s)$ is a function of $s$ which belongs to $\mathcal{C}(\mathcal{S})$, and if $\mu$ is a measure which belongs to the convex set, a measure $\nu$ which belongs to $\mathcal{M}(\mathcal{S})$ is defined by the equation

$$\int f d\nu = \int h f d\mu$$

for every function $f(s)$ of $s$ which belongs to $\mathcal{C}(\mathcal{S})$. The measure $\nu$ has real values if the function $h(s)$ of $s$ has real values.

When the measure $\mu$ has norm at most one and the inequalitics

$$0 \leq h(s) \leq 1$$

hold for all $s$ in $\mathcal{S}^\wedge$, a convex combination

$$\mu = (1-t)\mu_+ + t\mu_-$$

of elements $\mu_+$ and $\mu_-$ of the convex set of measures of norm at most one is defined by

$$2(1-t)\mu_+ = \mu + \nu$$

and

$$2t\mu_- = \mu - \nu$$

with $2(1-t)$ the norm of the measure $\mu + \nu$ and $2t$ the norm of the measure $\mu - \nu$.

If the integral

$$\int f d\mu = 0$$

vanishes for every function $f(s)$ of $s$ which belongs to $\mathcal{A}$, then the integrals

$$\int f d\mu_+ = 0$$

and

$$\int f d\mu_- = 0$$

vanish for every function $f(s)$ of $s$ which belong to $\mathcal{A}$.

Since

$$\mu_+ = \mu = \mu_-$$

when $\mu$ is an extreme element of the convex set and $0 < t < 1$, a number $\phi(h)$ in the interval $[0,1]$ is obtained which satisfies the identity

$$\int hf d\mu = \phi(h) \int f d\mu$$

for every function $f(s)$ of $s$ which belongs to $\mathcal{C}(\mathcal{S})$.

A homomorphism of the Banach algebra $\mathcal{C}(\mathcal{S})$ onto the algebra of complex numbers is defined by taking a function $h(s)$ of $s$ into the solution $\phi(h)$ of the equation

$$\int hf d\mu = \phi(h) \int f du$$

for every function $f(s)$ of $s$ which belongs to $\mathcal{C}(\mathcal{S})$.

An element $\eta$ of $\mathcal{S}^\wedge$ exists such that

$$\phi(h) = h(\eta)$$

for every function $h(s)$ of $s$ which belongs to $\mathcal{C}(\mathcal{S})$.

If a measure $\mu$ exists such that the integral

$$\int f d\mu$$

vanishes for every function $f(s)$ of $s$ which belongs to $\mathcal{A}$ and which does not vanish for every function $f(s)$ of $s$ which belongs to $\mathcal{C}(\mathcal{S})$, then such a measure $\mu$ exists such that for some element $\eta$ of $\mathcal{S}^\wedge$ the integral

$$\int f d\mu = f(\eta)$$

represents the value of the function at $\eta$ for every function $f(s)$ of $s$ which belongs to $\mathcal{C}(\mathcal{S})$.

An example of an algebra $\mathcal{A}$ is defined, when all subsets of $\mathcal{S}$ are measurable, to contain a function $f(s)$ of $s$ in $\mathcal{S}^\wedge$ if it vanishes outside of some finite subset of $\mathcal{S}$. The convex set is the set of measures of norm at most one with real values which vanish on finite subsets of $\mathcal{S}$. The measure problem is reduced to the problem of showing that $\mathcal{S}^\wedge$ is $\mathcal{S}$. An argument of Ulam [6] applies when $\mathcal{S}$ is the continuum.

Banach and Kuratowski treat the continuum as the set of all increasing sequences of nonnegative integers. A mapping of the continuum into the set of nonnegative integers is defined by taking a sequence into its $n$–th element. A nonnegative integer $\lambda_n$ exists such that the set of all sequences which have $\lambda_n$ as $n$–th element has measure one. The sequence which has $\lambda_n$ as $n$–th element for every nonnegative integer $n$ belongs to a set of measure one which contains no other element.

When the set $\mathcal{S}$ is arbitrary, argue by contradiction assuming that a nonnegative measure on the subsets of $\mathcal{S}$ exists which has one as its only nonzero value and which vanishes on all finite sets. This hypothesis permits the construction of Cohen models of set theory with remarkable properties. Solovay [5] constructs a model in which a nonnegative measure is defined on all subsets of the continuum which vanishes on all finite sets but does not vanish identically. A contradiction results since the argument which denies the existence of such measures applies in the Cohen model.

The origin of the contradiction is the hypothesis that a nonnegative measure exists which is defined on all subsets of a set, which has one as only nonzero value, and which vanishes on all finite sets.

A nonnegative measure which is defined on all subsets of a set is determined by its values on finite subsets. A countable subset exists whose complement has zero measure.

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