

The Measure Problem*

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A problem of Banach is to determine the structure of a nonnegative (countably additive) measure which is defined on all subsets of a set. The problem is trivial for countable sets. Banach and Kuratowski [1] show the existence of a countable set whose complement has zero measure when cardinality hypotheses are satisfied. These hypotheses are not verified generally for the classical continuum, which has the cardinality of the class of all subsets of a countably infinite set. Ulam [6] obtains the same conclusion for the classical continuum when the measure has no nonzero value other than one. The same conclusion is now obtained for the classical continuum for all nonnegative measures. According to a theorem of Solovay [5] the existence of a countable set whose complement has zero measure follows for all infinite sets. These results are valid in Zermelo–Fraenkel set theory with the axiom of choice.

Measure theory is treated in relation to the theory [4] of commutative algebras whose elements are continuous functions on a locally compact Hausdorff space with values in the complex numbers. The functions are defined on spaces with the discrete topology.

Theorem 1. *Assume that for every nonnegative measure which is defined on all subsets of a given infinite set \mathcal{S} and which has no nonzero value other than one, a countable subset of \mathcal{S} exists whose complement has zero measure. Then for every nonnegative measure which is defined on all subsets of \mathcal{S} , a countable subset exists whose complement has zero measure.*

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Proof. An algebra \mathcal{A} is defined to contain the convex span of the generating set all functions $f(s)$ of s in \mathcal{S} such that for every positive number ϵ the set of values of the function which do not satisfy the inequality

$$|f(s)| < \epsilon$$

is finite and which take the value zero on an infinite set if the number of values is finite. The algebra is given the metric topology defined by the norm

$$\|f\| = \sup |f(s)|$$

with the least upper bound taken over all elements s of \mathcal{S} .

If a signed measure μ of bounded variation $\|\mu\|_1$ is defined on all subsets of \mathcal{S} , a continuous linear functional L on \mathcal{A} is defined by taking f into the integral

$$Lf = \int f(s)d\mu(s)$$

with respect to μ . An element of the algebra is a function $f(s)$ of s in \mathcal{S} which has a countable number of values. The integral is an absolutely convergent sum of products of a value and the μ -measure of the set on which the value is taken. The total variation $\|\mu\|_1$ of the measure μ is the least upper bound of sums

$$|\mu(E_1)| + \dots + |\mu(E_r)|$$

taken are all finite classes of disjoint subsets E_1, \dots, E_r . Bounded variation for μ means that the least upper bound is finite. Continuity of the linear functional follows from the inequality

$$|Lf| \leq \|f\| \|\mu\|_1$$

which holds for all elements f of \mathcal{A} . The continuous linear functional defined on the algebra by Lebesgue integration is the unique continuous linear extension of its restriction to functions having only a finite number of values. Uniqueness is verified on elements of the generating set and follows by linearity for other elements.

Every continuous linear functional L on \mathcal{A} is represented by a signed measure μ of bounded variation which is defined on all subsets of \mathcal{S} . It is sufficient to make the verification when L is contractive: The inequality

$$|Lf| \leq \|f\|$$

holds for every element f of \mathcal{A} . It will be shown that the measure is of total variation at most one.

Define

$$\mu(E) = Lf$$

for a subset E of \mathcal{S} with f the element of \mathcal{A} which has value one on E and which vanishes elsewhere. For the verification of countable additivity assume that

$$E = \bigcup E_n$$

is a disjoint union of subsets E_n of \mathcal{S} indexed by the positive integers n . For every n

$$\mu(E_n) = Lf_n$$

is the action of L on the element f_n of \mathcal{A} which has value one on E_n and which vanishes elsewhere.

The identity

$$L\left(\sum c_n f_n\right) = \sum c_n Lf_n$$

holds by linearity and continuity for every sequence of complex numbers c_n which converges to zero. The inequality

$$\left|\sum c_n Lf_n\right| \leq \max |c_n|$$

holds by contractivity of L . The inequality

$$\sum |Lf_n| \leq 1$$

follows by the arbitrariness of the sequence of complex numbers c_n . The identity

$$\mu\left(\bigcup_{n=1}^r E_n\right) = \sum_{n=1}^r \mu(E_n)$$

holds for every positive integer r since it is permitted to choose $c_n = 1$ for $n \leq r$ and $c_n = 0$ for $n > r$. The inequality

$$\left|\mu\left(\bigcup_{n=r+1}^{\infty} E_n\right)\right| \leq \sum_{n=r+1}^{\infty} |\mu(E_n)|$$

is a consequence of contractivity which implies the limit

$$\mu \left(\bigcup_{n=1}^{\infty} E_n \right) = \lim \mu \left(\bigcup_{n=1}^r E_n \right)$$

and countable additivity

$$\mu \left(\bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \mu(E_n).$$

The identity

$$Lf = \int f(s) d\mu(s)$$

holds by linearity and continuity for every element f of \mathcal{S} since it holds whenever f has no nonzero value other than one.

A nonnegative measure σ which is defined on all subsets of \mathcal{S} is of bounded variation equal to the value $\sigma(\mathcal{S})$ of the measure on \mathcal{S} .

