

The Measure Problem*

Louis de Branges
Department of Mathematics
Purdue University
West Lafayette, IN 47907-2067, USA

A problem of Banach is to determine the structure of a nonnegative (countably additive) measure which is defined on all subsets of a set. The problem is trivial for countable sets. Banach and Kuratowski [1] show the existence of a countable set whose complement has zero measure when cardinality hypotheses are satisfied. These hypotheses are not verified generally for the classical continuum, which has the cardinality of the class of all subsets of a countably infinite set. Ulam [5] obtains the same conclusion for the classical continuum when the measure has no nonzero value other than one. The same conclusion is now obtained for the classical continuum for all nonnegative measures. According to a theorem of Solovay [4] the existence of a countable set whose complement has zero measure follows for all infinite sets. These results are valid in Zermelo–Fraenkel set theory with the axiom of choice.

The integral representation of continuous linear functionals on spaces of bounded measurable functions is a fundamental problem. Measures are assumed to be bounded and countably additive with domain the measurable subsets of a measure space. A bounded measure determines by integration a linear functional which is continuous for the topology of uniform convergence. Every continuous linear functional is represented by a unique bounded measure when the function space is suitably chosen. The construction of the space is the principal new contribution.

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An example is given to motivate the construction. Consider the set Z of nonnegative integers with all subsets accepted as measurable. A bounded measure μ on the subsets of Z is absolutely continuous with respect to counting measure and has derivative which is a summable function $\mu'(n)$ of elements n of Z . The value

$$\mu(E) = \sum \mu'(n)$$

of the measure on a set E is an absolutely convergent sum taken over the elements n of Z . The total variation

$$\|\mu\| = \sum |\mu'(n)|$$

is a convergent sum taken over the elements n of Z . The set of bounded measures is a Banach space in the total variation norm. The space is the dual space of a Banach algebra $\mathcal{C}(Z)$ of functions defined on Z .

By definition $\mathcal{C}(Z)$ is the set of functions $f(n)$ of n in Z with complex values which converge to zero in the limit of large n : For every positive number ϵ all but a finite number of elements n of Z satisfy the inequality

$$|f(n)| < \epsilon.$$

The set $\mathcal{C}(Z)$ is a commutative Banach algebra with norm

$$\|f\|_\infty = \max |f(n)|$$

defined as a maximum taken over the elements n of Z .

The Banach algebra $\mathcal{B}(Z)$ is defined as the set of all bounded functions $f(n)$ of n in Z with norm

$$\|f\|_\infty = \sup |f(n)|$$

a least upper bound taken over the elements n of Z .

A bounded measure μ on the subsets of Z defines a continuous linear functional L on $\mathcal{B}(Z)$ by the integral

$$Lf = \int f(n)d\mu(n)$$

which reduces to an absolutely convergent sum

$$Lf = \sum f(n)\mu'(n)$$

taken over the elements n of Z . The inequality

$$|Lf| \leq \|f\|_\infty \|\mu\|$$

is satisfied.

Every continuous linear functional L on $\mathcal{C}(Z)$ is represented as an integral with respect to a unique bounded measure μ . When a subset E of Z has only a finite number of elements, the value

$$\mu(E) = Lf$$

of the measure on E is the action of the linear functional on the element f of $\mathcal{C}(Z)$ which has value one on E and which vanishes elsewhere. The measure is determined on all subsets of Z since its derivative is determined on every element of Z .

An example of a continuous linear functional L on $\mathcal{C}(Z)$ is a linear functional which is contractive. The inequality

$$|Lf| \leq 1$$

holds for an element f of $\mathcal{C}(Z)$ which satisfies the inequality

$$|f(n)| \leq 1$$

for every element n of Z .

