THE RIEMANN HYPOTHESIS

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ABSTRACT. A proof of the Riemann hypothesis is to be obtained for the zeta functions constructed from a discrete vector space of finite dimension over the skew–field of quaternions with rational numbers as coordinates in hyperbolic analysis on locally compact Abelian groups obtained by completion. Zeta functions are generated by a discrete group of symplectic transformations. The coefficients of a zeta function are eigenfunctions of Hecke operators defined by the group. In the nonsingular case the Riemann hypothesis is a consequence of the maximal accretive property of a Radon transformation defined in Fourier analysis. In the singular case the Riemann hypothesis is a consequence of the maximal accretive property of the restriction of the Radon transformation to a subspace defined by parity. The Riemann hypothesis for the Euler zeta function is a corollary.

1. GENERALIZATION OF THE GAMMA FUNCTION

The Riemann hypothesis is the conjecture made by Riemann that the Euler zeta function has no zeros in a half–plane larger than the half–plane which has no zeros by the convergence of the Euler product. When Riemann made his conjecture, zeros were of interest for polynomials since a polynomial is a product of linear factors determined by zeros. Polynomials having no zeros in the upper or the lower half–plane appear when the vector space of all polynomials with complex coefficients is given a scalar product defined by integration with respect to a nonnegative measure on the real line.

The vector space has an orthogonal basis consisting of a polynomial $S_n(z)$ of degree n in z for every nonnegative integer n. The polynomials have only real zeros and can be chosen to have real coefficients. A linear combination $S_n(z) - iS_{n+1}(z)$ of consecutive polynomials is a polynomial which has either the upper or lower half–plane free of zeros.

Riemann was familiar with examples of orthogonal polynomials constructed from hypergeometric series. For these special functions it happens that linear combinations of consecutive orthogonal polynomials have no zeros in a half–plane which is larger than the upper or the lower half–plane. The boundary of the larger half–plane is shifted from the real axis by a distance one–half.

The Riemann hypothesis is contained in the issue of explaining the observed shift in zeros. There is no reason to restrict the study to orthogonal polynomials since the efforts of Hermite and Stieltjes create a larger context. The contribution of Hermite is a class of entire

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functions which like polynomials are essentially determined by zeros. The contribution of Stieltjes is a treatment of integration on the real line as a representation of positive linear functionals on polynomials. His death from tuberculosis prevented application to the Hermite class. The remaining step [1] is preparation for a continuation of their efforts on the Riemann hypothesis.

The Riemann hypothesis for Hilbert spaces of entire functions [2] is a condition on Stieltjes spaces of entire functions which explains the observed shift in zeros and which implies the Riemann conjecture if it can be applied to the Euler zeta function. Such application is not obvious since the Euler zeta function has a singularity in the proposed half–plane of analyticity. It will be seen that the function satisfies a parity condition which permits removal of the singularity. The Euler zeta function is an analogue of the Euler gamma function, whose properties indicate a successful treatment of the zeta function.

The gamma function is an analytic function of s in the complex plane with the exception of singularities at the nonpositive integers which satisfies the recurrence relation

$$s\Gamma(s) = \Gamma(s+1).$$

A generalization of the gamma function is obtained with the factor of s in the recurrence relation replaced by an arbitrary function of s which is analytic and has positive real part in the right half–plane.

An analytic weight function is defined as a function $W(z)$ of z which is analytic and without zeros in the upper half–plane.

Hilbert spaces of functions analytic in the upper half–plane were introduced in Fourier analysis by Hardy. The weighted Hardy space $\mathcal{F}(W)$ is defined as the Hilbert space of functions $F(z)$ of z analytic in the upper half–plane such that the least upper bound

$$\|F\|^2_{\mathcal{F}(W)} = \sup \int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 dx$$

taken over all positive y is finite. The least upper bound is attained in the limit as y decreases to zero. The classical Hardy space is obtained when $W(z)$ is identically one. Multiplication by $W(z)$ is an isometric transformation of the classical Hardy space onto the weighted Hardy space with analytic weight function $W(z)$.

An isometric transformation of the weighted Hardy space $\mathcal{F}(W)$ into itself is defined by taking a function $F(z)$ of z into the function

$$F(z)(z-w)/(z-w^-)$$

of z when w is in the upper half–plane. The range of the transformation is the set of elements of the space which vanish at w. A continuous linear functional on the weighted Hardy space $\mathcal{F}(W)$ is defined by taking a function $F(z)$ of z into its value $F(w)$ at w whenever w is in the upper half–plane. The function

$$W(z)W(w)^-/[2\pi i(w^- - z)]$$

of z belongs to the space when w is in the upper half–plane and acts as reproducing kernel function for function values at w.

A Hilbert space of functions analytic in the upper half–plane which has dimension greater than one is isometrically equal to a weighted Hardy space if an isometric transformation of the space onto the subspace of functions which vanish at w is defined by taking $F(z)$ into

$$F(z)(z - w)/(z - w^{-})$$

when w is in the upper half–plane and if a continuous linear functional is defined on the space by taking $F(z)$ into $F(w)$ for w of the upper half–plane.

Examples of weighted Hardy spaces are constructed from the Euler gamma function. An analytic weight function

$$W(z) = \Gamma(s)$$

is defined by

$$s = \tfrac{1}{2} - iz.$$

A maximal accretive transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z + i)$ whenever the functions of z belong to the space.

A linear relation with domain and range in a Hilbert space is said to be accretive if the sum

$$\langle a, b\rangle + \langle b, a\rangle \geq 0$$

of scalar products in the space is nonnegative whenever (a, b) belongs to the graph of the relation. A linear relation is said to be maximal accretive if it is not the proper restriction of an accretive linear relation with domain and range in the same Hilbert space. A maximal accretive transformation with domain and range in a Hilbert space is a transformation which is a maximal accretive relation with domain and range in the Hilbert space.

Theorem 1. *A maximal accretive transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space if, and only if, the function*

$$W(z - \tfrac{1}{2}i)/W(z + \tfrac{1}{2}i)$$

of z admits an extension which is analytic and has nonnegative real part in the upper half–plane.

Proof of Theorem 1. A Hilbert space $\mathcal{H}$ whose elements are functions analytic in the upper half–plane is constructed when a maximal accretive transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z + i)$ whenever the functions of z belong to the space. The space $\mathcal{H}$ is constructed from the graph of the adjoint of the transformation which takes $F(z)$ into $F(z + i)$ whenever the functions of z belong to the space.

An element

$$F(z) = (F_{+}(z), F_{-}(z))$$

of the graph is a pair of analytic functions of z, which belong to the space $\mathcal{F}(W)$, such that the adjoint takes $F_+(z)$ into $F_-(z)$. The scalar product

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{F}(W)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{F}(W)}$$

of elements $F(z)$ and $G(z)$ of the graph is defined as the sum of scalar products in the space $\mathcal{F}(W)$. Scalar self–products are nonnegative in the graph since the adjoint of a maximal accretive transformation is accretive.

An element $K(w, z)$ of the graph is defined by

$$K_+(w, z) = W(z)W(w - \tfrac{1}{2}i)^-/[2\pi i(w^- + \tfrac{1}{2}i - z)]$$

and

$$K_-(w, z) = W(z)W(w + \tfrac{1}{2}i)^-/[2\pi i(w^- - \tfrac{1}{2}i - z)]$$

when w is in the half–plane

$$1 < iw^- - iw.$$

The identity

$$F_+(w + \tfrac{1}{2}i) + F_-(w - \tfrac{1}{2}i) = \langle F(t), K(w, t)\rangle$$

holds for every element

$$F(z) = (F_+(z), F_-(z))$$

of the graph. An element of the graph which is orthogonal to itself is orthogonal to every element of the graph.

An isometric transformation of the graph onto a dense subspace of $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into the function

$$F_+(z + \tfrac{1}{2}i) + F_-(z - \tfrac{1}{2}i)$$

of z in the half–plane

$$1 < iz^- - iz.$$

The reproducing kernel function for function values at w in the space $\mathcal{H}$ is the function

$$[W(z + \tfrac{1}{2}i)W(w - \tfrac{1}{2}i)^- + W(z - \tfrac{1}{2}i)W(w + \tfrac{1}{2}i)^-]/[2\pi i(w^- - z)]$$

of z in the half–plane when w is in the half–plane.

Division by $W(z + \frac{1}{2}i)$ is an isometric transformation of the space $\mathcal{H}$ onto a Hilbert space $\mathcal{L}$ whose elements are functions analytic in the half–plane and which contains the function

$$[\varphi(z) + \varphi(w)^-]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is in the half–plane,

$$\varphi(z) = W(z - \tfrac{1}{2}i)/W(z + \tfrac{1}{2}i).$$

A Hilbert space with the same reproducing kernel functions is given an axiomatic characterization in the Poisson representation [1] of functions which are analytic and have positive real part in the upper half–plane. The argument applies to the present space $\mathcal{L}$ whose elements are functions analytic in the smaller half–plane.

The function

$$[F(z) - F(w)]/(z - w)$$

of z belongs to $\mathcal{L}$ whenever the function $F(z)$ of z belongs to $\mathcal{L}$ if w is in the smaller half–plane. The identity

$$0 = \langle F(t), [G(t) - G(\alpha)]/(t - \alpha)\rangle_{\mathcal{L}} - \langle [F(t) - F(\beta)]/(t - \beta), G(t)\rangle_{\mathcal{L}}$$
$$-(\beta - \alpha^-)\langle [F(t) - F(\beta)]/(t - \beta), [G(t) - G(\alpha)]/(t - \alpha)\rangle_{\mathcal{L}}$$

holds for all functions $F(z)$ and $G(z)$ which belong to $\mathcal{L}$ when α and β are in the smaller half–plane.

An isometric transformation of the space $\mathcal{L}$ into itself is defined by taking a function $F(z)$ of z into the function

$$F(z) + (w - w^-)[F(z) - F(w)]/(z - w)$$

of z when w is in the smaller half–plane.

The same conclusion holds when w is in the upper half–plane by the preservation of the isometric property under iterated compositions. The elements of $\mathcal{L}$ are functions which have analytic extensions to the upper half–plane. The computation of reproducing kernel functions applies when w is in the upper half–plane. The function $\varphi(z)$ of z has an analytic extension with nonnegative real part in the upper half–plane.

Since multiplication by $W(z + \frac{1}{2}i)$ is an isometric transformation of the space $\mathcal{L}$ onto the space $\mathcal{H}$, the elements of $\mathcal{H}$ have analytic extensions to the upper half–plane. The function

$$[W(z + \tfrac{1}{2}i)W(w - \tfrac{1}{2}i)^- + W(z - \tfrac{1}{2}i)W(w + \tfrac{1}{2}i)^-]/[2\pi i(w^- - z)]$$

of z belongs to the space when w is in the upper half–plane and acts as reproducing kernel function for function values at w.

The argument is reversed to construct a maximal accretive transformation in the weighted Hardy space $\mathcal{F}(W)$ when the function $\phi(z)$ of z admits an extension which is analytic and has positive real part in the upper half–plane. The Poisson representation constructs a Hilbert space $\mathcal{L}$ whose elements are functions analytic in the upper half–plane and which contains the function

$$[\phi(z) + \phi(w)^-]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is in the upper half–plane. Multiplication by $W(z + \frac{1}{2}i)$ acts as an isometric transformation of the space $\mathcal{L}$

onto a Hilbert space $\mathcal{H}$ whose elements are functions analytic in the upper half–plane and which contains the function

$$[W(z + \tfrac{1}{2}i)W(w - \tfrac{1}{2}i)^- + W(z - \tfrac{1}{2}i)W(w + \tfrac{1}{2}i)^-]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is in the upper half–plane.

A transformation is defined in the space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z + i)$ whenever the functions of z belong to the space. The graph of the adjoint is a space of pairs

$$F(z) = (F_+(z), F_-(z))$$

of elements of the space such that the adjoint takes the function $F_+(z)$ of z into the function $F_-(z)$ of z. The graph contains

$$K(w, z) = (K_+(w, z), K_-(w, z))$$

with

$$K_+(w, z) = W(z)W(w - \tfrac{1}{2}i)^- /[2\pi i(w^- + \tfrac{1}{2}i - z)]$$

and

$$K_-(w, z) = W(z)W(w + \tfrac{1}{2}i)^- /[2\pi i(w^- - \tfrac{1}{2}i - z)]$$

when w is in the half–plane

$$1 < iw^- - iw.$$

The elements $K(w, z)$ of the graph span the graph of a restriction of the adjoint. The transformation in the space $\mathcal{F}(W)$ is recovered as the adjoint of the restricted adjoint.

A scalar product is defined on the graph of the restricted adjoint so that an isometric transformation of the graph of the restricted adjoint into the space $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into

$$F_+(z + \tfrac{1}{2}i) + F_-(z - \tfrac{1}{2}i).$$

The identity

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{F}(W)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{F}(W)}$$

holds for all elements

$$F(z) = (F_+(z), F_-(z))$$

and

$$G(z) = (G_+(z), G_-(z))$$

of the graph of the restricted adjoint. The restricted adjoint is accretive since scalar self–products are nonnegative in its graph. The adjoint is accretive since the transformation in the space $\mathcal{F}(W)$ is the adjoint of its restricted adjoint.

The accretive property of the adjoint is expressed in the inequality

$$\|F_+(t) - \lambda^- F_-(t)\|_{\mathcal{F}(W)} \leq \|F_+(t) + \lambda F_-(t)\|_{\mathcal{F}(W)}$$

for elements

$$F(z) = (F_+(z), F_-(z))$$

of the graph when λ is in the right half–plane. The domain of the contractive transformation which takes the function

$$F_+(z) + \lambda F_-(z)$$

of z into the function

$$F_+(z) - \lambda^- F_-(z)$$

of z is a closed subspace of the space $\mathcal{F}(W)$. The maximal accretive property of the adjoint is the requirement that the contractive transformation be everywhere defined for some, and hence every, λ in the right half–plane.

Since $K(w, z)$ belongs to the graph when w is in the half–plane

$$1 < iw^- - iw,$$

an element $H(z)$ of the space $\mathcal{F}(W)$ which is orthogonal to the domain of the accretive transformation satisfies the identity

$$H(w - \tfrac{1}{2}i) + \lambda H(w + \tfrac{1}{2}i) = 0$$

when w is in the upper half–plane. The function $H(z)$ of z admits an analytic extension to the complex plane which satisfies the identity

$$H(z) + \lambda H(z + i) = 0.$$

A zero of $H(z)$ is repeated with period i. Since

$$H(z)/W(z)$$

is analytic and of bounded type in the upper half–plane, the function $H(z)$ of z vanishes everywhere if it vanishes somewhere.

The space of elements $H(z)$ of the space $\mathcal{F}(W)$ which are solutions of the equation

$$H(z) + \lambda H(z + i) = 0$$

for some λ in the right half–plane has dimension zero or one. The dimension is independent of λ.

If τ is positive, multiplication by

$$\exp(i\tau z)$$

is an isometric transformation of the space $\mathcal{F}(W)$ into itself which takes solutions of the equation for a given λ into solutions of the equation with λ replaced by

$$\lambda \exp(\tau).$$

A solution $H(z)$ of the equation for a given λ vanishes identically since the function

$$\exp(-i\tau z)H(z)$$

of z belongs to the space for every positive number τ and has the same norm as the function $H(z)$ of z.

The transformation which takes $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space $\mathcal{F}(W)$ is maximal accretive since it is the adjoint of its adjoint, which is maximal accretive.

This completes the proof of the theorem.

The theorem has an equivalent formulation. A maximal accretive transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ for some real number h by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space if, and only if, the function

$$W(z + \tfrac{1}{2}ih)/W(z - \tfrac{1}{2}ih)$$

of z admits an extension which is analytic and has nonnegative real part in the upper half–plane.

Another theorem is obtained in the limit of small h. A maximal accretive transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $iF'(z)$ whenever the functions of z belong to the space if, and only if, the function

$$iW'(z)/W(z)$$

of z has nonnegative real part in the upper half–plane. The proof of the theorem is similar to the proof of Theorem 1. A maximal accretive transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $iF'(z)$ whenever the functions of z belong to the space if, and only if, the modulus of $W(x+iy)$ is a nondecreasing function of positive y for every real number x.

An entire function $E(z)$ of z is said to be of Hermite class if it has no zeros in the upper half–plane and if the modulus of $E(x+iy)$ is a nondecreasing function of positive y for every real number x. The Hermite class is also known as the Pólya class. Entire functions of Hermite class are limits of polynomials having no zeros in the upper half–plane [1]. Such polynomials appear in the Stieltjes representation of positive linear functions on polynomials.

An Euler weight function is defined as an analytic weight function $W(z)$ such that a maximal accretive transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ whenever

h is in the interval $[-1, 1]$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space.

If a function $\phi(z)$ of z is analytic and has positive real part in the upper half–plane, a logarithm of the function is defined continuously in the half–plane with values in the strip of width π centered on the real line. The inequalities

$$-\pi \le i\log\phi(z)^- - i\log\phi(z) \le \pi$$

are satisfied. A function $\phi_h(z)$ of z which is analytic and has nonnegative real part in the upper half–plane is defined when h is in the interval $(-1, 1)$ by the integral

$$\log\phi_h(z) = \sin(\pi h)\int_{-\infty}^{+\infty} \frac{\log\phi(z-t)dt}{\cos(2\pi it) + \cos(\pi h)}.$$

An application of the Cauchy formula in the upper half–plane shows that the function

$$\frac{\sin(\pi h)}{\cos(2\pi iz) + \cos(\pi h)} = \int_{-\infty}^{+\infty} \exp(2\pi itz)\,\frac{\exp(\pi ht - \exp(-\pi ht)}{\exp(\pi t) - \exp(-\pi t)}\,dt$$

of z is the Fourier transform of a function

$$\frac{\exp(\pi ht) - \exp(-\pi ht)}{\exp(\pi t) - \exp(-\pi t)}$$

of positive t which is square integrable with respect to Lebesgue measure and is bounded by h when h is in the interval $(0, 1)$.