An equivalence relation is defined on the algebra: Elements f and g of \mathcal{A} are equivalent if they are equal almost everywhere with respect to a non-negative measure σ : The set of elements s of \mathcal{S} such that $f(s)$ and $g(s)$ are unequal has zero σ -measure.

The quotient space \mathcal{A}_σ of \mathcal{A} inherits the structure of an algebra. The canonical projection of \mathcal{A} onto \mathcal{A}_σ is an algebra homomorphism whose kernel is the closed ideal of elements of \mathcal{A} which vanish almost everywhere with respect to σ .

The quotient topology of \mathcal{A}_σ is a metric topology defined by the quotient norm

$$\|f\|_\sigma = \inf \|g\|$$

with the greatest lower bound taken over the elements g of \mathcal{A} which represent an element

$$f = \pi g$$

of \mathcal{A}_σ .

A continuous linear functional on \mathcal{A}_σ is treated as a continuous linear functional on \mathcal{A} which annihilates elements of \mathcal{A} which vanish almost everywhere with respect to σ .

An element f of \mathcal{A} is said to be nonnegative if its values are nonnegative numbers. An element f of \mathcal{A} is nonnegative if, and only if, the equation

$$f = g^*g$$

admits a solution for an element g of \mathcal{A} with g^* the element of \mathcal{A} with conjugate values: The identity

$$g^*(s) = g(s)^-$$

holds for every element s of \mathcal{S} .

A linear functional L on \mathcal{A} is said to be nonnegative if its action Lf on every nonnegative element f of \mathcal{A} is a nonnegative number.

The identity

$$Lf^* = (Lf)^-$$

holds for every element f of \mathcal{A} if L is a nonnegative linear functional on \mathcal{A} . Since the action of L on

$$(f + \lambda)^*(f + \lambda)$$

is nonnegative for every complex number λ (treated as an element of \mathcal{A} whose only value is λ), the inequality

$$(Lf)^-(Lf) \leq L(f^*f)L(1)$$

is satisfied. Since the action of L on the nonnegative element

$$\|f\|^2 - f^*f$$

of \mathcal{A} is a nonnegative number, the inequality

$$L(f^*f) \leq \|f\|^2 L(1)$$

holds for every element f of \mathcal{A} . A nonnegative linear functional L is continuous since the inequality

$$|Lf| \leq \|f\|L(1)$$

holds for every element f of \mathcal{A} .

A linear functional on \mathcal{A} is nonnegative if, and only if, it is represented by a nonnegative measure. The set of nonnegative and contractive linear functionals on \mathcal{A} is a convex set which is compact in the weak topology

induced by duality with \mathcal{A} . The set is the closed convex span of its extreme points by the Krein–Milman theorem [3]. A characterization of extreme points is applied in a proof of the Stone–Weierstrass theorem [2] from the Krein–Milman theorem.

An element L of the convex set is said to be an extreme point of the set if it is not a convex combination

$$L = (1 - t)L_- + tL_+$$

of distinct elements L_+ and L_- of the set for an element t of the interval $[0, 1]$ such that t and $1 - t$ are positive.

If L is an extreme point of the convex set and if

$$h = h^*$$

is a self-conjugate element of \mathcal{A} such that

$$\|h\| \leq 1$$

and such that

$$1 - t = L(1 - h)$$

and

$$1 + t = L(1 + h)$$

are positive, then elements L_+ and L_- of the convex set are defined by

$$(1 - t)L_- f = L[(1 - h)f]$$

and

$$(1 + t)L_+ f = L[(1 + h)f]$$

for every element f of \mathcal{A} . The identity

$$L_- f = L_+ f$$

holds for every element f of \mathcal{A} since

$$2L = (1 - t)L_- + (1 + t)L_+.$$

The resulting identity

$$L(hf) = L(h)L(f)$$

implies that L is a homomorphism of \mathcal{A} into the complex numbers.

A nonzero extreme point of the convex set is represented by a measure whose only nonzero value is one. An element s of \mathcal{S} exists by hypothesis such that the identity

$$Lf = f(s)$$

holds for every element f of \mathcal{A} .

The set has the discrete topology because of its identification with a set of nonzero extreme points of the convex set. If E is a subset of \mathcal{S} and if f is the element of \mathcal{A} which has value one on E and vanishes elsewhere, then the set of elements s of \mathcal{S} such that

$$f(s) = 1$$

is both open and closed. If f represents a nonzero element of \mathcal{A}_σ , then

$$\sigma(E) = \int f(s)d\sigma(s)$$

is positive.

If an element s of \mathcal{S} is a zero of every element of \mathcal{A} which vanishes almost everywhere with respect to σ , then every subset of \mathcal{S} which contains s has positive σ -measure. The set E of such elements s is countable.

If f is the element of \mathcal{A} which has value one on the complement of E and vanishes elsewhere, then the integral

$$\int f(s)d\mu(s)$$

vanishes by the Krein–Milman theorem since it belongs to the closed convex span of values $f(s)$ with s in E .

The complement of E is a set of zero σ -measure.

This completes the proof of the theorem. □

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