A linear functional L on $\mathcal{C}(Z)$ is said to be nonnegative if Lf is a nonnegative number whenever f is an element of $\mathcal{C}(Z)$ with nonnegative values:

$$f(n) \geq 0$$

for every element n of Z . A nonnegative linear functional on $\mathcal{C}(Z)$ is continuous. A continuous linear functional on $\mathcal{C}(Z)$ is nonnegative if, and only if, it is represented by a measure with nonnegative values.

A linear functional on $\mathcal{C}(Z)$ is nonnegative and contractive if, and only if, it is represented by a nonnegative measure whose value on Z is at most one. The value of the measure on Z is then equal to its total variation.

A measure space is a set \mathcal{S} with a σ -algebra of subsets called measurable. A countable union of measurable sets is assumed to be measurable. The complement of a measurable set is assumed to be measurable.

A mapping π of \mathcal{S} onto Z is said to be measurable if for every subset E of Z a measurable subset of \mathcal{S} is defined as the set of elements s of \mathcal{S} such that πs belongs to E .

A Banach algebra $\mathcal{B}(\mathcal{S})$ is defined as the set of all bounded measurable functions $f(s)$ of s in \mathcal{S} with norm

$$\|f\|_\infty = \sup |f(s)|$$

a least upper bound taken over the elements s of \mathcal{S} .

A function $f(s)$ of s in \mathcal{S} belongs to $\mathcal{B}(\mathcal{S})$ if it can be factored

$$f(s) = f_\pi(\pi s)$$

by a measurable mapping π of \mathcal{S} onto Z with a function $f_\pi(n)$ of n in Z which belongs to $\mathcal{B}(Z)$. A dense set of elements of $\mathcal{B}(\mathcal{S})$ are factorable.

A property of measurable mappings of \mathcal{S} onto Z is required for the verification that sums and products of factorable elements of $\mathcal{B}(\mathcal{S})$ are factorable. The conjugate function

$$f^-(s) = f(s)^-$$

of s in \mathcal{S} belongs to $\mathcal{B}(\mathcal{S})$ whenever the function $f(s)$ of s in \mathcal{S} belongs to $\mathcal{B}(\mathcal{S})$.

If α and β are measurable mappings of \mathcal{S} onto Z , then a measurable mapping γ of \mathcal{S} onto Z exists such that equality

$$\gamma(s_1) = \gamma(s_2)$$

for elements s_1 and s_2 of \mathcal{S} is equivalent to the equalities

$$\alpha(s_1) = \alpha(s_2)$$

and

$$\beta(s_1) = \beta(s_2).$$

Mappings α' and β' of Z onto Z exist such that

$$\alpha(s) = \alpha'(\gamma s)$$

and

$$\beta(s) = \beta'(\gamma s)$$

for every element s of \mathcal{S} .

Functions $f(s)$ and $g(s)$ of s in \mathcal{S} which can be factored

$$f(s) = f_\alpha(\alpha s)$$

and

$$g(s) = g_\beta(\beta s)$$

with f_α and g_β in $\mathcal{B}(Z)$ can be factored

$$f(s) = f_\gamma(\gamma s)$$

and

$$g(s) = g_\gamma(\gamma s)$$

with f_γ and g_γ in $\mathcal{B}(Z)$ with

$$f_\gamma(n) = f_\alpha(\alpha' n)$$

and

$$g_\gamma(n) = g_\beta(\beta' n)$$

for every element n of Z .

The functions

$$f(s) + g(s) = f_\gamma(\gamma s) + g_\gamma(\gamma s)$$

and

$$f(s)g(s) = f_\gamma(\gamma s)g_\gamma(\gamma s)$$

of s in \mathcal{S} can be factored with

$$f_\gamma(n) + g_\gamma(n)$$

and

$$f_\gamma(n)g_\gamma(n)$$

functions of n in Z which belong to $\mathcal{B}(Z)$.