The identity

$$\phi_{-h}(z) = \phi_h(z)^{-1}$$

is satisfied. The function

$$\phi(z) = \lim \phi_h(z)$$

of z is recovered in the limit as h increases to one. The identity

$$\phi_{a+b}(z) = \phi_a(z - \tfrac{1}{2}ib)\phi_b(z + \tfrac{1}{2}ia)$$

when a, b, and $a + b$ belong to the interval $(-1, 1)$ is a consequence of the trigonometric identity

$$\frac{\sin(\pi a + \pi b)}{\cos(2\pi iz) + \cos(\pi a + \pi b)}$$
$$= \frac{\sin(\pi a)}{\cos(2\pi iz + \pi b) + \cos(\pi a)} + \frac{\sin(\pi b)}{\cos(2\pi iz - \pi a) + \cos(\pi b)}.$$

An Euler weight function $W(z)$ is defined within a constant factor by the limit

$$iW'(z)/W(z) = \lim \frac{\log\phi_h(z)}{h} = \pi\int_{-\infty}^{+\infty} \frac{\log\phi(z-t)dt}{1 + \cos(2\pi it)}.$$

as h decreases to zero. The identity

$$W(z + \tfrac{1}{2}ih) = W(z - \tfrac{1}{2}ih)\phi_h(z)$$

applies when h is in the interval $(-1, 1)$. The identity reads

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z)$$

in the limit as h increases to one.

An Euler weight function $W(z)$ is constructed which satisfies the identity

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z)$$

for a given nontrivial function $\phi(z)$ of z which is analytic and has nonnegative real part in the upper half–plane.

A factorization of Euler weight functions is a consequence of the construction of Euler weight functions. If $W(z)$ is an Euler weight function, Euler weight functions $W_+(z)$ and $W_-(z)$ exist such that

$$W(z) = W_+(z)/W_-(z)$$

and such that the real part of

$$W_+(z - \tfrac{1}{2}ih)/W_+(z + \tfrac{1}{2}ih)$$

and of

$$W_-(z - \tfrac{1}{2}ih)/W_-(z + \tfrac{1}{2}ih)$$

is greater than or equal to one when z is in the upper half–plane and h is in the interval $[0, 1]$. The functions

$$|W_+(x + iy)|$$

and

$$|W_-(x + iy)|$$

of positive y are nondecreasing for every real number x.

If a maximal accretive transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space, then the identity

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z)$$

holds for a function $\phi(z)$ of z which is analytic and has nonnegative real part in the upper half–plane. The analytic weight function $W(z)$ is the product of an Euler weight function and an entire function which is periodic of period i and has no zeros.

If $W(z)$ is an Euler weight function, the maximal accretive transformation defined for h in the interval $[0, 1]$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space $\mathcal{F}(W)$ is subnormal: The transformation is the restriction to an invariant subspace of a normal transformation in the larger Hilbert space $\mathcal{H}$ of (equivalence classes of) Baire functions $F(x)$ of real x for which the integral

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt$$

converges. The passage to boundary value functions maps the space $\mathcal{F}(W)$ isometrically into the space $\mathcal{H}$.

A linear functional on polynomials with complex coefficients is said to be nonnegative if it has nonnegative values on polynomials whose values on the real axis are nonnegative. A positive linear functional on polynomials is a nonnegative linear functional on polynomials which does not vanish identically. A nonnegative linear functional on polynomials is represented as an integral with respect to a nonnegative measure μ on the Baire subsets of the real line. The linear functional takes a polynomial $F(z)$ into the integral

$$\int F(t) d\mu(t).$$

Stieltjes examines the action of a positive linear functional on polynomials of degree less than r for a positive integer r. A polynomial which has nonnegative values on the real axis is a product

$$F^*(z)F(z)$$

of a polynomial $F(z)$ and the conjugate polynomial

$$F^*(z) = F(z^-)^-.$$

If the positive linear functional does not annihilate

$$F^*(z)F(z)$$

for any nontrivial polynomial $F(z)$ of degree less than r, a Hilbert space exists whose elements are the polynomials of degree less than r and whose scalar product

$$\langle F(t), G(t) \rangle$$

is defined as the action of the positive linear functional on the polynomial

$$G^*(z)F(z).$$

Stieltjes shows that the Hilbert space of polynomials of degree less than r is contained isometrically in a weighted Hardy space $\mathcal{F}(W)$ whose analytic weight function $W(z)$ is a polynomial of degree r having no zeros in the upper half–plane.

An axiomatization of the Stieltjes spaces is stated in a general context [1]. Hilbert spaces are examined whose elements are entire functions and which have these properties:

(H1) Whenever an entire function $F(z)$ of z belongs to the space and has a nonreal zero w, the entire function

$$F(z)(z - w^-)/(z - w)$$

of z belongs to the space and has the same norm as $F(z)$.

(H2) A continuous linear functional on the space is defined by taking a function $F(z)$ of z into its value $F(w)$ at w for every nonreal number w.

(H3) The entire function

$$F^*(z) = F(z^-)^-$$

of z belongs to the space and has the same norm as $F(z)$ whenever the entire function $F(z)$ of z belongs to the space.

An example of a Hilbert space of entire functions which satisfies the axioms is obtained when an entire function $E(z)$ of z satisfies the inequality

$$|E(x - iy)| < |E(x + iy)|$$

for all real x when y is positive. A weighted Hardy space $\mathcal{F}(W)$ is defined with analytic weight function

$$W(z) = E(z).$$

A Hilbert space $\mathcal{H}(E)$ which is contained isometrically in the space $\mathcal{F}(W)$ is defined as the set of entire functions $F(z)$ of z such that the entire functions $F(z)$ and $F^*(z)$ of z belong to the space $\mathcal{F}(W)$. The entire function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

of z belongs to the space $\mathcal{H}(E)$ for every complex number w and acts as reproducing kernel function for function values at w.

A Hilbert space $\mathcal{H}$ of entire functions which satisfies the axioms (H1), (H2), and (H3) is isometrically equal to a space $\mathcal{H}(E)$ if it contains a nonzero element. The proof applies reproducing kernel functions which exist by the axiom (H2).

For every nonreal number w a unique entire function $K(w, z)$ of z exists which belongs to the space and acts as reproducing kernel function for function values at w. The function does not vanish identically since the axiom (H1) implies that some element of the space has a nonzero value at w when some element of the space does not vanish identically. The scalar self–product $K(w, w)$ of the function $K(w, z)$ of z is positive. The axiom (H3) implies the symmetry

$$K(w^-, z) = K(w, z^-)^-.$$

If λ is a nonreal number, the set of elements of the space which vanish at λ is a Hilbert space of entire functions which is contained isometrically in the given space. The function

$$K(w, z) - K(w, \lambda)K(\lambda, \lambda)^{-1}K(\lambda, z)$$

of z belongs to the subspace and acts as reproducing kernel function for function values at w. The identity

$$[K(w,z) - K(w,\lambda)K(\lambda,\lambda)^{-1}K(\lambda,z)](z-\lambda^-)(w^- - \lambda)$$
$$= [K(w,z) - K(w,\lambda^-)K(\lambda^-,\lambda^-)^{-1}K(\lambda^-,z)](z-\lambda)(w^- - \lambda^-)$$

is a consequence of the axiom (H1).

An entire function $E(z)$ of z exists such that the identity

$$K(w,z) = [E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

holds for all complex z when w is not real. The entire function can be chosen with a zero at λ when λ is in the lower half–plane. The function is then unique within a constant factor of absolute value one. A space $\mathcal{H}(E)$ exists and is isometrically equal to the given space $\mathcal{H}$.

Examples [1] of Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) are constructed from the analytic weight function

$$W(z) = \Gamma(\tfrac{1}{2} - iz)/\Gamma(h - iz)$$

when $h \geq \frac{1}{2}$. The space is contained isometrically in the weighted Hardy space $\mathcal{F}(W)$ and contains every entire function $F(z)$ such that the functions $F(z)$ and $F^*(z)$ of z belong to the space $\mathcal{F}(W)$. The space of entire functions is isometrically equal to a space $\mathcal{H}(E)$ whose defining function $E(z)$ is computed [1]. Properties of the space motivate the definition of a class of Hilbert spaces of entire functions.

An Euler space of entire functions is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) such that a maximal accretive transformation is defined in the space for every h in the interval $[-1,1]$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space.

Theorem 2. *A maximal accretive transformation is defined in a Hilbert space $\mathcal{H}(E)$ of entire functions for a real number h by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space if, and only if, a Hilbert space $\mathcal{H}$ of entire functions exists which contains the function*

$$[E(z+\tfrac{1}{2}ih)E(w-\tfrac{1}{2}ih)^- - E^*(z+\tfrac{1}{2}ih)E(w^- + \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$
$$+ [E(z-\tfrac{1}{2}ih)E(w+\tfrac{1}{2}ih)^- - E^*(z-\tfrac{1}{2}ih)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w for every complex number w.

Proof of Theorem 2. The space $\mathcal{H}$ is constructed from the graph of the adjoint of the transformation which takes $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space. An element

$$F(z) = (F_+(z), F_-(z))$$

of the graph is a pair of entire functions of z, which belong to the space $\mathcal{H}(E)$, such that the adjoint takes $F_+(z)$ into $F_-(z)$. The scalar product

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{H}(E)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{H}(E)}$$

of elements $F(z)$ and $G(z)$ of the graph is defined as a sum of scalar products in the space $\mathcal{H}(E)$. Scalar self–products are nonnegative since the adjoint of a maximal accretive transformation is accretive.

An element

$$K(w, z) = (K_+(w, z), K_-(w, z))$$

of the graph is defined for every complex number w by

$$K_+(w, z) = [E(z)E(w - \tfrac{1}{2}ih)^- - E^*(z)E(w^- + \tfrac{1}{2}ih)]/[2\pi i(w^- + \tfrac{1}{2}ih - z)]$$

and

$$K_-(w, z) = [E(z)E(w + \tfrac{1}{2}ih)^- - E^*(z)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - \tfrac{1}{2}ih - z)].$$

The identity

$$F_+(w + \tfrac{1}{2}ih) + F_-(w - \tfrac{1}{2}ih) = \langle F(t), K(w, t)\rangle$$

holds for every element

$$F(z) = (F_+(z), F_-(z))$$

of the graph. An element of the graph which is orthogonal to itself is orthogonal to every element of the graph.

A partially isometric transformation of the graph onto a dense subspace of the space $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into the entire function

$$F_+(z + \tfrac{1}{2}ih) + F_-(z - \tfrac{1}{2}ih)$$

of z. The reproducing kernel function for function values at w in the space $\mathcal{H}$ is the function

$$\begin{aligned}&[E(z + \tfrac{1}{2}ih)E(w - \tfrac{1}{2}ih)^- - E^*(z + \tfrac{1}{2}ih)E(w^- + \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]\\ &+ [E(z - \tfrac{1}{2}ih)E(w + \tfrac{1}{2}ih)^- - E^*(z - \tfrac{1}{2}ih)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]\end{aligned}$$

of z for every complex number w.

This completes the construction of a Hilbert space $\mathcal{H}$ of entire functions with the desired reproducing kernel functions when the maximal accretive transformation exists in the space $\mathcal{H}(E)$. The argument is reversed to construct the maximal accretive transformation in the space $\mathcal{H}(E)$ when the Hilbert space of entire functions with the desired reproducing kernel functions exists.

A transformation is defined in the space $\mathcal{H}(E)$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space. The graph of the adjoint is a space of pairs

$$F(z) = (F_+(z), F_-(z))$$

such that the adjoint takes the function $F_+(z)$ of z into the function $F_-(z)$ of z. The graph contains

$$K(w,z) = (K_+(w,z), K_-(w,z))$$

with

$$K_+(w,z) = [E(z)E(w - \tfrac{1}{2}ih)^- - E^*(z)E(w^- + \tfrac{1}{2}ih)]/[2\pi i(w^- + \tfrac{1}{2}ih - z)]$$

and

$$K_-(w,z) = [E(z)E(w + \tfrac{1}{2}ih)^- - E^*(z)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - \tfrac{1}{2}ih - z)]$$

for every complex number w. The elements $K(w,z)$ of the graph span the graph of a restriction of the adjoint. The transformation in the space $\mathcal{H}(E)$ is recovered as the adjoint of its restricted adjoint.

A scalar product is defined on the graph of the restricted adjoint so that an isometric transformation of the graph of the restricted adjoint into the space $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into

$$F_+(z + \tfrac{1}{2}ih) + F_-(z - \tfrac{1}{2}ih).$$

The identity

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{H}(E)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{H}(E)}$$

holds for all elements

$$F(z) = (F_+(z), F_-(z))$$

of the graph of the restricted adjoint. The restricted adjoint is accretive since scalar self–products are nonnegative in its graph. The adjoint is accretive since the transformation in the space $\mathcal{H}(E)$ is the adjoint of its restricted adjoint.

The accretive property of the adjoint is expressed in the inequality

$$\|F_+(t) - \lambda^- F_-(t)\|_{\mathcal{H}(E)} \le \|F_+(t) + \lambda F_-(t)\|_{\mathcal{H}(E)}$$

for elements

$$F(z) = (F_+(z), F_-(z))$$

of the graph when λ is in the right half–plane. The domain of the contractive transformation which takes the function

$$F_+(z) + \lambda F_-(z)$$

of z into the function

$$F_+(z) - \lambda^- F_-(z)$$

of z is a closed subspace of the space $\mathcal{H}(E)$. The maximal accretive property of the adjoint is the requirement that the contractive transformation be everywhere defined for some, and hence every, λ in the right half–plane.

Since $K(w, z)$ belongs to the graph for every complex number w, an entire function $H(z)$ of z which belongs to the space $\mathcal{H}(E)$ and is orthogonal to the domain is a solution of the equation

$$H(z) + \lambda H(z + i) = 0.$$

The function vanishes identically if it has a zero since zeros are repeated periodically with period i and since the function

$$H(z)/E(z)$$

of z is of bounded type in the upper half–plane. The space of solutions has dimension zero or one. The dimension is zero since it is independent of λ.

The transformation which takes $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space $\mathcal{H}(E)$ is maximal accretive since it is the adjoint of its adjoint, which is maximal accretive.

This completes the proof of the theorem.

The defining function $E(z)$ of an Euler space of entire functions is of Hermite class since the function

$$E(z - \tfrac{1}{2}ih)/E(z + \tfrac{1}{2}ih)$$

of z is of bounded type and of nonpositive mean type in the upper half–plane when h is in the interval $(0, 1)$. Since the function is bounded by one on the real axis, it is bounded by one in the upper half–plane. The modulus of $E(x + iy)$ is a nondecreasing function of positive y for every real x. An entire function $F(z)$ of z which belongs to the space $\mathcal{H}(E)$ is of Hermite class if it has no zeros in the upper half–plane and if the inequality

$$|F(x - iy)| \le |F(x + iy)|$$

holds for all real x when y is positive.

In a given Stieltjes space $\mathcal{H}(E)$ multiplication by z is the transformation which takes $F(z)$ into $zF(z)$ whenever the functions of z belong to the space. Multiplication by z need not be a densely defined transformation in the space, but if it is not, the orthogonal complement of the domain of multiplication by z has dimension one. If $E(z) = A(z)-iB(z)$ for entire functions $A(z)$ and $B(z)$ of z which are real for real z, an entire function

$$S(z) = A(z)u + B(z)v$$

of z which belongs to the orthogonal complement of the domain of multiplication by z is a linear combination of $A(z)$ and $B(z)$ with complex coefficients u and v. This result is a consequence of the identity

$$[K(w, z)S(w) - K(w, w)S(z)]/(z - w) = [K(w^-, z)S(w^-) - K(w^-, w^-)S(z)]/(z - w^-)$$

which characterizes functions $S(z)$ of z which belong to the space and are linear combinations of u and v. The identity

$$v^- u = u^- v$$

is then satisfied.

When multiplication by z is not densely defined in a Stieltjes space with defining function

$$E(b,z) = A(b,z) - iB(b,z)$$

and when the domain of multiplication by z contains a nonzero element, the closure of the domain of multiplication by z is a Stieltjes space with defining function

$$E(a,z) = A(a,z) - iB(a,z)$$

which is contained isometrically in the given space. The defining function can be chosen so that the matrix equation

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z)) \begin{pmatrix} 1 - \pi u v^- z & \pi u u^- z \\ -\pi v v^- z & 1 + \pi v u^- z \end{pmatrix}$$

holds for complex numbers u and v such that

$$v^- u = u^- v.$$

A Stieltjes space of dimension r whose elements are the polynomials of degree less than r has a polynomial

$$E(r,z) = A(r,z) - iB(r,z)$$

of degree r as defining function. A Stieltjes space of dimension n whose elements are the polynomials of degree less than n and which is contained isometrically in the given space exists for every positive integer n less than r. The defining function

$$E(n,z) = A(n,z) - iB(n,z)$$

of the space can be chosen so that the matrix equation

$$(A(n+1,z), B(n+1,z)) = (A(n,z), B(n,z)) \begin{pmatrix} 1 - \pi u_n v_n^- z & \pi u_n u_n^- z \\ -\pi v_n v_n^- z & 1 + \pi v_n u_n^- z \end{pmatrix}$$

is satisfied. The initial defining function can be chosen so that the equation holds when n is zero with

$$(A(0,z), B(0,z)) = (1,0).$$

A Stieltjes space is defined by the function

$$E(t,z) = (n+1-t)E(n,z) + (t-n)E(n+1,z)$$

when $n \leq t \leq n+1$. The space is contained contractively in the Stieltjes space with defining function $E(n+1, z)$ and contains isometrically the Stieltjes space with defining function $E(n, z)$.