The total variation of a bounded measure μ on the measurable subsets of \mathcal{S} is the least upper bound

$$\|\mu\| = \sup(|\mu(E_1)| + \dots + |\mu(E_r)|)$$

taken over all finite choices of disjoint measurable subsets E_1, \dots, E_r of \mathcal{S} . The set of all bounded measures is a Banach space $\mathcal{M}(\mathcal{S})$ with total variation as norm.

If π is a measurable mapping of \mathcal{S} onto Z , then for every bounded measure μ on the measurable subsets of \mathcal{S} a bounded measure μ_π on the subsets of Z is defined whose value $\mu_\pi(E)$ on a set E is the value of μ on the set of elements s of \mathcal{S} such that πs belongs to E . The linear transformation which takes μ into μ_π is contractive in the total variation norm: The inequality

$$\|\mu_\pi\| \leq \|\mu\|$$

holds for every element μ of $\mathcal{M}(\mathcal{S})$.

A continuous linear functional on $\mathcal{M}(\mathcal{S})$ determines a function $h(s)$ of s in \mathcal{S} which belongs to $\mathcal{B}(\mathcal{S})$ such that

$$L\mu = \int h(s)d\mu(s)$$

for every element μ of $\mathcal{M}(\mathcal{S})$. A construction of the function is made if a measurable mapping π of \mathcal{S} onto Z exists such that L annihilates every element of $\mathcal{M}(\mathcal{S})$ which is mapped by π into the origin of $\mathcal{M}(Z)$. A continuous linear functional on $\mathcal{M}(Z)$ is then defined which takes ν into $L\mu$ whenever π takes μ into ν . The function

$$h(s) = h_\pi(\pi s)$$

of s in \mathcal{S} is then factored by π with $h_\pi(n)$ a function of n in Z which belongs to $\mathcal{B}(Z)$. The function $h(s)$ of s in \mathcal{S} is otherwise obtained by approximation.

The set of continuous linear functionals L on $\mathcal{M}(\mathcal{S})$ is a Banach space $\mathcal{B}^\wedge(\mathcal{S})$ with norm

$$\|L\|_\infty = \sup |L\mu|$$

a least upper bound taken over the elements μ of $\mathcal{M}(\mathcal{S})$ of total variation

$$\|\mu\| \leq 1$$

at most one. The canonical isometric transformation of $\mathcal{B}(\mathcal{S})$ into $\mathcal{B}^\wedge(\mathcal{S})$ takes a function $h(s)$ of s in \mathcal{S} into the linear functional L whose action

$$L\mu = \int h(s)d\mu(s)$$

on an element μ of $\mathcal{M}(\mathcal{S})$ is defined by integration. It needs to be shown that every element of $\mathcal{B}^\wedge(\mathcal{S})$ belongs to the canonical image of $\mathcal{B}(\mathcal{S})$.

If π is a measurable mapping of \mathcal{S} onto Z , a closed subspace of $\mathcal{B}^\wedge(\mathcal{S})$ is defined as the set of elements which annihilate all elements of $\mathcal{M}(\mathcal{S})$ which are mapped by π into the origin of $\mathcal{M}(Z)$. The subspace is a Banach space with the inherited norm. An isometric transformation of $\mathcal{B}(Z)$ onto the quotient Banach space is defined by taking a function $h(s)$ of s in \mathcal{S} into the linear functional L which takes μ into

$$L\mu = \int h(s)d\mu(s)$$

whenever

$$h(s) = h_\pi(\pi s)$$

for a function $h_\pi(n)$ of n in Z which belongs to $\mathcal{B}(Z)$. Every element of the quotient space of $\mathcal{B}^\wedge(\mathcal{S})$ is represented by an element of $\mathcal{B}^\wedge(\mathcal{S})$ which belongs to the canonical image of $\mathcal{B}(\mathcal{S})$.

Since no nonzero element of $\mathcal{M}(\mathcal{S})$ is mapped into the origin of $\mathcal{M}(Z)$ by every measurable mapping of \mathcal{S} onto Z , elements of $\mathcal{B}^\wedge(\mathcal{S})$ are equal if they have equal images in every quotient space defined by a measurable mapping. If L is an element of $\mathcal{B}^\wedge(\mathcal{S})$, a measurable mapping π of \mathcal{S} onto Z can be chosen so that the difference between the norm of L in $\mathcal{B}^\wedge(\mathcal{S})$ and the norm of the image of L in the quotient space is less than any given positive number ϵ . If a function $h(s)$ of s in \mathcal{S} belongs to $\mathcal{B}(\mathcal{S})$ and determines an element L_h of $\mathcal{B}^\wedge(\mathcal{S})$ which has the same image as L in the quotient space, then

$$\|L - L_h\|_\infty \leq \epsilon.$$

Since $\mathcal{B}(\mathcal{S})$ is a Banach space whose canonical isometric image is dense in $\mathcal{B}^\wedge(\mathcal{S})$, every element of $\mathcal{B}^\wedge(\mathcal{S})$ belongs to the canonical image of $\mathcal{B}(\mathcal{S})$.

An algebra $\mathcal{C}(\mathcal{S})$ is defined as the set of elements of $\mathcal{B}(\mathcal{S})$ which are annihilated by every continuous linear functional on $\mathcal{B}(\mathcal{S})$ which annihilates a function

$$f(s) = f_\pi(\pi s)$$

of s in \mathcal{S} whenever it is factored by a measurable mapping π of \mathcal{S} onto Z with a function $f_\pi(n)$ of n in Z which belongs to $\mathcal{C}(Z)$. By the Hahn–Banach theorem $\mathcal{C}(\mathcal{S})$ is the closed span in $\mathcal{B}(\mathcal{S})$ of functions which can be so factored.

The conjugate function

$$f^-(s) = f(s)^-$$

is a function of s in \mathcal{S} which belongs to $\mathcal{C}(\mathcal{S})$ whenever the function $f(s)$ of s in \mathcal{S} belongs to $\mathcal{C}(\mathcal{S})$. For the verification that $\mathcal{C}(\mathcal{S})$ is an algebra it is sufficient to verify that the product

$$h(s) = f(s)g(s)$$

is a function of s in \mathcal{S} with the desired factorization whenever

$$f(s) = f_\alpha(\alpha s)$$

and

$$g(s) = g_\beta(\beta s)$$

are functions of s in \mathcal{S} with the desired factorization: α and β are measurable mappings of \mathcal{S} onto Z and $f_\alpha(n)$ and $g_\beta(n)$ are functions of n in Z which belong to $\mathcal{C}(Z)$.

A measurable mapping γ of \mathcal{S} onto Z exists such that equality

$$\gamma(s_1) = \gamma(s_2)$$

for elements s_1 and s_2 of \mathcal{S} is equivalent to the equalities

$$\alpha(s_1) = \alpha(s_2)$$

and

$$\beta(s_1) = \beta(s_2).$$

Mappings α' and β' of Z onto Z exist such that

$$\alpha(s) = \alpha'(\gamma s)$$

and

$$\beta(s) = \beta'(\gamma s)$$

for every element s of \mathcal{S} . Factorizations

$$f(s) = f_\gamma(\gamma s)$$

and

$$g(s) = g_\gamma(\gamma s)$$

are obtained with

$$f_\gamma(n) = f_\alpha(\alpha' n)$$

and

$$g_\gamma(n) = g_\beta(\beta' n)$$

functions of n in Z which belong to $\mathcal{C}(Z)$. The function

$$h(s) = h_\gamma(\gamma s)$$

of s in \mathcal{S} is factored with

$$h_\gamma(n) = f_\gamma(n)g_\gamma(n)$$

a function of n in Z which belongs to $\mathcal{C}(Z)$.

A good relationship between measure and integration follows for every measure space \mathcal{S} .