A nondecreasing matrix function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

of t in the interval $[0, r]$ is defined by

$$m(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$m(n+1) - m(n) = \begin{pmatrix} \pi u_n u_n^- & \pi v_n u_n^- \\ \pi v_n u_n^- & \pi v_n v_n^- \end{pmatrix}$$

for every nonnegative integer n less than r, and

$$m(t) = (n+1-t)m(n) + (t-n)m(n+1)$$

when $n < t < n+1$.

The differential equation

$$(A'(t,z), B'(t,z))I = z(A(t,z), B(t,z))m'(t)$$

is satisfied when t is in an interval $(n, n+1)$ with the prime indicating differentiation with respect to t and with

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since $A(t,z)$ and $B(t,z)$ are continuous functions of t, the integral equation

$$(A(b,z), B(b,z))I - (A(a,z), B(a,z))I = z \int_a^b (A(t,z), B(t,z)) dm(t)$$

is satisfied when a and b are in the interval $[0, r]$.

The integral equation for Stieltjes spaces of finite dimension admits a generalization to Stieltjes spaces of infinite dimension. The generalization applies a continuous function of positive t whose values are matrices

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

with real entries such that the matrix inequality

$$m(a) \leq m(b)$$

holds when a is less than b. It is assumed that $\alpha(t)$ is positive when t is positive, that

$$\lim \alpha(t) = 0$$

as t decreases to zero, and that the integral

$$\int_0^1 \alpha(t) d\gamma(t)$$

converges.

When a is positive, the integral equation

$$M(a,b,z)I - I = z \int_a^b M(a,t,z) dm(t)$$

admits a unique continuous solution

$$M(a,b,z) = \begin{pmatrix} A(a,b,z) & B(a,b,z) \\ C(a,b,z) & D(a,b,z) \end{pmatrix}$$

as a function of b greater than or equal to a for every complex number z. The entries of the matrix are entire functions of z which are self–conjugate and of Hermite class for every b. The matrix has determinant one. The identity

$$M(a,c,z) = M(a,b,z)M(b,c,z)$$

holds when $a \leq b \leq c$.

A bar is used to denote the conjugate transpose

$$M^- = \begin{pmatrix} A^- & C^- \\ B^- & D^- \end{pmatrix}$$

of a square matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with complex entries and also for the conjugate transpose

$$c^- = (c_+^-, c_-^-)$$

of a column vector

$$c = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

with complex entries. The space of column vectors with complex entries is a Hilbert space of dimension two with scalar product

$$\langle u, v \rangle = v^- u = v_+^- u_+ + v_-^- u_- .$$

When a and b are positive with a less than or equal to b, a unique Hilbert space $\mathcal{H}(M(a,b))$ exists whose elements are pairs

$$F(z) = \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

of entire functions of z such that a continuous transformation of the space into the Hilbert space of column vectors is defined by taking $F(z)$ into $F(w)$ for every complex number w and such that the adjoint takes a column vector c into the element

$$[M(a,b,z)IM(a,b,w)^- - I]c/[2\pi(z - w^-)]$$

of the space.

An entire function

$$E(c,z) = A(c,z) - iB(c,z)$$

of z which is of Hermite class exists for every positive number c such that the self–conjugate entire functions $A(c,z)$ and $B(c,z)$ satisfy the identity

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))M(a,b,z)$$

when a is less than or equal to b and such that the entire functions

$$E(c,z)\exp[\beta(c)z]$$

of z converge to one uniformly on compact subsets of the complex plane as c decreases to zero.

A space $\mathcal{H}(E(c))$ exists for every positive number c. The space $\mathcal{H}(E(a))$ is contained contractively in the space $\mathcal{H}(E(b))$ when a is less than or equal to b. The inclusion is isometric on the orthogonal complement in the space $\mathcal{H}(E(a))$ of the elements which are linear combinations

$$A(a,z)u + B(a,z)v$$

with complex coefficients u and v. These elements form a space of dimension zero or one since the identity

$$v^-u = u^-v$$

is satisfied.

A positive number b is said to be singular with respect to the function $m(t)$ of t if it belongs to an interval (a,c) such that equality holds in the inequality

$$[\beta(c) - \beta(a)]^2 \leq [\alpha(c) - \alpha(a)][\gamma(c) - \gamma(a)]$$

with $m(b)$ unequal to $m(a)$ and unequal to $m(c)$. A positive number is said to be regular with respect to $m(t)$ if it is not singular with respect to the function of t.

If a and c are positive numbers such that a is less than c and if an element b of the interval (a, c) is regular with respect to $m(t)$, then the space $\mathcal{H}(M(a, b))$ is contained isometrically in the space $\mathcal{H}(M(a, c))$ and multiplication by $M(a, b, z)$ is an isometric transformation of the space $\mathcal{H}(M(b, c))$ onto the orthogonal complement of the space $\mathcal{H}(M(a, b))$ in the space $\mathcal{H}(M(a, c))$.

If a and b are positive numbers such that a is less than b and if a is regular with respect to $m(t)$, then the space $\mathcal{H}(E(a))$ is contained isometrically in the space $\mathcal{H}(E(b))$ and an isometric transformation of the space $\mathcal{H}(M(a, b))$ onto the orthogonal complement of the space $\mathcal{H}(E(a))$ in the space $\mathcal{H}(E(b))$ is defined by taking

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

into

$$\sqrt{2}\ [A(a, z)F_+(z) + B(a, z)F_-(z)].$$

A function $\tau(t)$ of positive t with real values exists such that the function

$$m(t) + Iih(t)$$

of positive t with matrix values is nondecreasing for a function $h(t)$ of t with real values if, and only if, the functions

$$\tau(t) - h(t)$$

and

$$\tau(t) + h(t)$$

of positive t are nondecreasing. The function $\tau(t)$ of t, which is continuous and nondecreasing, is called a greatest nondecreasing function such that

$$m(t) + Ii\tau(t)$$

is nondecreasing. The function is unique within an added constant.

If a and b are positive numbers such that a is less than b, multiplication by

$$\exp(ihz)$$

is a contractive transformation of the space $\mathcal{H}(E(a))$ into the space $\mathcal{H}(E(b))$ for a real number h, if, and only if, the inequalities

$$\tau(a) - \tau(b) \le h \le \tau(b) - \tau(a)$$

are satisfied. The transformation is isometric when a is regular with respect to $m(t)$.

An analytic weight function $W(z)$ may exist such that multiplication by

$$\exp(i\tau(c)z)$$

is an isometric transformation of the space $\mathcal{H}(E(c))$ into the weighted Hardy space $\mathcal{F}(W)$ for every positive number c which is regular with respect to $m(t)$. The analytic weight function is unique within a constant factor of absolute value one if the function

$$\alpha(t) + \gamma(t)$$

of positive t is unbounded in the limit of large t. The function

$$W(z) = \lim E(c, z) \exp(i\tau(c)z)$$

can be chosen as a limit as c increases to infinity uniformly on compact subsets of the upper half–plane.

If multiplication by

$$\exp(i\tau z)$$

is an isometric transformation of a space $\mathcal{H}(E)$ into the weighted Hardy space $\mathcal{F}(W)$ for some real number τ and if the space $\mathcal{H}(E)$ contains an entire function $F(z)$ whenever its product with a nonconstant polynomial belongs to the space, then the space $\mathcal{H}(E)$ is isometrically equal to the space $\mathcal{H}(E(c))$ for some positive number c which is regular with respect to $m(t)$.

The Hilbert spaces of entire functions constructed from an Euler weight function are Euler spaces of entire functions.

Theorem 3. *A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains a nonzero element is an Euler space of entire functions if it contains an entire function whenever its product with a nonconstant polynomial belongs to the space and if multiplication by* $\exp(i\tau z)$ *is for some real number* τ *an isometric transformation of the space into the weighted Hardy space* $\mathcal{F}(W)$ *of an Euler weight function* $W(z)$.

Proof of Theorem 3. It can be assumed that τ vanishes since the function

$$\exp(-i\tau z)W(z)$$

is an Euler weight function whenever the function $W(z)$ of z is an Euler weight function.

The given Hilbert space of entire functions is isometrically equal to a space $\mathcal{H}(E)$ for an entire function $E(z)$ which has no real zeros since an entire function belongs to the space whenever its product with a nonconstant polynomial belongs to the space.

An accretive transformation is defined in the space $\mathcal{H}(E)$ when h is in the interval $[0, 1]$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space since the space is contained isometrically in the space $\mathcal{F}(W)$ and since an accretive transformation is defined in the space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space. It remains to prove the maximal accretive property of the transformation in the space $\mathcal{H}(E)$.

The ordering theorem for Hilbert spaces of entire functions applies to spaces which satisfy the axioms (H1), (H2), and (H3) and which are contained isometrically in a weighted Hardy space $\mathcal{F}(W)$ when a space contains an entire function whenever its product with a nonconstant polynomial belongs to the space. One space is properly contained in the other when the two spaces are not identical.

A Hilbert space $\mathcal{H}$ of entire functions which satisfies the axioms (H1) and (H2) and which contains a nonzero element need not satisfy the axiom (H3). Multiplication by $\exp(iaz)$ is for some real number a an isometric transformation of the space onto a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3).

A space $\mathcal{H}$ which satisfies the axioms (H1) and (H2) and which is contained isometrically in the space $\mathcal{F}(W)$ is defined as the closure in the space $\mathcal{F}(W)$ of the set of those elements of the space which functions $F(z+ih)$ of z for functions $F(z)$ of z belonging to the space $\mathcal{H}(E)$. An example of a function $F(z+ih)$ of z is obtained for every element of the space $\mathcal{H}(E)$ which is a function $F(z)$ of z such that the function $z^2F(z)$ of z belongs to the space $\mathcal{H}(E)$. The space $\mathcal{H}$ contains an entire function whenever its product with a nonconstant polynomial belongs to the space.

The function

$$E(z)/W(z)$$

of z is of bounded type in the upper half–plane and has the same mean type as the function

$$E(z+ih)/W(z+ih)$$

of z which is of bounded type in the upper half–plane. Since the function

$$W(z+ih)/W(z)$$

of z is of bounded type and has zero mean type in the upper half–plane, the function

$$E(z+ih)/E(z)$$

of z is of bounded type and of zero mean type in the upper half–plane.

If a function $F(z)$ of z is an element of the space $\mathcal{H}(E)$ such that the functions

$$F(z)/W(z)$$

and

$$F^*(z)/W(z)$$

of z have equal mean type in the upper half–plane, and such that the functions

$$G(z) = F(z+ih)$$

and

$$G^*(z) = F^*(z-ih)$$

of z belong to the space $\mathcal{F}(W)$, then the functions

$$G(z)/W(z)$$

and

$$G^*(z)/W(z)$$

of z have equal mean type in the upper half–plane. It follows that the space $\mathcal{H}$ satisfies the axiom (H3).

Equality of the spaces $\mathcal{H}$ and $\mathcal{H}(E)$ follows when the space $\mathcal{H}$ is contained in the space $\mathcal{H}(E)$ and when the space $\mathcal{H}(E)$ is contained in the space $\mathcal{H}$.

The function $F(z+ih)$ of z belongs to the space $\mathcal{H}(E)$ whenever the function $F(z)$ of z belongs to the space $\mathcal{H}$ and the function $F(z+ih)$ of z belongs to the space $\mathcal{F}(W)$ since the spaces $\mathcal{H}$ and $\mathcal{H}(E)$ satisfy the axiom (H3). If the space $\mathcal{H}(E)$ is contained in the space $\mathcal{H}$, then the space $\mathcal{H}$ is contained in the space $\mathcal{H}(E)$. If the space $\mathcal{H}$ is contained in the space $\mathcal{H}(E)$, then the space $\mathcal{H}(E)$ is contained in the space $\mathcal{H}$.

Since the transformation T which takes $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space $\mathcal{H}(E)$ is subnormal, the domain of the adjoint T^* of T contains the domain of T. A dense subspace of the graph of T^* is determined by elements of the domain of T^* which belong to the domain of T. The accretive property of T implies the accretive property of T^*. The maximal accretive property of T follows since T is the adjoint of T^*.

This completes the proof of the theorem.

An associated Euler space of entire functions exists for every Euler weight function: If $W(z)$, is an Euler weight function, a nontrivial entire function $F(z)$ of z exists such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

belong to the weighted Hardy space $\mathcal{F}(W)$ for some positive number τ. The set of such an entire functions is then an Euler space of entire functions which is mappeded isometrically in the weighted Hardy space $\mathcal{F}(W)$ on multiplication by $\exp(\pi i\tau z)$.

The construction of an associated Euler space of entire functions reduces to the case in which the function

$$|W(x+iy)|$$

of positive y, or its reciprocal, is nondecreasing for every real number x. If the function is nondecreasing, an entire function $E_0(z)$ of Hermite class exists such that the real part of

$$E_0(z)/W(z)$$

is nonnegative in the upper half–plane. The desired Euler space of entire functions is easily constructed.

If the reciprocal is nondecreasing, then

$$W(z) = \Gamma(h - iz) W_0(z)$$

for an Euler weight function $W_0(z)$ for which the reciprocal is nondecreasing and which satisfies an additional condition:

$$y^{-1} \frac{\partial}{\partial y} \log |W_0(x + iy)|$$

converges to zero in the limit of large y for every real number x. Since $h \geq \frac{1}{2}$, the Euler weight function

$$\Gamma(h - iz)$$

defines computable Euler spaces of entire functions. The construction of Euler spaces of entire functions reduces to the case in which the additional hypothesis is satisfied. An entire function $E_0(z)$ of Hermite class exists such that the real part of

$$W_0(z) E_0(z)$$

is nonnegative in the upper half–plane. A theorem of Beurling and Malliavin [1] is applied for the construction of the desired Euler space of entire function.

An entire function $E_1(z)$ of Hermite class exists such that

$$E_1^*(z) E_1(z) = 1 + E_0^*(z) E_0(z)$$

and such that the inequalities

$$|E_0(z)| < |E_1(z)$$

and

$$1 < |E_1(z)|$$

hold in the upper half–plane.

The entire function $E_1(z)$ is of bounded type in the upper half–plane. The function is of exponential type τ with τ the mean type of the function in the upper half–plane. It defines a τ–local operator on Fourier transforms in the Wiener operational calculus. For every positive number a the domain of the operator contains a square integrable function which vanishes outside of the interval $(-a, a)$ and which does not vanish identically.

The Fourier transform is an entire function $F(z)$ of exponential type at most a such that the functions

$$F(z)$$

and

$$E_1(z) F(z)$$

of z are square integrable on the real axis. The entire function

$$E_0(z) F(z)$$

of z is of exponential type at most $\tau + a$ and is square integrable on the real axis.

The desired Euler space of entire function is easily constructed.

Computable examples of Euler spaces of entire functions are constructed from the Euler weight function

$$W(z) = \Gamma(h - iz)/\Gamma(k - iz)$$

when $h \geq \frac{1}{2}$ and $k \geq \frac{1}{2}$. A Stieltjes space of entire functions is defined by the function

$$E(a, z) = A(a, z) - iB(a, z)$$

of z for every positive number a as the set of entire functions $F(z)$ of z such that the functions

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

of z in the upper half–plane belong to the weighted Hardy space $\mathcal{F}(W)$.

The space with parameter a is contained in the space with parameter b when $a < b$. The spaces are contained isometrically in the weighted Hardy space. The spaces are symmetric about the origin: The function $A(a, z)$ of z is even and the function $B(a, z)$ of z is odd for every positive number a.

The integral equation

$$(A(a, z), B(a, z))I - (A(b, z), B(b, z))I = z \int_a^b (A(t, z), B(t, z))dm(t)$$

applies with

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

a nonincreasing matrix valued function whose off–diagonal entry

$$\beta(t) = 0$$

vanishes identically. The parametrization is made so that

$$\alpha'(t)\gamma'(t) = 1.$$

The function

$$(k - iz)F(z + i)/(h - iz)$$

of z belongs to the weighted Hardy space $\mathcal{F}(W)$ whenever the function $F(z)$ of z belongs to the space. The identity

$$\langle (k - it)F(t + i)/(h - it), G(t)\rangle = \langle F(t), (k - it)G(t + i)/(h - it)\rangle$$

holds for all functions $F(z)$ and $G(z)$ of z which belong to the space with the scalar product taken in the space.

The function

$$(k - iz)F(z + i)/(h - iz)$$

of z belongs to the Stieltjes space of parameter a whenever the function $F(z)$ of z belongs to the space and vanishes at $i - ih$. The identity

$$\langle (k - it)F(t + i)/(h - it), G(t)\rangle = \langle F(t), (k - it)G(t + i)/(h - it)\rangle$$

holds for all functions $F(z)$ and $G(z)$ of z which belongs to the space and vanishes at $i - ih$.