Theorem 1. *Every continuous linear functional L on $\mathcal{C}(\mathcal{S})$ is represented by a unique bounded measure μ on the measurable subsets of \mathcal{S} . The action*

$$Lf = \int f(s)d\mu(s)$$

of the linear functional on a function $f(s)$ of s in \mathcal{S} which belongs to $\mathcal{C}(\mathcal{S})$ is equal to the integral of the function with respect to the measure.

Proof of Theorem 1. It can be assumed without loss of generality that the given linear functional on $\mathcal{C}(\mathcal{S})$ is contractive: The inequality

$$|Lf| \leq \|f\|_\infty$$

holds for every function $f(s)$ of s in \mathcal{S} which belongs to $\mathcal{C}(\mathcal{S})$. By the Hahn–Banach theorem the linear functional L on $\mathcal{C}(\mathcal{S})$ is the restriction of a contractive linear functional L on $\mathcal{B}(\mathcal{S})$: The inequality

$$|Lf| \leq \|f\|_\infty$$

holds for every function $f(s)$ of s in \mathcal{S} which belongs to $\mathcal{B}(\mathcal{S})$.

If π is a measurable mapping of \mathcal{S} into Z , a contractive linear functional L_π on $\mathcal{B}(Z)$ is defined by

$$L_\pi f = L f_\pi$$

for every function $f(n)$ of n in Z which belongs to $\mathcal{B}(Z)$ with

$$f_\pi(s) = f(\pi s)$$

the composed function of s in \mathcal{S} which belongs to $\mathcal{B}(\mathcal{S})$.

A unique measure μ_π on the subsets of Z exists such that

$$L_\pi f = \int f(n) d\mu_\pi(n)$$

for every function $f(n)$ of n in Z which belongs to $\mathcal{C}(Z)$. The measure μ_π has total variation at most one since L_π is contractive. If a function

$$f(s) = f_\pi(\pi s)$$

of s in \mathcal{S} can be factored with $f_\pi(n)$ a function of n in Z which belongs to $\mathcal{C}(Z)$, then

$$L f = \int f_\pi(n) d\mu_\pi(n).$$

A subset E of \mathcal{S} is said to be saturated with respect to π if an image subset E_π of Z exists such that an element s of \mathcal{S} belongs to E if, and only if, if image πs in Z belongs to E_π . The measure μ is defined by

$$\mu(E) = \mu_\pi(E_\pi)$$

for every subset E of \mathcal{S} which is saturated with respect to π .

The definition of $\mu(E)$ applies to every measurable subset E of \mathcal{S} since the set is saturated with respect to some measurable mapping of \mathcal{S} onto Z , but it needs to be shown that the definition is independent of the measurable mapping π with respect to which E is saturated.

If α and β are measurable mappings of \mathcal{S} onto Z with respect to which E is saturated, then a measurable mapping γ of \mathcal{S} onto Z with respect to which E is saturated exists such that equality

$$\gamma(s_1) = \gamma(s_2)$$

for elements s_1 and s_2 of \mathcal{S} is equivalent to the equalities

$$\alpha(s_1) = \alpha(s_2)$$

and

$$\beta(s_1) = \beta(s_2).$$

Mappings α' and β' of Z onto Z then exist such that

$$\alpha(s) = \alpha'(\gamma s)$$

and

$$\beta(s) = \beta'(\gamma s)$$

for every element s of \mathcal{S} .

Continuous linear functionals L_α , L_β , and L_γ on $\mathcal{B}(Z)$ are defined by

$$L_\alpha f = L f_\alpha$$

and

$$L_\beta f = L f_\beta$$

and

$$L_\gamma f = L f_\gamma$$

for every element f of $\mathcal{B}(Z)$ with

$$f_\alpha(s) = f(\alpha s)$$

and

$$f_\beta(s) = f(\beta s)$$

and

$$f_\gamma(s) = f(\gamma s)$$

functions of s in \mathcal{S} which belong to $\mathcal{B}(\mathcal{S})$. It follows that

$$L_\alpha f = L_\gamma f'_\alpha$$

and

$$L_\beta f = L_\gamma f'_\beta$$

for every function $f(n)$ of n in Z which belongs to $\mathcal{B}(Z)$ with

$$f'_\alpha(n) = f(\alpha' n)$$

and

$$f'_{\beta}(n) = f(\beta'n)$$

functions of n in Z which belong to $\mathcal{B}(Z)$.

Measures μ_{α} , μ_{β} , and μ_{γ} on the subsets of Z are defined by

$$L_{\alpha}f = \int f(n)d\mu_{\alpha}(n) = \sum f(n)\mu'_{\alpha}(n)$$

and

$$L_{\beta}f = \int f(n)d\mu_{\beta}(n) = \sum f(n)\mu'_{\beta}(n)$$

and

$$L_{\gamma}f = \int f(n)d\mu_{\gamma}(n) = \sum f(n)\mu'_{\gamma}(n)$$

for every function $f(n)$ of n in Z which belongs to $\mathcal{C}(Z)$.

If functions $f(n)$ and $g(n)$ of n in Z belong to $\mathcal{C}(Z)$, then the function

$$h(n) = f(\alpha'n)g(\beta'n)$$

of n in Z belongs to $\mathcal{C}(Z)$. The action

$$L_{\gamma}h = \sum f(\alpha'n)g(\beta'n)\mu'_{\gamma}(n)$$

on the product function is a sum over the elements n of Z .

When the functions $f(n)$ and $g(n)$ of n in Z have no nonzero values other than one and the function $f(\alpha'n)$ of n in Z vanishes whenever the function $g(\beta'n)$ of n in Z vanishes, the function

$$h(n) = f(\alpha'n)$$

of n in Z has no nonzero value other than one. Since

$$L_{\alpha}f = L_{\gamma}h,$$

the identity

$$\sum f(n)\mu'_{\alpha}(n) = \sum f(\alpha'n)\mu'_{\gamma}(n)$$

holds with summation over the elements n of Z . If $g(\beta'n)$ does not vanish for some element n of Z , the choice of f can be made so that the identity reads

$$\mu'_{\alpha}(n) = \sum \mu'_{\gamma}(k)$$

with summation over the elements k of Z such that

$$n = \alpha'k.$$

The identity holds for every element n of Z by the arbitrariness of g .

A subset E of \mathcal{S} is saturated with respect to γ if it is saturated with respect to α and β . The identity

$$\mu_\alpha(E_\alpha) = \mu_\gamma(E_\gamma)$$

follows since E_α is the set of elements n of Z such that

$$n = \alpha'k$$

for an element k of E_γ . The identity

$$\mu_\beta(E_\beta) = \mu_\gamma(E_\gamma)$$

is obtained by a similar argument. It follows that

$$\mu_\alpha(E_\alpha) = \mu_\beta(E_\beta).$$

The identity

$$Lf = \int f_\pi(n)d\mu_\pi(n) = \int f(s)d\mu(s)$$

holds for every function

$$f(s) = f_\pi(\pi s)$$

of s in \mathcal{S} which can be factored by a measurable mapping π of \mathcal{S} onto Z with $f_\pi(n)$ a function of n in Z which belongs to $\mathcal{C}(Z)$. Since a dense set of elements f of $\mathcal{C}(\mathcal{S})$ can be so factored and since L is contractive, the total variation

$$\|\mu\| = \sup \|\mu_\pi\| \leq 1$$

of the measure μ is at most one. The identity

$$Lf = \int f(s)d\mu(s)$$

follows for every function $f(s)$ of s in \mathcal{S} which belongs to $\mathcal{C}(\mathcal{S})$.

This completes the proof of the theorem.