The Euler weight function

$$W_-(z) = \Gamma(h - iz)/\Gamma(k - iz)$$

is related to the Euler weight function

$$W_+(z) = \Gamma(h + 1 - iz)/\Gamma(k - iz)$$

by the recurrence relation

$$W_+(z) = (h - iz)W_-(z)$$

for the gamma function. Multiplication by $h - iz$ is an isometric transformation of the weighted Hardy space $\mathcal{F}(W_-)$ into the weighted Hardy space $\mathcal{F}(W_+)$.

For every positive number a multiplication by $h - iz$ is an isometric transformation of the Stieltjes space defined by

$$E_-(a, z) = A_-(a, z) - iB_-(a, z)$$

onto the subspace of the Stieltjes space defined by

$$E_+(a, z) = A_+(a, z) - iB_+(a, z)$$

whose elements vanish at $-ih$.

The identities

$$\begin{gathered}(h - iz)[B_-(a, z)A_+(a, ih)^- - A_-(a, z)B_+(a, ih)^-] \\ = B_+(a, z)A_+(a, ih)^- - A_+(a, z)B_+(a, h)^-\end{gathered}$$

and

$$\begin{gathered}(h + iz)[B_-(a, z)A_+(a, ih) - A_-(a, z)B_+(a, ih)] \\ = B_+(a, z)A_+(a, ih) - A_+(a, z)B_+(a, h)\end{gathered}$$

are satisfied.

The differential equations

$$\frac{\partial}{\partial t} \; B_-(t,z) = zA_-(t,z)\alpha'_-(t)$$

and

$$-\frac{\partial}{\partial t} \; A_-(t,z) = zB_-(t,z)\gamma'_-(t)$$

and the differential equations

$$\frac{\partial}{\partial t} \; B_+(t,z) = zA_+(t,z)\alpha'_+(t)$$

and

$$-\frac{\partial}{\partial t} \; A_+(t,z) = zB_+(t,z)\gamma'_+(t)$$

imply the equations

$$\begin{aligned} &z\;[A_-(t,z)\alpha'_-(t)A_+(t,ih) + B_-(t,z)\gamma'_-(t)B_+(t,ih)] \\ &-ih[A_-(t,z)\alpha'_+(t)A_+(t,ih) + B_-(t,z)\gamma'_+(t)B_+(t,ih)] \\ &= -i[A_+(t,z)\alpha'_+(t)A_+(t,ih) + B_+(t,z)\gamma'_+(t)B_+(t,ih)] \end{aligned}$$

and

$$\begin{aligned} &z\;[A_-(t,z)\alpha'_-(t)A_+(t,ih)^- + B_-(t,z)\gamma'_-(t)B_+(t,ih)^-] \\ &+ih[A_-(t,z)\alpha'_+(t)A_+(t,ih)^- + B_-(t,z)\gamma'_+(t)B_+(t,ih)^-] \\ &= i[A_+(t,z)\alpha'_+(t)A_+(t,ih)^- + B_+(t,z)\gamma'_+(t)B_+(t,ih)^-]. \end{aligned}$$

Since

$$\begin{aligned} &[B_+(t,ih)A_+(t,ih)^- - A_+(t,ih)B_+(t,ih)^-]A_+(t,z) \\ &= (h-iz)[B_-(t,z)A_+(t,ih)^- - A_-(t,z)B_+(t,ih)^-]A_+(t,ih) \\ &-(h+iz)[B_-(t,z)A_+(t,ih) - A_-(t,z)B_+(t,ih)]A_+(t,ih)^- \end{aligned}$$

and

$$\begin{aligned} &[B_+(t,ih)A_+(t,ih)^- - A_+(t,ih)B_+(t,ih)^-]B_+(t,z) \\ &= (h-iz)[B_-(t,z)A_+(t,ih)^- - A_-(t,z)B_+(t,ih)^-]B_+(t,ih) \\ &-(h+iz)[B_-(t,z)A_+(t,ih) - A_-(t,z)B_+(t,ih)]B_+(t,ih)^- \end{aligned}$$

and since the functions $A_-(t,z)$ and $B_-(t,z)$ of z satisfy no nontrivial linear equation with linear functions of z as coefficients, the equations

$$\begin{aligned} &A_+(t,ih)\alpha'_-(t)A_+(t,ih)^- + B_+(t,ih)\gamma'_-(t)B_+(t,ih)^- \\ &= A_+(t,ih)\alpha'_+(t)A_+(t,ih)^- + B_+(t,ih)\gamma'_+(t)B_+(t,ih)^- \end{aligned}$$

and

$$\begin{aligned} &A_+(t,ih)\alpha'_-(t)A_+(t,ih) + B_+(t,ih)\gamma'_-(t)B_+(t,ih) \\ &= -A_+(t,ih)\alpha'_+(t)A_+(t,ih) - B_+(t,ih)\gamma'_+(t)B_+(t,ih) \end{aligned}$$

are satisfied.

Since $A_+(t, ih)$ is real and $B_+(t, ih)$ is imaginary, the equations read

$$A_+(t, ih)\alpha'_-(t)A_+(t, ih)^- = B_+(t, ih)\gamma'_+(t)B_+(t, ih)^-$$

and

$$B_+(t, ih)\gamma'_-(t)B_+(t, ih)^- = A_+(t, ih)\alpha'_+(t)A_+(t, ih)^-.$$

The differential equations

$$\frac{\partial}{\partial t}\, B_+(t, ih) = ihA_+(t, ih)\alpha'_+(t)$$

and

$$-\frac{\partial}{\partial t}\, A_+(t, ih) = ihB_+(t, ih)\gamma'_+(t)$$

can be rewritten

$$\frac{\partial}{\partial t}\, A_+(t, -ih)^{-1} = ihB_+(t, -ih)^{-1}\alpha'_-(t)$$

and

$$-\frac{\partial}{\partial t}\, B_+(t, -ih)^{-1} = ihA_+(t, -ih)^{-1}\gamma'_-(t).$$

Since

$$\frac{\partial}{\partial t}\, B_-(t, -ih) = -ihA_-(t, -ih)\alpha'_-(t)$$

and

$$\frac{\partial}{\partial t}\, A_-(t, -ih) = ihB_-(t, -ih)\gamma'_-(t),$$

the derivative of the function

$$A_-(t, -ih)/A_+(t, -ih) + B_-(t, -ih)/B_+(t, -ih)$$

of t vanishes identically. The function is a constant which is computed in the limit if large t.

When the functions $E_-(t, z)$ of z are normalized as usual with value one at the origin, the functions $E_+(t, z)$ of z have the unusual normalization of value h at the origin. The identity

$$A_-(t, -ih)/A_+(t, -ih) + B_-(t, -ih)/B_+(t, -ih) = h^{-1}$$

is satisfied.

The entire function

$$L(a, z) = A_-(a, z)u(a) + B_-(a, z)v(a)$$

of z defined by

$$u(a) = h/A_+(a, -ih)$$

and

$$v(a) = h/B_+(a, -ih)$$

has value one at $-ih$. The function

$$[F(z+i) - L(a, z)F(i - ih)]/(h - iz)$$

of z belongs to the Stieltjes space defined by

$$E_-(a, z) = A_-(a, z) - iB_-(a, z)$$

whenever the function $F(z)$ of z belongs to the space.

The identity

$$\langle F(t+i) + (k-h)[F(t+i) - L(a, t)F(i - ih)]/(h - it), G(t)\rangle$$
$$= \langle F(t), G(t+i) + (k-h)[G(t+i) - L(a, t)G(i - ih)]/(h - it)\rangle$$

holds for all functions $F(z)$ and $G(z)$ of z which belong to the space with the scalar product taken in the space.

This obtains the information from the plus family of Stieltjes spaces which is relevant to the minus family of Stieltjes spaces. The subscript minus is omitted when the minus family of spaces is treated by itself. The identity is applied with suppression of the parameter a.

The identity reads

$$F(\beta + i) + (k-h)[F(\beta + i) - L(\beta)F(i - ih)]/(h - i\beta)$$
$$= G(\alpha + i)^- + (k-h)[G(\alpha + i)^- - L(\alpha)^- G(i - ih)^-]/(h + i\alpha^-)$$

when

$$F(z) = K(\alpha, z)$$

and

$$G(z) = K(\beta, z)$$

for complex numbers α and β. Explicitly it reads

$$K(w, z+i) + (k-h)[K(w, z+i) - L(z)K(w, i - ih)]/(h - iz)$$
$$= K(w+i, z) + (k-h)[K(w+i, z) - L(w)^- K(i - ih, z)]/(h + iw^-)$$

for complex numbers z and w. It can be written

$$\{B(z+i) + (k-h)[B(z+i) - L(z)B(i - ih)]/(h - iz)$$
$$+(k-h)u^-[B(z)A(i - ih)^- - A(z)B(i - ih)^-]/(1 - h - iz)\}A(w)^-$$
$$-\{A(z+i) + (k-h)[A(z+i) - L(z)A(i - ih)]/(h - iz)$$
$$-(k-h)v^-[B(z)A(i - ih)^- - A(z)B(i - ih)^-]/(1 - h - iz)\}B(w)^-$$
$$= B(z)\{A(w+i)^- + (k-h)[A(w+i)^- - L(w)^- A(i - ih)^-]/(h + iw^-)$$
$$+(k-h)v[B(i - ih)A(w)^- - A(i - ih)B(w)^-]/(1 - h + iw^-)\}$$
$$-A(z)\{B(w+i)^- + (k-h)[B(w+i)^- - L(w)^- B(i - ih^-)]/(h + iw^-)$$
$$-(k-h)u[B(i - ih)A(w)^- - A(i - ih)B(w)^-]/(1 - h + iw^-)\}$$

Assume that h and k are unequal. Since the functions $A(a,z)$ and $B(a,z)$ of z are linearly independent, complex numbers $p(a), r(a)$, and $s(a)$ exist such that

$$\begin{aligned} A(a,z+i) + (k-h)[A(a,z+i) - L(a,z)A(a,i-ih)]/(h-iz) \\ -(k-h)v(a)^-[B(a,z)A(a,i-ih)^- - A(a,z)B(a,i-ih)^-]/(1-h-iz) \\ = A(a,z)s(a) - B(a,z)ir(a) \end{aligned}$$

and

$$\begin{aligned} B(a,z+i) + (k-h)[B(a,z+i) - L(a,z)B(a,i-ih)]/(h-iz) \\ -(k-h)u(a)^-[B(a,z)A(a,i-ih)^- - A(a,z)B(a,i-ih)^-]/(1-h-iz) \\ = A(a,z)ip(a) + B(a,z)s(a). \end{aligned}$$

The numbers $p(a), r(a)$, and $s(a)$ are real since the functions $A(a,z)$ and $B(a,z)$ of z are self–conjugate.

The equations

$$(k-iz)A(a,z+i)/(h-iz) = A(a,z)P(a,z) + B(a,z)R(a,z)$$

and

$$(k-iz)B(a,z+i)/(h-iz) = A(a,z)Q(a,z) + B(a,z)S(a,z)$$

are satisfied with

$$\begin{aligned} P(a,z) + s(a) \\ = A(a,i-ih)u(a)(k-h)/(h-iz) - B(a,i-ih)^-v(a)^-(k-h)/(1-h-iz) \end{aligned}$$

and

$$\begin{aligned} Q(a,z) + ip(a) \\ = B(a,i-ih)u(a)(k-h)/(h-iz) + B(a,i-ih)^-u(a)^-(k-h)/(1-h-iz) \end{aligned}$$

and

$$\begin{aligned} R(a,z) - ir(a) \\ = A(a,i-ih)v(a)(k-h)/(h-iz) + A(a,i-ih)^-v(a)^-(k-h)/(1-h-iz) \end{aligned}$$

and

$$\begin{aligned} S(a,z) + s(a) \\ = B(a,i-ih)v(a)(k-h)/(h-iz) - A(a,i-ih)^-u(a)^-(k-h)/(1-h-iz). \end{aligned}$$

The equations

$$(1-k-iz)A(a,z)/(1-h-iz) = A(a,z+i)S(a,z) - B(a,z+i)R(a,z)$$

and

$$(1-k-iz, B(a,z)/(1-h-iz) = -A(a,z+i)Q(a,z) + B(a,z+i)P(a,z)$$

are obtained when z is eplaced by $-i-z$.

Consistency of the two sets of equations imposes the condition

$$P(a,z)S(a,z) - Q(a,z)R(a,z) = \frac{(k-iz)(1-k-iz)}{(h-iz)(1-h-iz)}$$

The consistency condition is equivalent to the equations

$$s(a)^2 - p(a)r(a) = 1$$

and

$$\begin{aligned}&(h+k-1)-(k-h)L(a,ih-i)^2\\ &= (2h-1)[A(a,ih-i)s(a) - iB(a,ih-i)r(a)]u(a)\\ &-(2h-1)i[A(a,ih-i)p(a) - iB(a,ih-i)s(a)]v(a).\end{aligned}$$

Differentiation with respect to the parameter a, elimination of functions of $z+i$ in favor of functions of z, and comparison of the coefficients of $A(a,z)$ and $B(a,z)$ gives the equations

$$P'(a,z) = -(z+i)Q(a,z)\gamma'(a) - zR(a,z)\alpha'(a)$$

and

$$Q'(a,z) = (z+i)P(a,z)\alpha'(a) - zS(a,z)\alpha'(a)$$

and

$$R'(a,z) = -(z+i)S(a,z)\gamma'(a) + zP(a,z)\gamma'(a)$$

and

$$S'(a,z) = (z+i)R(a,z)\alpha'(a) + zQ(a,z)\gamma'(a)$$

where the prime denotes differentiation with respect to the parameter.

The differential equations

$$u'(a) = ihv(a)\alpha'(a)$$

and

$$v'(a) = -ihu(a)\gamma'(a)$$

and

$$p'(a) = s(a)\alpha'(a) + 2(h-k)L(a,ih-i)\alpha'(a)$$

and

$$r'(a) = s(a)\gamma'(a) - 2(h-k)L(a,ih-i)\gamma'(a)$$

and

$$p(a)\gamma'(a) = s'(a) = r(a)\alpha'(a)$$

are satisfied.

2. Zeta Functions

The original conjecture known as the Riemann hypothesis applies to the zeta function whose functional identity was obtained by Euler as an application of hypergeometric series. Fourier analysis gives another proof of the functional identity which generalizes to analogous zeta functions for which there is a conjecture also known as the Riemann hypothesis.

Fourier analysis is formulated for any locally compact Abelian group. The groups which produce a zeta function are completions of an initial discrete group in topologies which are compatible with additive structure. The discrete field of rational numbers is commonly used for its additive group to create zeta functions. The discrete skew–plane of quaternions with rational numbers as coordinates is now used because a proof of the Riemann hypothesis is offered for these zeta functions.

The construction of zeta functions is motivated by the Euclidean algorithm. An element of the discrete skew–plane is said to be integral if its coordinates are all integers or all halves of odd integers. Sums and products of integral elements are integral. The conjugate ξ^- of an integral element ξ is integral.

The Euclidean algorithm applies to pairs of integral elements of the discrete skew–plane. The Cartesian product of the discrete skew–plane with itself is treated as a left vector space over the discrete skew–plane with indefinite scalar product

$$\langle(\alpha,\beta),(\gamma,\delta)\rangle = \delta^-\alpha + \gamma^-\beta$$

whose values are elements of the discrete skew–plane.

A linear transformation of the vector space into itself is defined by a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

whose entries are elements of the discrete skew–plane. The transformation takes an element (α,β) of the vector space into the element

$$(\gamma,\delta) = (\alpha,\beta)\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The matrix is said to be symplectic if the transformation is isometric for the indefinite scalar product. The identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} A^- & C^- \\ B^- & D^- \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

characterizes a symplectic matrix.

The transformation is said to be integral if it takes pairs of integral elements of the discrete skew–plane into pairs of integral elements of the discrete skew–plane. The transformation is integral if, and only if, the entries of its matrix are integral elements of the discrete skew–plane.

The integral symplectic matrices are a group used to define an equivalence relation on the vector space. Equivalence of (α, β) and (γ, δ) means that

$$(\gamma, \delta) = (\alpha, \beta) \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for a matrix belonging to the group.

A formulation of the Euclidean algorithm applies in the presence of an indefinite scalar product. A pair (α, β) of elements of the discrete skew–plane whose scalar self–product

$$\beta^- \alpha + \alpha^- \beta = 0$$

vanishes is equivalent to a pair (γ, δ) for which

$$\delta = 0$$

vanishes.

Since α and β can be multiplied by a positive integer, it is sufficient to give a verification when α and β are integral. If α vanishes, the required integral symplectic matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

When neither α nor β vanishes, it can be assumed by the use of the same matrix that

$$\beta^- \beta \le \alpha^- \alpha.$$

By hypothesis the integral element

$$\beta^- \alpha$$

of the discrete skew–plane is skew–conjugate.

A skew–conjugate integral element γ of the discrete skew–plane exists such that the inequality

$$(\beta^- \alpha - \beta^- \beta \gamma)^- (\beta^- \alpha - \beta^- \beta \gamma) < (\beta^- \beta)^2$$

is satisfied. The inequality

$$(\alpha - \beta \gamma)^- (\alpha - \beta \gamma) < \beta^- \beta$$

is then satisfied by an equivalent pair

$$(\alpha - \beta \gamma, \beta) = (\alpha, \beta) \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

since the integral matrix

$$\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

is symplectic.

The desired pair of elements of the discrete skew–plane is found by iteration.

Hecke operators are linear transformations on homogeneous functions $f(\xi, \eta)$ of pairs of elements ξ and η of the discrete skew–plane which have equal values at equivalent pairs: The identity

$$f(\alpha, \beta) = f(\gamma, \delta)$$

holds whenever

$$(\gamma, \delta) = (\alpha, \beta) \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for an integral symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Homogeneous means that the identity

$$f(\xi, \eta) = f(\omega\xi, \omega\eta)$$

holds for every positive rational number ω.

A Hecke operator $\Delta(r)$ is defined for every positive integer r. The definition of the transformation applies a representation of r by integral matrices.

An integral matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is said to represent a positive integer r if

$$\begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A^- & C^- \\ B^- & D^- \end{pmatrix}.$$

Two such matrices are considered equivalent for the definition of the Hecke operator if they are obtained from each other on multiplication on the left by an integral symplectic matrix. Every equivalence class contains a diagonal matrix

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$$

such that

$$r = \omega^- \omega.$$

Two diagonal matrices are equivalent if, and only if, they are obtained from each other on multiplication on the left by a diagonal matrix with equal entries on the diagonal which are integral elements of the discrete skew–plane with integral inverse.

The Hecke operator $\Delta(r)$ takes a function $f(\xi, \eta)$ of pairs of elements of the discrete skew–plane into the function

$$g(\xi, \eta) = \sum f(\xi\omega, \eta\omega)$$

of pairs of elements of the discrete skew–plane defined as a sum over the equivalence classes of integral elements ω of the discrete skew–plane which represent

$$r = \omega^- \omega.$$

Equivalent elements are obtained from each other on multiplication on the left by an integral element of the discrete skew–plane with integral inverse.

The identity

$$\Delta(m)\Delta(n) = \sum \Delta(mn/k^2)$$

holds for all positive integers m and n with summation over the common odd divisors k of m and n.

Hecke operators are applied to functions belonging to a Hilbert space of finite dimension such that an isometric transformation of the space into itself is defined by taking a function $f(\xi, \eta)$ of pairs of integral elements of the discrete skew–plane into the function $f(\xi\omega, \eta\omega)$ of pairs of integral elements of the discrete skew–plane for every nonzero element ω of the discrete skew–field.

Hecke operators are commuting self–adjoint transformations. The Hilbert space is the orthogonal sum of invariant subspaces which are determined by eigenvalues of Hecke operators. The eigenvalue $\tau(r)$ of $\Delta(r)$ in an invariant subspace is a real number. The identity

$$\tau(m)\tau(n) = \sum \tau(mn/k^2)$$

holds for all positive integers m and n with summation over the common odd divisors k of m and n.

The zeta function defined by an invariant subspace is the Dirichlet series

$$Z(s) = \sum \tau(n)n^{-s}$$

whose coefficients are the eigenfunctions of Hecke operators. The Euler product

$$1/Z(s) = [1 - \tau(2)2^{-s}] \prod [1 - \tau(p)p^{-s} + p^{-2s}]$$

is taken over the odd primes p with an exceptional factor for the even prime.

A preliminary to the Riemann hypothesis is the Ramanujan hypothesis

$$-2 \leq \tau(p) \leq 2$$

for every odd prime p and the analogous hypothesis

$$-1 \leq \tau(2) \leq 1$$

for the even prime. When these inequalities are satisfied, the Dirichlet series and its Euler product converge in the half–plane $\mathcal{R}s > 1$ and define an analytic function of s which has no zeros in the half–plane.

Another preliminary to the Riemann hypothesis is the analytic extension of the function to the half–plane $\mathcal{R}s > \frac{1}{2}$ with the possible exception of a simple pole at $s = 1$. When these preliminaries are completed, the Riemann hypothesis states that the analytic extension has no zeros in the half–plane.

The Hilbert spaces of functions on which Hecke operators act are constructed in Fourier analysis on completions of the vector space of pairs of elements of the discrete skew–plane in topologies which are compatible with vector space structure. The elements of a completion are pairs of elements of the completion of the discrete skew–plane in topologies which are compatible with additive and multiplicative structure. A completion of the discrete skew–plane is a space of quaternions whose coordinates are completions of the field of rational numbers which are compatible with additive and multiplicative structure.

It is sufficient to treat two completions of the field of rational numbers. The completion in the Dedekind topology is the field of real numbers. The canonical measure for the additive group of real numbers is Lebesgue measure on the Baire subsets of real numbers. The complex skew–plane is the algebra of quaternions whose coordinates are real numbers. The canonical measure for the complex skew–plane is the Cartesian product measure of the canonical measures for four coordinate lines. The complex projective skew–plane is the space of pairs of elements of the complex skew–plane. The canonical measure for the complex projective skew–plane is the Cartesian product measure of the canonical measures of two complex skew–planes.

For Hecke operators the functions $f(\xi, \eta)$ of pairs of elements ξ and η of the complex skew–plane are homogeneous in the sense that the identity

$$f(\xi, \eta) = f(\omega\xi, \omega\eta)$$

holds for every positive real number ω. The condition of square integrability with respect to the canonical measure is applied to the function

$$f(\xi, \eta)/(\xi^- \eta + \eta^- \xi)$$

over a fundamental domain.

There are two regions to be treated according to the sign of the scalar self–product

$$\xi^- \eta + \eta^- \xi,$$

one in which the scalar self–product is positive and one in which the scalar self–product is negative. The set of which the scalar self–product vanishes is ignored since it has zero measure. The set on which

$$\eta + \eta^-$$

vanishes is ignored for the same reason. Again there are two regions depending on a choice of sign. When positive signs are chosen, a fundamental region is defined by

$$\eta + \eta^- = 2.$$

The adic topology of the rational numbers applies integrality for its definition. The ring of integral adic numbers is isomorphic to the Cartesian product of the rings of integral p–adic numbers taken over all primes p. The ring of adic numbers is the ring of quotients of the ring of integral adic numbers with positive integers as denominators. An invertible adic number ω is the unique product of a positive rational number $\lambda(\omega)$ and an integral adic number with integral inverse. The ring of integral adic numbers is compact and has measure one with respect to the canonical measure for the ring of adic numbers.

The adic skekw–plane is the algebra of quaternions whose coordinates are adic numbers. An element of the adic skew–plane is integral if its coordinates are all integral or all nonintegral halves of integral adic numbers. The ring of integral elements of the adic skew–plane is compact and has measure one with respect to the canonical measure for the adic skew–plane. The adic projective skew–plane is the space of pairs of elements of the adic skew–plane. The canonical measure for the adic projective skew–plane is the Cartesian product measure of the canonical measures of two adic skew–planes.

For Hecke operators the function $f(\xi, \eta)$ of pairs of elements ξ and η in the adic skew–plane are homogeneous in the sense that the identity

$$f(\xi, \eta) = f(\omega\xi, \omega\eta)$$

holds for every positive rational number ω. Square integrability with respect to the canonical measure is applied to the function

$$f(\xi, \eta)/\lambda(\xi^- \eta + \eta^- \xi)$$

over a fundamental domain. The set on which

$$\xi^- \eta + \eta^- \xi$$

is noninvertible is ignored since it has zero measure. The set on which

$$\eta + \eta^-$$

is noninvertible is ignored for the same reason. A fundamental domain is defined by

$$\lambda(\eta + \eta^-) = 2.$$

The adelic skew–plane is the Cartesian product of the complex skew–plane and the adic skew–plane. An element ξ of the adelic skew–plane has a component ξ_+ in the complex skew–plane and a component ξ_- in the adic skew–plane. The canonical measure for the adelic skew–plane is the Cartesian product measure of the canonical measure for the complex skew–plane and the canonical measure for the adic skew–plane.

Since an element ω of the discrete skew–plane is an element of the complex skew–plane and an element of the adic skew–plane, the product

$$\eta = \xi\omega$$

of an element ξ of the adelic skew–plane and an element ω of the discrete skew–plane is defined by

$$\eta_+ = \xi_+\omega$$

and

$$\eta_- = \xi_-\omega.$$

Multiplication by ω is a measure preserving transformation with respect to the canonical measure for the adelic skew–plane.

Hecke opeators are applied in Hilbert spaces whose elements are homogeneous functions $f(\xi, \eta)$ of pairs of elements ξ and η of the adelic skew–plane such that the identity

$$f(\gamma, \delta) = f(\alpha, \beta)$$

holds whenever

$$(\gamma, \delta) = (\alpha, \beta)\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for an integral symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

with entries in the discrete skew–field.

The Hecke operator $\Delta(r)$ takes a function $f(\xi, \eta)$ of pairs of elements of the adelic skew–plane into the function

$$g(\xi, \eta) = \sum f(\xi\omega, \eta\omega)$$

of pairs of elements of the adelic skew–plane defined by summation over the equivalence classes of elements ω of the discrete skew–plane which represent

$$r = \omega^-\omega.$$

Equivalence of nonzero elements of the discrete skew–plane means that they are obtained from each other on multiplication on the left by an integral element of the discrete skew–plane with integral inverse.

Hilbert spaces of finite dimension on which Hecke operators act are obtained as invariant subspaces for a compact group of commuting isometric transformations. The transformations are defined by elements ω of the adelic skew–plane with conjugate as inverse which take a function $f(\xi, \eta)$ of pairs of elements of the adelic skew–plane into the functions $f(\omega\xi, \eta)$ and $f(\xi, \omega\eta)$ of pairs of elements of the adelic skew–plane.

The Riemann hypothesis is a consequence of a phenomenon in Fourier analysis which Fourier called the flow of heat. Since the infinitessimal generator of heat flow is a differential operator, an inverse integral operator is needed for application to locally compact Abelian groups which do not have a differential structure.

The generalization of heat flow is made by a Radon transformation. The relevant features of the transformation are that it is maximal accretive and has a subnormal adjoint.

Subnormal means that a transformation is the restriction of a normal transformation to an invariant subspace. A spectral expansion of a subnormal transformation is expected from the spectral expansion of a normal transformation. A spectral expansion of a Radon transformation is indeed obtained in this way and is called a Laplace transformation. The Radon transformation is then unitarily equivalent to multiplication by some function in a Hilbert space of square integrable functions with respect to a nonnegative measure.

A Laplace transformation for the complex skew–plane applies a Hilbert space of functions analytic in the upper half–plane which are square integrable with respect to a nonnegative measure. The adjoint of the Radon transformation is unitarily equivalent to multiplication by

$$i/z$$

where z is the independent variable for the complex plane. The accretive property is immediate since the multiplier has positive real part in the upper half–plane. The maximal accretive property is verified by showing that the adjoint is accretive. The adjoint is computed on the elements of the Hilbert space whose scalar product with a function assigns the function value at an element of the upper half–plane.

A Laplace transformation for the adic skew–plane imitates the spectral expansion for the Radon transformation for the complex skew–plane. All computations for the adic skew–plane are reduced to computations in Hilbert spaces of finite dimension. When the maximal accretive property is verified in Hilbert spaces of arbitrary finite dimension, it follows immediately in Hilbert spaces of infinite dimension.

A Laplace transformation for the adelic skew–plane combines a Laplace transformation for the complex skew–plane and a Laplace transformation for the adic skew–plane. The range of the Laplace transformation for the adelic skew–plane is a tensor product of the range of a Laplace transformation for the complex skew–plane and the range of a Laplace transformation for the adic skew–plane. The functions in the adelic case are defined on a space which is the Cartesian product of the space on which functions are defined in the complex case and the space on which the functions are defined in the adic case. The multiplier in the adelic case has values which are products of the values of the multiplier in the complex case and the values of the multiplier in the adic case. Since the values of the multiplier for the complex skew–plane have positive real part and since the values of the multiplier for the adic skew–plane are positive, the values of the multiplier for the adelic skew–plane have positive real part. Verification of the maximal accretive property for the adelic skew–plane is made in a finite calculation.

There is no obstacle to the proof of the Riemann hypothesis when the zeta function has no singularity at the unit of the complex plane. The argument fails in the singular case because the Radon transformation for the adic algebra fails to be maximal accretive. In this case the Radon transformation commutes with an isometric transformation which is its own inverse. The Hilbert space has an orthogonal decomposition into a subspace of functions of even parity and a subspace of functions of odd parity. The isometric transformation multiplies a function of even parity by one and a function of odd parity by minus one. The graph of the Radon transformation decomposes into the graph of the Radon transformation in functions of even parity and the graph of the Radon transformation on functions of odd parity. The Radon transformation acts as a maximal accretive transformation on functions of even parity. This information is sufficient for a proof of the Riemann hypothesis in the singular case.

The proof of the Riemann hypothesis creates a canonical mechanical system to whose spectral function the zeta function contributes a factor. A fundamental problem is to determine whether a mechanical system is determined by its spectral function. The issue is treated in certain Hilbert spaces of entire functions which originate with Stieltjes. He treated mechanical systems which are finite. An axiomatic treatment of the Stieltjes spaces removes finiteness conditions. A fundamental theorem [1] states that the mechanical system is indeed determined by its spectral properties.

Applications of the Riemann hypothesis require the mechanical system determined by a zeta function together with gamma function factors in a spectral function. The proof of the Riemann hypothesis is treated as a computation of the mechanical system.

3. Harmonic Analysis on a Complex Skew–Plane

Euler weight functions and associated Stieltjes spaces of entire functions are constructed in harmonic analysis on a complex skew–plane.

The complex skew–plane is the skew–field of quaternions whose coordinates are real numbers. The topology of the complex skew–plane is the Cartesian product topology of the topologies of four coordinate lines. Addition is continuous as a transformation of the Cartesian product of the complex skew–plane with itself into the complex skew–plane. Multiplication by an element of the complex skew–plane on left or right is a continuous transformation of the complex skew–plane into itself. Conjugation is a continuous transformation of the complex skew–plane into itself.

The complex skew–plane is a locally compact Abelian group. The Baire subsets of a locally compact Abelian group are the smallest class of sets which contains the open sets and the closed sets, which contains every countable union of sets of the class, and which contains every complement of a set of the class. The transformation which takes ξ into $\xi + \eta$ takes Baire sets into Baire sets for every element η of the group.

A canonical measure for a locally compact Abelian group is a nonnegative measure on its Baire subsets which is finite on compact sets, which is positive on nonempty open sets, such that the transformation which takes ξ into $\xi + \eta$ is measure preserving for every element η of the group.

Canonical measures exist and are unique within a constant factor. There is usually agreement about the choice of canonical measure. The canonical measure for the real line is Lebesgue measure, which has value one on the interval $(0,1)$. The canonical measure for the complex skew–plane is the Cartesian product measure of the canonical measures of four coordinate lines.

The modulus of an element ξ of the complex skew–plane is the nonnegative real number $\lambda(\xi)$ such that

$$\lambda(\xi)^2 = \xi^- \xi.$$

Multiplication on left or right by an element ξ of the complex skew–plane multiplies the canonical measure of the complex skew–field by a factor of

$$\lambda(\xi)^4.$$

The function

$$\exp(2\pi i \xi)$$

of self–conjugate elements ξ of the complex skew–plane is a continuous homomorphism of the additive group of the field of self–conjugate elements into the multiplicative group of complex numbers of absolute value one.

The Fourier transformation for a complex skew–plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^- \eta + \eta^- \xi)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^- \eta + \eta^- \xi)) g(\eta) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The Fourier transformation for the complex skew–plane commutes with the isometric transformations of the Hilbert space into itself defined by taking a function $f(\xi)$ of ξ into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ for every element ω of the complex skew–plane with conjugate as inverse. The Hilbert space decomposes into the orthogonal sum of irreducible invariant subspaces under the action of the transformations.

A homomorphism of the multiplicative group of nonzero elements of the complex skew–plane onto the multiplicative group of the positive half–line is defined by taking ξ into $\xi^- \xi$. The identity

$$\int |f(\xi^- \xi)|^2 d\xi = \pi^2 \int |f(\xi)|^2 \xi d\xi$$

holds with integration on the left with respect to the canonical measure for the complex skew–plane and with integration on the right with respect to Lebesgue measure for every Baire function $f(\xi)$ of ξ in the positive half–line.

The Hilbert space of homogeneous polynomials of degree ν is the set of functions

$$\xi = t + ix + jy + kz$$

of ξ in the complex skew–plane which are linear combinations of monomials

$$x^a y^b z^c t^d$$

whose exponents are nonnegative integers with sum

$$\nu = a + b + c + d.$$

The monomials are an orthogonal set with

$$\frac{a!b!c!d!}{\nu!}$$

as the scalar self–product of the monomial with exponents a, b, c, and d. The function

$$2^{-\nu}(\eta^- \xi + \xi^- \eta)^\nu$$

of ξ in the complex skew–field belongs to the space for every element η of the complex skew–plane and acts as reproducing kernel function for function values at η.

Isometric transformations of the Hilbert space of homogeneous polynomials of degree ν into itself are defined by taking a function $f(\xi)$ of ξ in the complex skew–plane into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the complex skew–plane for every element ω of the complex skew–plane with conjugate as inverse.

The Laplacian

$$\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

takes homogeneous polynomials of degree ν into homogeneous polynomials of degree $\nu - 2$ when ν is greater than one and annihilates polynomials of smaller degree. The Laplacian commutes with the transformations which take a function $f(\xi)$ of ξ in the complex skew–plane into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the complex skew–plane for every element ω of the complex skew–plane with conjugate as inverse.

Multiplication by $\xi^- \xi$ is a continuous transformation of the Hilbert space of homogeneous polynomials of degree ν into the Hilbert space of homogeneous polynomials of degree $\nu + 2$ for every nonnegative integer ν. The adjoint has the same kernel as the Laplacian as a transformation of the Hilbert space of homogeneous polynomials of degree $\nu + 2$ into the Hilbert space of homogeneous polynomials of degree ν.