The Stone–Weierstrass theorem states that an algebra of continuous functions on a compact Hausdorff space is uniformly dense in the algebra of all continuous functions on the space if the algebra contains constants, if the algebra contains the complex conjugate of every function which it contains, and if distinct elements of the space are mapped into distinct complex numbers by some element of the algebra.

The Stone–Weierstrass theorem applies also to a locally compact Hausdorff space since it is a subspace of a compact Hausdorff space which contains only one other element, said to be at infinity. A continuous function on the locally compact space is said to vanish at infinity if it admits a continuous extension to the compact space which vanishes at infinity: For every positive number ϵ the function is bounded by ϵ outside of some compact set. An algebra of continuous functions which vanish at infinity is uniformly dense in the algebra of all continuous functions which vanish at infinity if it contains the complex conjugate of every function it contains and if distinct elements of the space are mapped into distinct complex numbers by some element of the algebra.

A proof of the Stone–Weierstrass theorem [2] is given from the Krein–Milman theorem [3].

The convex set of all nonnegative elements μ of $\mathcal{M}(\mathcal{S})$ of total variation

$$\mu(\mathcal{S}) \leq 1$$

at most one is a compact Hausdorff space in the weakest topology with respect to which the integral

$$\int f(s)d\mu(s)$$

is a continuous function of μ for every element f of $\mathcal{C}(\mathcal{S})$. The convex set is the closed convex span of its extreme points by the Krein–Milman theorem.

If μ is an element of the convex set, a convex decomposition

$$\mu = \frac{1}{2}\mu_+ + \frac{1}{2}\mu_-$$

into elements μ_+ and μ_- of the convex set is defined by a function $h(s)$ of s in \mathcal{S} which belongs to $\mathcal{C}(\mathcal{S})$ and satisfies the inequalities

$$-1 \leq h(s) \leq 1$$

for every element s of \mathcal{S} if the integral

$$\int h(s)d\mu(s) = 0$$

vanishes. The measures μ_+ and μ_- are defined on every measurable subset E of \mathcal{S} by integration

$$\mu_+(E) = \int_E [1 + h(s)]d\mu(s)$$

and

$$\mu_-(E) = \int_E [1 - h(s)]d\mu(s).$$

The measures

$$\mu_+ = \mu = \mu_-$$

are equal when μ is an extreme point of the convex set. The function $h(s)$ of s in \mathcal{S} vanishes almost everywhere with respect to μ . An element of $\mathcal{C}(\mathcal{S})$ is equal to a constant almost everywhere with respect to μ . The measure μ has no nonzero value other than one.

The Stone representation of the σ -algebra of measurable subsets of \mathcal{S} is an isomorphism into the Boolean algebra of regular open subsets of a compact Hausdorff space. A computation of the Stone representation is made from the set \mathcal{S}^\wedge of nonzero extreme points of the convex set of nonnegative elements of $\mathcal{M}(\mathcal{S})$ of total variation at most one. The set of all extreme points is a compact Hausdorff space in the weakest topology with respect to which the integral

$$\int f(s)d\mu(s)$$

is a continuous function of μ for every function $f(s)$ of s in \mathcal{S} which belongs to $\mathcal{C}(\mathcal{S})$. The set \mathcal{S}^\wedge is a locally compact Hausdorff space with the measure which is identically zero at infinity.

When E is a measurable subset of \mathcal{S} , a function $f(s)$ of s in \mathcal{S} is defined to have value one on E and value zero on the complement of E . If the function belongs to $\mathcal{C}(\mathcal{S})$, the function $f^\wedge(\mu)$ of μ in \mathcal{S}^\wedge has no nonzero value other than one. The set E^\wedge on which f^\wedge has the value one is both open and closed. Every subset of \mathcal{S}^\wedge which is both open and compact is the image of such a measurable subset of \mathcal{S} .