A homogeneous polynomial of degree ν is said to be harmonic if it is annihilated by the Laplacian. Homogeneous polynomials $f(\omega\xi)$ and $f(\xi\omega)$ of degree ν are harmonic for every element ω of the complex skew–plane with conjugate as inverse if the homogeneous polynomial $f(\xi)$ of degree ν is harmonic.

The Hilbert space of homogeneous harmonic polynomials of degree ν is the orthogonal complement in the space of homogeneous polynomials of degree ν of products of $\xi^-\xi$ with homogeneous polynomials of degree $\nu - 2$ when r is greater than one. The space of homogeneous polynomials of degree ν has dimension

$$(1+\nu)(2+\nu)(3+\nu)/6.$$

The space of homogeneous harmonic polynomials of degree ν has dimension

$$(1+\nu)^2.$$

The function

$$2^{-\nu}(\xi^-\eta+\eta^-\xi)^\nu+\sum_{k=1}^n \frac{(\nu-k)!}{k!(\nu-2k)!}\ 2^{\nu-2k}(\xi^-\eta+\eta^-\xi)^{\nu-2k}\ (-\frac{\xi^-\xi}{4})^k$$

of ξ in the complex skew–plane belongs to the space of homogeneous harmonic polynomials of degree ν for every element η of the complex skew–plane and acts as reproducing kernel function for function values at η, $2n$ equal to ν when r is even and equal to $\nu - 1$ when ν is odd.

The boundary of the disk, which is the set of elements of the complex skew–plane with conjugate as inverse, is a compact Hausdorff space in the subspace topology inherited from the complex skew–plane. The canonical measure for the space is the essentially unique nonnegative measure on its Baire subsets such that measure preserving transformations are defined on multiplication left or right by an element of the space. Uniqueness is obtained by stipulating that the full space has measure

$$\pi^2.$$

A homomorphism of the multiplicative group of nonzero elements of the complex skew–plane onto the positive half–line defined by taking ξ into $\xi^-\xi$ maps the canonical measure for the complex skew–plane into the measure whose value on a Baire set E is the Lebesgue integral

$$\pi^2\int tdt$$

over the set E. The canonical measure for the complex skew–plane is the Cartesian product measure of the canonical measure for the boundary of the disk and a measure on Baire subsets of the positive half–line.

The complex plane is the field whose elements are the elements of the complex skew–plane which commute with i. The complementary space to the complex plane in the complex skew–plane is the set of elements ξ of the complex skew–plane which satisfy the identity

$$\xi\eta = \eta^- \xi$$

for every element ξ of the complex plane. The complex plane and its complementary space are locally compact Abelian groups whose canonical measures are the Cartesian product measures of the canonical measures of two coordinate lines. The complex skew–plane is the Cartesian product of the complex plane and its complementary space. The canonical measure for the complex skew–plane is the Cartesian product measure of the canonical measure for the complex plane and the canonical measure for its complementary space.

Multiplication on left or right by an element ξ of the complex plane multiplies the canonical measure of the complex plane by a factor of

$$\lambda(\xi)^2$$

and the canonical measure of the complementary space by the same factor. Multiplication on left or right by an element ξ of the complementary space takes the canonical measure of the complex plane into the canonical measure of the complementary space multiplied by the same factor and takes the canonical measure for the complementary space into the canonical measure for the complex plane multiplied by the same factor.

The Fourier transformation for a complex plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp\langle \pi i(\xi^- \eta + \eta^- \xi)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^- \eta + \eta^- \xi)) g(\eta) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

A Radon transformation of harmonic ϕ is defined for the complex skew–plane when a nontrivial homogeneous harmonic polynomial $\phi(\xi)$ of ξ in the complex skew–plane has degree ν. The domain and range of the transformation are contained in the Hilbert space of functions $f(\xi)$ of ξ in the complex skew–plane which are square integrable with respect to the canonical measure for the complex skew–plane and satisfy the identity

$$\phi(\xi) f(\omega\xi) = \phi(\omega\xi) f(\xi)$$

for every element ω of the complex skew–plane with conjugate as inverse.

The Radon transformation of harmonic ϕ takes a function $f(\xi)$ of ξ in the complex skew–plane into the function $g(\xi)$ of ξ in the complex skew–plane when the identity

$$g(\omega\xi)/\phi(\omega\xi) = \int f(\omega\xi + \omega\eta)/\phi(\omega\xi + \omega\eta) d\eta$$

holds for ξ in the complex plane for every element ω of the complex skew–plane with conjugate as inverse with integration with respect to the canonical measure for the complementary space to the complex plane in the complex skew–plane.

The integral is a limit of integrals over compact subsets of the complementary space which contain $\omega\xi$ for every element ω of the complex skew–plane with conjugate as inverse whenever they contain ξ. Convergence is in the weak topology of the Hilbert space of square integrable functions with respect to the canonical measure for the complex skew–plane.

The upper half–plane is the set of elements $z = x + iy$ of the complex plane such that $y > 0$ is positive. The upper half–plane is an open subset of the complex plane. The upper half–plane is a locally compact Hausdorff space in the subspace topology inherited from the complex plane. The canonical measure for the upper half–plane is the restriction to Baire subsets of the upper half–plane of the canonical measure for the complex plane.

The continuous function $\exp(\pi i\xi)$ of self–conjugate elements of the complex plane has a unique continuous extension to the closure of the upper half–plane which is analytic and bounded by one in the upper half–plane.

The function

$$\phi(\xi) \exp(\pi i\eta\xi^{-}\xi)$$

of ξ in the complex skew–plane is an eigenfunction of the Radon transformation of harmonic ϕ for the eigenvalue

$$i/\eta$$

for every element η of the upper half–plane.

The Laplace transformation of harmonic ϕ is a spectral decomposition of the adjoint of the Radon transformation of harmonic ϕ. The harmonic polynomial is assumed to have norm one in the Hilbert space of homogeneous polynomials of degree ν.

The domain of the Laplace transformation of harmonic ϕ is the set of functions

$$f(\xi) = \phi(\xi)h(\xi^{-}\xi)$$

of ξ in the complex skew–plane which are square integrable with respect to the canonical measure and which are parametrized by Baire functions $h(\xi)$ of ξ in the upper half–plane satisfying the identity

$$h(\xi) = h(\eta)$$

whenever ξ and η are elements of the upper half–plane satisfying the constraint

$$\xi^{-}\xi = \eta^{-}\eta.$$

The identity

$$\int |f(\xi)|^2 d\xi = \pi \int \lambda(\xi)^{\nu} |h(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the complex skew–plane and integration on the right with respect to the canonical measure for the upper half–plane.

Every Baire function $h(\xi)$ of ξ in the upper half–plane satisfying the constraint for which the integral on the right converges parametrizes an element of the domain of the Laplace transformation of harmonic ϕ.

The Laplace transform of harmonic ϕ of the function $f(\xi)$ of ξ in the complex skew–plane is the analytic function

$$h^\wedge(\eta) = \int \lambda(\xi)^\nu h(\xi) \exp(\pi i \xi \eta) d\xi$$

of η in the upper half–plane defined by integration with respect to the canonical measure for the upperx half–plane.

The identity

$$\int |h^\wedge(\xi)|^2 \lambda(\xi - \xi^-)^\nu d\xi = \pi \pi^{-\nu} \Gamma(1+\nu) \int \lambda(\xi)^\nu |h(\xi)|^2 d\xi$$

holds with integration with respect to the canonical measure for the upper half–plane.

Every analytic function $h^\wedge(\xi)$ of ξ in the upper half–plane for which the integral on the right converges belongs to the range of the Laplace transformation of harmonic ϕ.

The adjoint of the Radon transformation of harmonic ϕ is unitarily equivalent under the Laplace transformation of harmonic ϕ to multiplication by the function

$$i/\xi$$

of ξ in the upper half–plane in the Hilbert space of analytic functions which is the range of the Laplace transformation of harmonic ϕ.

Properties of Laplace transforms are stated in the notation for analytic functions of one complex variable.

The analytic function

$$K(w,z) = \pi \int_0^\infty t^\nu \exp(-\pi i t w^-) \exp(\pi i t z) t dt = \frac{\pi \Gamma(2+\nu)}{[\pi i (w^- - z)]^{2+\nu}}$$

of z belongs to the space when w is in the upper half–plane and acts as reproducing kernel function for function values at w.

An isometric transformation of the space onto itself is defined when a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

has real entries and determinant one by taking an analytic function $F(z)$ of z in the upper half–plane into the analytic function

$$\frac{1}{(Cz+D)^{2+\nu}} F\left(\frac{Az+B}{Cz+D}\right)$$

of z in the upper half–plane.

The adjoint of the Radon transformation of harmonic ϕ takes a function $f(\xi)$ of ξ in the complex skew–plane into a function $g(\xi)$ of ξ in the complex skew–plane when the identity

$$\int \phi(\xi)^- g(\xi) \exp(\pi i z \xi^- \xi) d\xi = (i/z) \int \phi(\xi)^- f(\xi) \exp(\pi i z \xi^- \xi) d\xi$$

holds when z is in the upper half–plane with integration with respect to the canonical measure for the complex skew–plane. The transformation is maximal accretive.

The Fourier transform for the complex skew–plane of the function

$$\phi(\xi) \exp(\pi i z \xi^- \xi)$$

of ξ in the complex skew–plane is the function

$$i^\nu (i/z)^{2+\nu} \phi(\xi) \exp(-\pi i z^{-1} \xi^- \xi)$$

of ξ in the complex skew–plane when z is in the upper half–plane.

Since the Fourier transformation commutes with the transformations which take a function $f(\xi)$ of ξ into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ for every element ω of the complex skew–plane with conjugate as inverse, it is sufficient to make the verification when

$$\phi(t + ix + jy + kz) = (t + ix)^\nu.$$

The verification reduces to showing that the Fourier transform for the complex plane of the function

$$\xi^\nu \exp(\pi i z \xi^- \xi)$$

of ξ in the complex plane is the function

$$i^\nu (i/z)^{1+\nu} \xi^\nu \exp(-\pi i z^{-1} \xi^- \xi)$$

of ξ in the complex plane. It is sufficient by analytic continuation to make the verification when z lies on the imaginary axis. It remains by a change of variable to show that the Fourier transform of the function

$$\xi^\nu \exp(-\pi \xi^- \xi)$$

of ξ in the complex plane is the function

$$i^\nu \xi^\nu \exp(-\pi \xi^- \xi)$$

of ξ in the complex plane.

The desired identity follows since

$$i^\nu \xi^\nu \exp(-\pi\xi^-\xi) = \sum_{k=0}^{\infty} \xi^\nu \int \frac{(\pi i\xi^-\xi)^k (\pi i\eta^-\eta)^{\nu+k}}{k!(\nu+k)!} \exp(-\pi\eta^-\eta) d\eta$$

where

$$\exp(\pi i(\xi^-\eta + \eta^-\xi)) = \sum_{n=0}^{\infty} \frac{(\pi i\xi^-\eta + \pi i\eta^-\xi)^n}{n!}$$

and

$$(\pi i\xi^-\eta + \pi i\eta^-\xi)^n = \sum_{k=0}^{n} \frac{(\pi i\eta^-\xi)^{n-2k}(\pi i\xi^-\xi)^k(\pi i\eta^-\eta)^k}{k!(n-k)!}$$

where

$$\int (\pi i\eta^-\eta)^{\nu+k} \exp(-\pi\eta^-\eta) d\eta = i^{\nu+k}(\nu+k)!$$

and

$$i^\nu \exp(-\pi\xi^-\xi) = \sum_{k=0}^{\infty} i^{\nu+k} \frac{(\pi i\xi^-\xi)^k}{k!}$$

Integrations are with respect to the canonical measure for the complex plane. Interchanges of summation and integration are justified by absolute convergence.

If a function $f(\xi)$ of ξ in the complex skew–plane is square integrable with respect to the canonical measure and satisfies the identity

$$\phi(\xi)f(\omega\xi) = \phi(\omega\xi)f(\xi)$$

for every element ω of the complex skew–plane with conjugate as inverse, then its Fourier transform is a function $g(\xi)$ of ξ in the complex skew–plane which is square integrable with respect to the canonical measure and which satisfies the identity

$$\phi(\xi)g(\omega\xi) = \phi(\omega\xi)g(\xi)$$

for every element ω of the complex skew–plane with conjugate as inverse. The Laplace transforms of harmonic ϕ are functions $F(z)$ and $G(z)$ of z in the upper half–plane which satisfy the identity

$$G(z) = i^\nu (i/z)^{2+\nu} F(-1/z).$$

The Fourier transform for the complex skew–plane of a function

$$f(\xi) = \phi(\xi)h(\xi^-\xi)$$

of ξ in the complex skew–plane is a function

$$f'(\xi) = \phi(\xi)h'(\xi^-\xi)$$

of ξ in the complex skew–plane which satisfies the identity

$$\pi \int_0^\infty t^\nu \exp(\pi itz) h'(t) t dt = i^\nu (i/z)^{2+\nu} \pi \int_0^\infty t^\nu \exp(-\pi it/z) h(t) t dt$$

for z in the upper half–plane.

A computation of the function $h'(t)$ of t is made from the function $h(t)$ of t by the Hankel transformation of order $1 + \nu$, whose classical definition is made from a Bessel function. An equivalent definition is made from the confluent hypergeometric series

$$F(1 + \nu; z) = \sum_{n=0}^\infty \frac{z^n}{n!(1 + \nu + n)!}.$$

The identity

$$h'(t) = i^\nu \pi^{2+\pi} \int_0^\infty x^\nu F(1 + \nu; -\pi^2 tx) h(x) x dx$$

holds when the integral is absolutely convergent. The transformation is otherwise defined so as to maintain the identity

$$\pi \int_0^\infty t^\nu |h'(t)|^2 t dt = \pi \int_0^\infty t^\nu |h(t)|^2 t dt.$$

For the verification that the transformation computes the Fourier transformation it is sufficient to show that

$$h'(t) = i^\nu \exp(\pi itz)$$

when

$$h(t) = (i/z)^{2+\nu} \exp(-\pi it/z).$$

It is sufficient by analytic continuation to verify the identity when $z = iy$ lies on the imaginary axis. The desired identity reduces by changes of variable to the case $y = \pi$ and $t = 1$. The identity is verified after an interchange of summation and integration from the identities

$$(1 + \nu + n)! = \int_0^\infty t^{1+\nu+n} \exp(-t) dt$$

and

$$\exp(-t) = \sum_{n=0}^\infty \frac{(-t)^n}{n!}.$$

As Laplace transforms a function and its Fourier transform become a pair of analytic functions $f(z)$ and $g(z)$ of z in the upper half–plane such that

$$g(z) = i^\nu (i/z)^{2+\nu} f(-1/z).$$

A transformation which is its own inverse is obtained by omitting the factor of i^ν. A Laplace transform of order ν is the unique sum of the Laplace transform of a self–reciprocal function, which is its own Hankel transform of order ν, and the Laplace transform of a skew–reciprocal function, which is minus its own Hankel transform of order ν. The computation of Hankel transforms is a computation of self–reciprocal functions and skew–reciprocal functions.

A decomposition of the Hilbert space of Laplace transforms of order ν is a consequence of the expansion

$$\frac{\pi\Gamma(2+\nu)}{[\pi i(w^- - z)]^{2+\nu}} = \sum_{k=0}^{2+\nu} \frac{(2+\nu)!}{k!(2+\nu-k)!} \frac{1}{(zw^-)^{2+\nu}} \frac{\pi\Gamma(2+\nu)}{[\pi i(w^- - 1/\omega^- - z + 1/z)]^{2+\nu}}$$

For every $k = 0, \ldots, 2+\nu$ a Hilbert space of functions analytic in the upper half–plane appears which has the function

$$\frac{1}{(zw^-)^k} \frac{\pi\Gamma(2+\nu)}{[\pi i(w^- - 1/w^- - z + 1/z)]^{2+\nu}}$$

of z as reproducing kernel function for function values at w when w is in the upper half–plane. An isometric transformation of the Hilbert space of Laplace transforms of order ν onto the space is defined by taking an analytic function $f(z)$ of z in the upper half–plane into the analytic function

$$(i/z)^k f(z - 1/z)$$

of z in the upper half–plane.

An analytic function

$$h_k^\wedge(z) = (i/z)^k \pi \int_0^\infty t^\nu \exp(\pi i t(z - 1/z)) h_k(t) t dt$$

of z in the upper half–plane which belongs to the space is represented by a Baire function $h_k(t)$ of positive t such that the integral

$$\pi \int_0^\infty t^\nu |h_k(t)|^2 dt$$

converges.

The range of the Laplace transformation of order $1+\nu$ is the nonorthogonal sum of the component spaces defined for $k = 0, \ldots, 2+\nu$. The decomposition is made by a generalization of the concept of an orthogonal complement.

If a Hilbert space $\mathcal{P}$ is contained contractively in a Hilbert space $\mathcal{H}$, a unique Hilbert space $\mathcal{Q}$, which is contained contractively in $\mathcal{H}$, exists such that every element

$$c = a + b$$

of $\mathcal{H}$ is the sum of an element a of $\mathcal{P}$ and an element b of $\mathcal{Q}$, such that the inequality

$$\|c\|^2_{\mathcal{H}} \leq \|a\|^2_{\mathcal{P}} + \|b\|^2_{\mathcal{Q}}$$

holds for every such decomposition of c, and such that some decomposition exists for which equality holds.

The minimal decomposition is unique and is obtained when a is obtained from c under the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$ and b is obtained from c under the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$.