The measure problem concerns the structure of a nonnegative measure μ which belongs to $\mathcal{M}(\mathcal{S})$ when all subsets of \mathcal{S} are measurable. An element s of \mathcal{S} determines a measure which has value one on sets which contain s and which has value zero on sets which do not contain s . The measure is an element of \mathcal{S}^\wedge . The measure problem admits a solution when every element of \mathcal{S}^\wedge is determined by an element of \mathcal{S} .

If a nonnegative measure μ belongs to $\mathcal{M}(\mathcal{S})$, define E to be the set of elements of \mathcal{S} which belong to some finite set of zero measure. An element s of \mathcal{S} which does not belong to E defines a set of positive measure whose only element is s . The complement of E is countable since these measures have a convergent sum. It remains to show that E has zero measure.

It is sufficient to treat the case in which E is the full space \mathcal{S} . It needs to be shown that an element μ of $\mathcal{M}(\mathcal{S})$ vanishes identically if all finite subsets of \mathcal{S} have zero measure. The set of all nonnegative elements of $\mathcal{M}(\mathcal{S})$ of total variation at most one which have value zero on every finite subset of \mathcal{S} is convex. The set of measures μ is compact in the weakest topology with respect to which the integral

$$\int f(s)d\mu(s)$$

is a continuous function of μ for every function $f(s)$ of s in \mathcal{S} which belongs to $\mathcal{C}(\mathcal{S})$ since every function which vanishes outside of a finite set belongs to $\mathcal{C}(\mathcal{S})$.

The convex set is the closed convex span of its extreme points by the Krein–Milman theorem. A previous argument shows that the extreme points of the convex set are measures whose only nonzero value is one. No nonzero extreme point exists since these elements of \mathcal{S}^\wedge are represented by elements of \mathcal{S} and since elements of \mathcal{S} are denied by the action of the measure on finite sets. The convex set contains no nonzero element by the Krein–Milman theorem. No nonzero element of $\mathcal{M}(\mathcal{S})$ exists which is nonnegative and which has value zero on all finite sets.

The hypothesis which permits a solution of the measure problem is always satisfied.

Theorem 2. *Every element of \mathcal{S}^\wedge is represented by an element of \mathcal{S} when all subsets of \mathcal{S} are measurable.*

Proof of Theorem 2. The theorem is due to Ulam [5] when the set \mathcal{S} is the class of all subsets of Z : If a nontrivial measure μ is defined on all subsets of \mathcal{S} and has no nonzero value other than one, then an element s of \mathcal{S} exists such that μ has value one on sets which contain s and value zero on sets which do not contain s . Since the class of finite subsets of Z is countable, the desired conclusion is immediate if this class is a subset of \mathcal{S} of positive measure. Otherwise an element k of Z defines a measurable mapping π of \mathcal{S} onto Z which takes an infinite subset into its k -th element in the inherited ordering and which is defined arbitrarily on finite subsets. Since π maps the measure μ into a nontrivial measure on the subsets of Z whose only nonzero value is one, an element n_k of Z exists such that the measure μ has value one on the class of infinite subsets of Z whose k -th element is n_k and has value zero on the class of infinite subsets of Z which do not have n_k as k -th element. The infinite subset of Z whose k -th element is n_k for every element k of Z is then an element of \mathcal{S} which determines μ .

A theorem of Solovay [4] reduces the proof of the theorem to the case in which the Ulam argument applies. Argue by contradiction assuming that some set \mathcal{S} exists with all subsets measurable and some element of \mathcal{S}^\wedge not represented by an element of \mathcal{S} . Under this hypothesis Solovay constructs a Cohen model of set theory which contradicts the Ulam theorem: A non-negative measure is constructed which is defined on all subsets of the same set as Ulam which has value zero on all finite subsets and which does not vanish identically. The origin of the contradiction is the hypothesis that a nontrivial measure exists which is defined on all subsets of a set and whose only nonzero value is one, other than a measure determined by an element of the set.

This completes the proof of the theorem.

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