The Hilbert space $\mathcal{H}_k$ of analytic functions constructed for every $k = 0, \dots, 2+\nu$ is contained continuously in the Hilbert space $\mathcal{H}$ which is the range of the Laplace transformation of order ν. If an analytic function $F_k(z)$ of z is chosen in the space $\mathcal{H}_k$ for every $k = 0, \dots, 2+\nu$, the analytic function

$$F(z) = \sum_{k=0}^{2+\nu} \frac{(2+\nu)!}{k!(2+\nu-k)!} \; F_k(z)$$

of z belongs to the space $\mathcal{H}$ and satisfies the inequality

$$\|F\|^2_{\mathcal{H}} \leq \sum_{k=0}^{2+\nu} \frac{(2+\nu)!}{k!(2+\nu-k)!} \; \|F_k\|^2_{\mathcal{H}_k}.$$

Every analytic function $F(z)$ of z which belongs to the space $\mathcal{H}$ is a sum of analytic functions $F_k(z)$ of z for which equality holds.

When w is in the upper half–plane, the minimal decomposition of

$$F(z) = \frac{\pi\Gamma(2+\nu)}{[\pi i(w^- - z)]^{2+\nu}}$$

is obtained with

$$F_k(z) = \frac{1}{(zw^-)^k} \; \frac{\pi\Gamma(2+\nu)}{[\pi i(w^- - 1/w^- - z + 1/z)]^{2+\nu}}$$

for every $k = 0, \dots, 2+\nu$.

A computation of minimal decompositions applies the Sonine construction of functions which vanish in a neighborhood of the origin and whose Hankel transform of order ν vanishes in the same neighborhood. The Sonine transformation of order ν is a variant of the Hankel transformation of order ν.

If $h(t)$ is a Baire function of positive t such that the integral

$$\pi \int_0^\infty t^\nu |h(t)|^2 t dt$$

converges and if the analytic function

$$h^\wedge(z) = \pi \int_0^\infty t^\nu \exp(\pi itz) h(t) t dt$$

of z is its Laplace transform of order ν, then the minimal decomposition

$$h^\wedge(z) = \sum_{k=0}^{2+\nu} \frac{(2+\nu)!}{k!(2+\nu-k)!} \left(\frac{i}{z}\right)^k h_k(z - 1/z)$$

is obtained with

$$h_k^\wedge(z) = \pi \int_0^\infty t^\nu \exp(\pi itz) h_k(t) t dt$$

where

$$h_0(t) = h(t) - \pi^2 t \int_t^\infty F(1; -\pi^2 t(x-t)) h(x) dx$$

and

$$h_k(t) = \pi^k \int_t^\infty (x-t)^{k-1} F(k-1; -\pi^2 t(x-t)) h(x) dx$$

when $k > 0$. The equation

$$\frac{dh_k}{dt}(t) = -\pi h_{1+k}(t) - \pi h_{k-1}(t)$$

holds when $0 < k < 2 + \nu$. The identity

$$\pi \int_0^\infty t^\nu |h(t)|^2 t dt = \sum_{k=0}^{2+\nu} \frac{(2+\nu)!}{k!(2+\nu-k)!} \pi \int_0^\infty t^\nu |h_k(t)|^2 t dt$$

is satisfied.

The Mellin transformation is the Fourier transformation for the positive half–line. A Mellin transform of order ν is defined when an analytic function

$$h^\wedge(z) = \pi \int_0^\infty t^\nu \exp(\pi itz) h(t) t dt$$

is the Laplace transform of order ν of a Baire function $h(t)$ of positive t, such that the integral

$$\pi \int_0^\infty t^\nu |h(t)|^2 t dt$$

converges, which vanishes in the interval $(0, a)$ for some positive number a.

The Mellin transform of order ν is the analytic function $F(z)$ of z in the upper half–plane defined by the integral

$$\pi F(z) = \int_0^\infty h^\wedge(it) t^{\frac{1}{2}\nu - iz} dt$$

or equivalently by the integral

$$F(z) = W(z) \int_0^\infty h(t) t^{\frac{1}{2}\nu + iz} dt$$

where

$$W(z) = \pi^{-\frac{1}{2}\nu - 1 + iz} \Gamma(\tfrac{1}{2}\nu + 1 - iz)$$

is an Euler weight function.

The analytic function

$$a^{-iz} F(z)$$

of z in the upper half–plane belongs to the weighted Hardy space $\mathcal{F}(W)$. Every function which belongs to the weighted Hardy space is so obtained.

A Stieltjes space of entire functions which is mapped isometrically in the weighted Hardy space by taking an entire function $F(z)$ of z into the analytic function

$$a^{-iz} F(z)$$

of z in the upper half–plane is defined as the set of all entire functions $F(z)$ of z such that the analytic functions

$$a^{-iz} F(z)$$

and

$$a^{-iz} F^*(z)$$

belong to the weighted Hardy space.

The elements of the Stieltjes space are the Mellin transforms of order ν of functions in the domain of the Laplace transformation of order ν which vanish in the interval $(0, a)$ and whose Hankel transform of order ν vanishes in the same interval. The functions are constructed from the decomposition of the space of Laplace transforms of order ν.

If $h(t)$ and

$$h'(t) = t \ \frac{d}{dt} \ h(t)$$

are Baire functions of positive t such that the integrals

$$\pi \int_0^\infty t^\nu |h(t)|^2 t dt$$

and

$$\pi \int_0^\infty t^\nu |h'(t)|^2 t dt$$

converge, if

$$h_0(t) = h(t) - \pi^2 t \int_t^\infty F(1; -\pi^2 t(x - t)) h(x) dx$$

and

$$h'_0(t) = h'(t) - \pi^2 t \int_t^\infty F(1; -\pi^2 t(x-t))h'(x)dx,$$

and if

$$h_k(t) = \pi^k \int_t^\infty (x-t)^{k-1} F(k-1; -\pi^2 t(x-t))h(x)dx$$

and

$$h'_k(t) = \pi^k \int_t^\infty (x-t)^{k-1} F(k-1; -\pi^2 t(x-t))h'(x)dx$$

when $k > 0$, then

$$-h'_k(t) = t \frac{d}{dt} h_k(t) + kh_k(t) + 2\pi t h_{k-1}(t)$$

when $k > 0$ and

$$h'_k(t) = t \frac{d}{dt} h_k(t) - kh_k(t) + 2\pi t h_{1+k}(t)$$

when $k < 2+\nu$.

The identity

$$\sum_{k=0}^{2+0} \frac{(2+\nu)!}{k!(2+\nu-k)!} \int_a^\infty t^\nu [h'_k(t)^- h_k(t) + h_k(t)^- h'_k(t) + (2+\nu)h_k(t)^- h_k(t)]tdt$$

$$= \sum_{k=0}^{1+\nu} \frac{(1+\nu)!}{k!(1+\nu-k)!} a^{2+\nu}[h_{2+\nu-k}(a)^- h_{2+\nu-k}(a) - h_k(a)^- h_k(a)]$$

holds when $a > 0$.

A function $\Phi_k(t,z)$ of positive t is defined for all complex a when $k = 0, \dots, 2+\nu$ by the integral

$$\pi\Phi_k(a,z) = \int_0^\infty t^{-k} \exp(-\pi a(t+t^{-1}))t^{\frac{1}{2}\nu - iz}dt.$$

The differential equations

$$-(iz - \tfrac{1}{2}\nu - 1)\Phi_k(t,z) = t \tfrac{d}{dt} \Phi_k(t,z) + k\Phi_k(t,z) + 2\pi t\Phi_{k-1}(t,z)$$

are satisfied when $k > 0$ and

$$(iz - \tfrac{1}{2}\nu - 1)\Phi_k(t,z) = t \tfrac{d}{dt}\Phi_k(t,z) - k\Phi_k(t,z) + 2\pi t\Phi_{1+k}(t,z)$$

when $k < 2+\nu$.

The identity

$$\sum_{k=0}^{2+\nu} \frac{(2+\nu)!}{k!(2+\nu-k)!} \int_a^\infty t^\nu \Phi_k(t,z)\Phi_k(t,w)^- tdt$$

$$= \sum_{k=0}^{1+\nu} \frac{(1+\nu)!}{k!(1+\nu-k)!} a^{2+\nu}[\Phi_k(a,z)\Phi_k(a,w)^- - \Phi_{2+\nu-k}(a,z)\Phi_{2+\nu-k}(a,w)^-]/[i(w^- - z)]$$

holds for all complex z and w when $a > 0$.

If the function $h_k(t)$ of t is the Sonine transform of order k of a function $h(t)$ of t for $k = 0, \dots, 2+\nu$, then

$$\int_0^\infty t^\nu \exp(-\pi t y) h(t) t dt = \sum_{k=0}^{2+\nu} \frac{(2+\nu)!}{k!(2+\nu-k)!} \, y^{-k} \int_0^\infty t^\nu \exp(-\pi t(y+y^{-1})) h_k(t) t dt$$

for all positive y.

If the function $h(t)$ of t vanishes in a neighborhood of the origin, its Mellin transform of order ν is an analytic function $F(z)$ of z in the upper half–plane defined by the integral

$$F(w) = \int_0^\infty \int_0^\infty h(t) \exp(-\pi t y) y^{\frac{1}{2}\nu - iw} t dt dy$$

when w is in the upper half–plane.

When the function $h_k(t)$ of t vanishes in a neighborhood of the origin for every $k = 0, \dots, 2+\nu$, the function $F(z)$ of z has an analytic extension to the complex plane. The identity

$$F(z) = \sum_{k=0}^{2+\nu} \frac{(2+\nu)!}{k!(2+\nu-k)!} \pi \int_0^\infty t^\nu \Phi_k(t,z) h_k(t) t dt$$

is satisfied.

A computable example is obtained in the case $\nu = -1$, which does not appear in Fourier analysis on the complex skew–plane. The functions

$$\pi \Phi_0(a,z) = \int_0^\infty \exp(-\pi a(t+t^{-1})) t^{-\frac{1}{2}-iz} dt$$

and

$$\pi \Phi_1(a,z) = \int_0^\infty t^{-1} \exp(-\pi a(t+t^{-1})) t^{-\frac{1}{2}-iz} dt$$

satisfy the equation

$$\int_a^\infty [\Phi_0(t,z)\Phi_0(t,w)^- + \Phi_1(t,z)\Phi_1(t,w)^-] dt$$
$$= a[\Phi_0(a,z)\Phi_0(a,w)^- - \Phi_1(a,z)\Phi_1(a,w)^-]/[i(w^- - z)]$$

for all complex z and w when $a > 0$. The symmetry conditions

$$\Phi_1(a,z) = \Phi_0(a,-z) = \Phi_0^*(a,z)$$

are satisfied.

A Stieltjes space of entire functions exists with defining function

$$E(a,z) = A(a,z) - iB(a,z)$$

where

$$A(a,z) = \tfrac{1}{2}a^{\frac{1}{2}} \int_0^\infty \exp(-\pi a(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2)(t^{\frac{1}{2}} + t^{-\frac{1}{2}})t^{-iz}t^{-1}dt$$

and

$$B(a,z) = \tfrac{1}{2}ia^{\frac{1}{2}} \int_0^\infty \exp(-\pi a(t^{\frac{1}{2}} + t^{-\frac{1}{2}})^2)(t^{\frac{1}{2}} - t^{-\frac{1}{2}})t^{-iz}t^{-1}dt.$$

The differential equations

$$a \frac{d}{da} A(a,z) = zB(a,z)\exp(4\pi a)$$

and

$$a \frac{d}{da} B(a,z) = -zA(a,z)\exp(-4\pi a)$$

are satisfied.

A construction of Euler weight functions is obtained on applying the Mellin transformation. The Mellin transformation reformulates the Fourier transformation for the real line on the multiplicative group of the positive half–line. Analytic weight functions constructed from the gamma function appear when the Mellin transformation is adapted to the domain of the Laplace transformation of harmonic ϕ.

The Mellin transform of harmonic ϕ of the function $f(\xi)$ of ξ in the complex skew–field is an analytic function $F(z)$ of z in the upper half–plane which is defined when $f(\xi)$ vanishes in the disk $\xi^-\xi < a$ for some positive number a. The function is defined by the integral

$$\pi F(z) = \int_0^\infty g(it)t^{\frac{1}{2}\nu - iz}dt.$$

Since

$$g(iy) = \pi \int_0^\infty t^{\frac{1}{2}\nu}h(t)\exp(-\pi ty)tdt$$

when y is positive, the identity

$$F(z)/W(z) = \int_0^\infty h(t)t^{\frac{1}{2}\nu + iz}dt$$

holds with Euler weight function

$$W(z) = \pi^{-\frac{1}{2}\nu - 1 + iz}\Gamma(\tfrac{1}{2}\nu + 1 - iz).$$

The identity

$$\int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 dx = 2\pi \int_0^\infty |h(t)|^2 t^{\nu - 2y} t dt$$

holds when y is positive.

The analytic function

$$a^{-iz}F(z)$$

of z in the upper half–plane belongs to the weighted Hardy space $\mathcal{F}(W)$ since the function $f(\xi)$ of ξ in the complex skew–field vanishes when $\xi^-\xi < a$. An analytic function $F(z)$ of z in the upper half–plane such that the function

$$a^{-iz}F(z)$$

of z belongs to the space $\mathcal{F}(W)$ is the Mellin transform of a function $f(\xi)$ of ξ in the complex skew–field which belongs to the domain of the Laplace transformation of harmonic ϕ and vanishes in the disk $\xi^-\xi < a$.

The adjoint of the Radon transformation for the complex skew–field takes elements of the domain of the Laplace transformation of harmonic ϕ which vanish in the disk $\xi^-\xi < a$ into elements of the space which vanish in the disk. The transformation acts as a maximal accretive transformation on the subspace. The adjoint is a maximal accretive transformation which is unitarily equivalent to multiplication by

$$i/z$$

in the Hilbert space which is the image of the subspace under the Laplace transformation of harmonic ϕ. When $\xi^-\xi < 1$ is the unit disk, the transformation is unitarily equivalent to the transformation which takes $F(z)$ into $F(z - i)$ whenever the functions of z belong to the space $\mathcal{F}(W)$.

4. Harmonic Analysis on an adic Skew–Plane

The adic skew–plane is constructed as a locally compact Abelian group which is the completion of the discrete skew–plane in an adic topology. The adic topology is initially defined on the group of integral elements.

Multiplication by a positive integer r maps the additive group of integral elements of the discrete skew–plane onto a subgroup whose quotient group is the group of integral elements of the discrete skew–plane modulo r.

The discrete topology is the unique Hausdorff topology of the group of integral elements of the discrete skew–plane modulo r. Addition is continuous as a transformation of the Cartesian product of the group with itself into the group. Conjugation is continuous as a transformation of the group into itself.

The adic topology of the group of integral elements of the discrete skew–plane is the least topology with respect to which the homomorphism onto the group of integral elements of the discrete skew–plane modulo r is continuous for every positive integer. Addition is continuous as a transformation of the Cartesian product of the group with itself into the group. Conjugation is continuous as a transformation of the group into itself.

The group of integral elements of the adic skew–plane is the completion of the group of integral elements of the discrete skew–plane in the adic topology made uniform by additive structure. Addition extends continuously as a transformation of the Cartesian product of the group with itself into the group. Conjugation extends continuously as a transformation of the group into itself.

The homomorphism of the group of integral elements of the discrete skew–plane onto the group of integral elements of the discrete skew–plane modulo r extends continuously as a homomorphism of the group of integral elements of the adic skew–plane onto the group of integral elements of the discrete skew–plane modulo r for every positive integer r.

The images of an element of the group of integral elements of the adic skew–plane satisfy a constraint. If r and s are positive integers such that r is a divisor of s, a canonical homomorphism exists of the group of integral elements of the discrete skew–plane modulo s onto the group of integral elements of the discrete skew–plane modulo r which takes the image modulo s of an integral element of the discrete skew-plane into its image modulo r. The homomorphism takes the image of an element of the group of integral elements of the adic skew–plane in the group of integral elements of the discrete skew–plane modulo s into its image in the group of integral elements of the discrete skew–plane modulo r.

If an element of the group of integral elements of the discrete skew–plane modulo r is chosen for every positive integer r satisfying the constraint, then a unique element of the group of integral elements of the adic skew–plane exists whose image in the group of integral elements of the discrete skew–plane modulo r is the chosen element.

The group of integral elements of the adic skew–plane is a compact Hausdorff space.

Multiplication by a positive integer r is an isomorphism of the group of integral elements of the discrete skew–plane into itself which is continuous for the adic topology. Multiplication by r has a continuous extension as an isomorphism of the group of integral elements of the adic skew–plane into itself. The inverse isomorphism is continuous when the subgroup is given its subspace topology inherited from the full group. The subgroup is a compact open subset of the full group.

The adic skew–plane is a locally compact Abelian group which contains the group of integral elements as a compact open subgroup. Multiplication by a positive integer is a continuous isomorphism of the adic skew–plane into itself whose inverse isomorphism is continuous. The discrete skew–plane is a dense subgroup of the adic skew–plane.

The canonical measure for the adic skew–plane is chosen to have value one on the subgroup of integral elements.

Multiplication on left or right by a nonzero element ξ of the discrete skew–plane is a continuous homomorphism of the discrete skew–plane into itself for the adic topology and extends continuously as a homomorphism of the adic skew–plane into itself. The transformation multiplies the canonical measure of the adic skew–plane by a factor of

$$\lambda(\xi)^{-4}.$$

The homomorphism of the additive group of the field of rational numbers into the multiplicative group of complex numbers of absolute value one defined by taking ξ into

$$\exp(2\pi i\xi)$$

is continuous for the adic topology since its kernel is the set of integers. The transformation has a unique continuous extension as a homomorphism of the adic completion of the rational numbers into the multiplicative group of complex numbers of absolute value one.

The Fourier transformation for the adic skew–plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the adic skew–plane into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^- \eta + \eta^- \xi)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure for the adic skew–plane. Fourier inversion states that the integral

$$f(\xi) = \int \exp(-\pi i(\xi^- \eta + \eta^- \xi)) g(\eta) d\eta$$

with respect to the canonical measure for the adic skew–plane represents the function $f(\xi)$ of ξ when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous. The Fourier transformation for the adic skew–plane commutes with the transformation which takes a function $f(\xi)$ of ξ into the functions $f(\omega\xi)$ of ξ for every element ω of the adic completion of the Gauss skew–field with conjugate as inverse.

The p–adic skew–plane is defined for a prime p in the same way as the adic skew–plane except that only those positive integers r are accepted which are powers of p. The p–adic skew–plane is a locally compact Abelian group which is the image of the adic skew–plane under a canonical homomorphism taking integral elements into integral elements. The homomorphism is defined on the group of integral elements of the adic skew–plane as the unique continuous extension of the canonical homomorphism of the group of integral elements of the discrete skew–plane modulo s onto the group of integral elements of the discrete skew–plane modulo r which exists when r is a divisor of s such that r is a power of p and s/r is not divisible by p.

A canonical homomorphism exists of the group of integral elements of the discrete skew–plane modulo s onto the group of integral elements of the discrete skew–plane modulo s/r. The Chinese remainder states that an isomorphism results of the group of integral elements of the discrete skew–plane modulo s onto the Cartesian product of the group of integral elements of the discrete skew–plane modulo r and the group of integral elements of the discrete skew–plane modulo s/r.

The group of integral elements of the adic skew–plane is isomorphic to the Cartesian product of the group of integral elements of the p–adic skew–plane and a group which is defined in the same way as the adic skew–plane except that only those positive integers r are accepted which are not divisible by p.

The group of integral elements of the adic skew–plane is isomorphic to the Cartesian product of the group of integral elements of the p–adic skew–plane taken over all primes p. The topology of the group of integral elements of the adic skew–plane is the Cartesian product topology of the group of integral elements of the p–adic skew–plane taken over all primes p. The canonical measure of the group of integral elements of the adic skew–plane is the Cartesian product measure of the canonical measure of the group of integral elements of the p–adic skew–plane taken over all primes p.

A theorem, accredited to Diophantus but whose earliest known proof is due to Lagrange, states that every positive integer is the sum of four squares of integers. An integral element ι_p of the Gauss skew–field exists for every prime p such that

$$p = \iota_p^- \iota_p.$$

When some such representation of p is chosen, a p–adic plane is defined as the set of elements ξ of the p–adic skew–plane which commute

$$\iota_p \xi = \xi \iota_p$$

with ι_p. The p–adic plane which appears here is a vector space of dimension two over the field of p–adic numbers.

The complementary space to the p–adic plane in the p–adic skew–plane is the set of elements ξ of the p–adic skew–plane which satisfy the identity

$$\xi\eta = \eta^- \xi$$

for every element η of the p–adic plane. Every element of the p–adic skew–plane is the unique sum of an element of the p–adic plane and an element of the complementary space when p is odd. The elements of the p–adic plane and of the complementary space are integral if the element of the p–adic skew–plane is integral.

The p–adic plane and its complementary space are locally compact Abelian group whose canonical measures are chosen so that the compact subgroups of integral elements have measure one. The canonical measure for the p–adic skew–plane is the Cartesian product measure product measure of the canonical measure for the p–adic plane and the canonical measure for the complementary space to the p–adic plane in the p–adic skew–plane.

The adic plane is the set of elements of the adic skew–plane whose p–adic component belongs to the p–adic plane for every prime p. The complementary space to the adic plane in the adic skew–plane is the set of elements of the adic skew–plane whose p–adic component belongs to the complementary space to the p–adic plane in the p–adic skew–plane for every prime p.

The adic plane and the adic skew–plane are locally compact Abelian groups whose canonical measures are chosen so hat the compact subgroups of integral elements have measure one. The canonical measure for the adic skew–plane is the Cartesian product measure of the canonical measure for the adic plane and the canonical measure for the complementary space to the adic plane in the adic skew–plane.

The Fourier transformation for the adic plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the adic plane into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^- \eta + \eta^- \xi)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^- \eta + \eta^- \xi)) g(\eta) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The irreducible representations of the group of elements with conjugate as inverse are computed for the complex skew–plane by means of harmonic functions. The analogous computation for an adic skew–plane is made by means of functions defined on a quotient group of the group of nonzero elements of the discrete skew–plane.

A normal subgroup of the group of nonzero elements of the discrete skew–plane is the group of nonzero rational numbers. The integral group for the discrete skew–plane is the quotient group. Nonzero integral elements α and β of the discrete skew–plane are equivalent with respect to the subgroup if, and only if,

$$\alpha\delta = \beta\gamma$$

for nonzero rational numbers γ and δ.

A fundamental domain for the equivalence relation is the set of nonzero integral elements of the discrete skew–plane which are divisible by no nonzero integer which does not have an integer as inverse. Elements α and β of a fundamental domain are equivalent if, and only if,

$$\alpha = \beta\gamma$$

for an integer with integer as inverse.

The integral group of the discrete skew–plane has normal subgroups whose quotient groups are finite. A normal subgroup is generated for every positive integer r by the set of nonzero integral elements ξ of the discrete skew–plane such that for every prime divisor p of r the greatest power of p which is a divisor of $\xi^- \xi$ is a power of the greatest power of p which is a divisor of r. A fundamental domain for the equivalence relation defined by the subgroup is the set of elements ξ of the fundamental domain for the full group such that for every prime divisor p of r the greatest power of p which is a divisor of $\xi^- \xi$ is less than the greatest power of p which is a divisor of r. The intersection of the normal subgroups is the set of integral elements with integral inverse.

The adic skew–plane is an algebra whose group of invertible elements contains the closed normal subgroup whose elements are self–conjugate. The quotient group is isomorphic to the integral group of the discrete skew–plane. A function defined on the integral group of the discrete skew–plane is treated as a function defined on the group of invertible elements of the adic skew–plane which has equal values at elements obtained from each other on multiplication of the independent variable by a self–conjugate element. The function is defined to vanish at noninvertible elements of the adic skew–plane.

The integral group of the discrete skew–plane has normal subgroups which are comparable. If ρ and σ are positive integers, the subgroup defined by σ is contained in the subgroup defined by ρ if, and only if, for every prime p the greatest power of p which is a divisor of ρ is a power of the greatest power of p which is a divisor of σ. A canonical projection acts as a homomorphism of the quotient group defined by σ onto the quotient group defined by ρ. A function defined on the quotient group defined by σ is treated as a function defined on the quotient group defined by ρ which has equal values at elements which project into the same element of the quotient group defined by σ.

The quotient groups defined by ρ and by σ are compact Hausdorff spaces in the discrete topology and have canonical measures. The canonical measure is counting measure divided by the number of elements in the group. A function defined on the group is square integrable with respect to the canonical measure. The Hilbert space of square integrable functions with respect to the canonical measure for the quotient group defined by σ is contained isometrically in the Hilbert space of square integrable functions with respect to the canonical measure for the quotient group defined by ρ.

A harmonic function of order ρ is defined as a function which is square integrable with respect to the canonical measure for the quotient group defined by ρ which is orthogonal to every square integrable function with respect to the quotient group defined by σ for every positive integer σ less than ρ such that for every prime divisor p of σ greatest power of p which is a divisor of ρ is a power of the greatest power of p which is a divisor of σ.

The Radon transformation of harmonic ϕ for the adic skew–plane is a transformation with domain and range in the Hilbert space of functions $f(\xi)$ of ξ in the adic skew–plane which are square integrable with respect to the canonical measure for the adic skew–plane, which vanish at noninvertible elements, and which satisfy the identity

$$\phi(\xi)f(\omega\xi) = \phi(\omega\xi)f(\xi)$$

for every element ω of the adic skew–plane with conjugate as inverse.

The Radon transformation of harmonic ϕ takes a function $f(\xi)$ of ξ into the function $g(\xi)$ of ξ when the identity

$$g(\omega\xi)/\phi(\omega\xi) = \int f(\omega\xi + \omega\eta)/\phi(\omega\xi + \omega\eta) d\eta$$

holds for every element ω of the adic skew–plane with conjugate as inverse when ξ is in the adic plane with integration with respect to the canonical measure for the complementary space to the adic plane in the adic skew–plane.

The integral is interpreted as a limit of the integrals over the set of elements ξ of the complementary space such that

$$\lambda(\xi^-\xi) \leq n$$

for a positive integer n. Convergence is in the weak topology of the Hilbert space of square integrable functions with respect to the canonical measure for the adic skew–plane as n becomes eventually divisible by every positive integer whose prime divisors are divisors of r.

The Laplace transformation of harmonic ϕ for the adic skew–plane gives a spectral analysis of the adjoint of the Radon transformation of harmonic ϕ for the adic skew–plane. The domain of the Laplace transformation of harmonic ϕ is the Hilbert space of functions $f(\xi)$ of ξ in the adic skew–plane which are square integrable with respect to the canonical measure for the adic skew–plane, which vanish at noninvertible elements, and which satisfy the identity

$$\phi(\xi)f(\omega\xi) = \phi(\omega\xi)f(\xi)$$

for every element ω of the adic skew–plane with conjugate as inverse. An adic half–plane is applied in the parametrization of these functions.

A p–adic half–plane is an analogue for the p–adic skew–plane of the upper half–plane for the complex skew–plane.

A p–adic half–plane is a maximal subfield of the p–adic skew–plane such that every nonzero element is the product of a nonzero p–adic number and an integral element of the field with integral inverse.

When p is the even prime, the elements of the p–adic half–plane are quaternions which commute with

$$-\tfrac{1}{2} + \tfrac{1}{2}\ i + \tfrac{1}{2}\ j + \tfrac{1}{2}\ k.$$

When p is an odd prime, the elements of the p–adic half–plane are quaternions which commute with an integral quaternion

$$i\alpha + j\beta + k\gamma$$

with integral inverse which belongs to the complementary space to the p–adic plane in the p–adic skew–plane.

The integral p–adic numbers α, β, γ are chosen so that their residue classes modulo p determine a subfield of the algebra of quaternions whose coordinates are integers modulo p: The elements of the algebra which commute with the given element is a field.

The canonical measure for the adic half–plane is chosen so that the set of integral elements has measure one.

A Laplace transformation of harmonic ϕ is defined when a harmonic function ϕ of order ν for the adic skew–plane has norm one in the Hilbert space of harmonic functions of order ν. The domain of the Laplace transformation of harmonic ϕ is the set of functions $f(\xi)$ of ξ in the adic skew–plane which are square integrable with respect to the canonical measure for the adic skew–plane, which vanish at noninvertible elements of the adic skew–plane, and which satisfy the identity

$$\phi(\xi)f(\omega\xi) = \phi(\omega\xi)f(\xi)$$

for every element ω of the adic skew–plane with conjugate as inverse.

The parametrization of functions defined on the adic skew–plane applies properties of signatures for the adic half–plane. A signature modulo I for the adic half–plane is defined with respect to a closed ideal I of the ring of integral elements of the adic half–plane with finite quotient ring.

A signature σ modulo I is a function $\sigma(\xi)$ of ξ in the adic half–plane which vanishes at nonintegral elements and at integral elements which are not invertible modulo I, which has equal values at integral elements which are congruent modulo I, which satisfies the identity

$$\sigma(\xi\eta) = \sigma(\xi)\sigma(\eta)$$

for all integral elements ξ and η, and whose values are fourth roots of unity on elements with conjugate as inverse.

A signature modulo I is said to be primitive modulo I if it agrees on integral elements which are invertible modulo I with no signature modulo an ideal which properly contains I.

A function $\chi(\xi)$ of ξ in the adic line is the restriction of a signature modulo I if it vanishes at nonintegral elements and at integral elements which are not invertible modulo I, if it has equal values at elements which are congruent modulo I, if it satisfies the identity

$$\chi(\xi\eta) = \chi(\xi)\chi(\eta)$$

for all integral elements ξ and η, and if its values are square roots of unity on elements with self inverse.

An element ω of the adic half–plane is said to be stable if any two signatures which agree at all elements of the adic line with self inverse agree at ω.

An element of the adic half–plane is stable if it is the product of an element of the adic line and a stable element of the adic half–plane.

A function $h(\xi)$ of ξ in the adic half–plane is said to be analytic of signature σ if the identity

$$h(\omega\xi) = \sigma(\omega)h(\xi)$$

holds for every stable element ω of the adic half–plane with conjugate as inverse and if the function vanishes at unstable elements of the adic half–plane. The σ–extension of the analytic function is the function $\mathrm{ext}h(\xi)$ of ξ in the adic half–plane which agrees with $h(\xi)$ when ξ is a stable element of the adic half–plane and which satisfies the identity

$$\mathrm{ext}h(\omega\xi) = \sigma(\omega)\mathrm{ext}h(\xi)$$

for every element ω of the adic half–plane with conjugate as inverse.

A function $h(\xi)$ of ξ in the adic half–plane is said to be analytic if it is a linear combination of analytic functions of signatures σ. The signatures are chosen to agree at all elements of the adic half–plane with conjugate as inverse if they agree on elements of the adic line with self inverse. An extension of the analytic function is the function $\text{ext}h(\xi)$ of ξ which is a sum of σ–extensions of σ–components of the function $h(\xi)$ of ξ.

The Fourier transformation for the adic half–plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^- \eta + \eta^- \xi))f(\eta)d\eta$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^- \eta + \eta^- \xi))g(\eta)d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

Since the Fourier transformation for the adic half–plane takes a function $h(\xi)$ of ξ which satisfies the identity

$$h(\omega\xi) = \sigma(\omega)h(\xi)$$

for every element ω of the adic half–plane with conjugate as inverse into a function $g(\xi)$ of ξ which satisfies the identity

$$g(\omega\xi) = \sigma(\omega)g(\xi)$$

for every element ω of the adic half–plane with conjugate as inverse, the Fourier transformation takes analytic functions into analytic functions.

A Laplace transformation of harmonic ϕ is defined for the adic skew–plane when a harmonic ϕ is normalized

$$\int |\phi(\xi)|^2 d\xi = 1$$

by integration with respect to the canonical measure for the adic skew–plane over the set of integral elements of the adic skew–plane.

A function

$$f(\xi) = \phi(\xi)h(\xi^- \xi)$$

of ξ in the adic skew–plane which is square integrable with respect to the canonical measure for the adic skew–plane and which satisfies the identity

$$\phi(\xi)f(\omega\xi) = \phi(\omega\xi)f(\xi)$$

for every element ω of the adic skew–plane with conjugate as inverse is parametrized by an analytic function $h(\xi)$ of ξ in the adic half–plane which is square integrable with respect to the canonical measure for the adic half–plane.

The Laplace transform of harmonic ϕ of the function $f(\xi)$ of ξ in the adic skew–plane is the analytic function $g(\xi)$ of ξ in the adic half–plane which is the Fourier transform of the function $h(\xi)$ of ξ in the adic half–plane.

The identity

$$\int |f(\xi)|^2 d\xi = 2 \int |\mathrm{ext} g(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the adic skew–plane and with integration on the right with respect to the canonical measure for the adic half–plane.

Fourier inversion

$$h(\xi) = \int \exp(-\pi i(\xi^- \eta + \eta^- \xi)) g(\eta) d\eta$$

applies with integration with respect to the canonical measure for the adic half–plane when the function $g(\xi)$ of ξ is integrable and the function $h(\xi)$ of ξ is continuous.

The integral representation

$$f(\omega\xi + \omega\eta)/\phi(\omega\xi + \omega\eta) = \int \exp(-\pi i \xi^- \xi(\gamma + \gamma^-)) \exp(-\pi i \eta^- \eta(\gamma + \gamma^-)) g(\gamma) d\gamma$$

applies with integration with respect to the canonical measure for the adic half–plane when ξ is in the adic plane, η is in the complementary space to the adic plane in the adic skew–plane, and ω is an element of the adic skew–plane with conjugate as inverse.

If ρ is a positive integer and if γ is an element of the adic half–plane such that $\rho(\gamma+\gamma^-)^{-1}$ is integral, the identity

$$\int \exp(-\pi i \eta^- \eta(\gamma + \gamma^-)) d\eta = \lambda(\gamma + \gamma^-)^{-1}$$

holds with integration with respect to the canonical measure for the complementary space to the adic plane in the adic skew–plane over the set of elements η such that

$$\rho\eta^- \eta$$

is integral.

The identity

$$\int f(\omega\xi + \omega\eta)/\phi(\omega\xi + \omega\eta) d\eta = \int \exp(-\pi i \xi^- \xi(\gamma + \gamma^-)) \lambda_r(\gamma + \gamma^-)^{-1} g(\gamma) d\gamma$$

holds with integration on the left with respect to the canonical measure for the complementary space to the adic plane in the adic skew–plane and with integration on the right with respect to the canonical measure for the adic half–plane.

When it converges, the integral on the right represents the square integrable analytic function $h'(\xi)$ of ξ in the adic half–plane such that

$$\int u(\xi)^- h'(\xi) d\xi = \int v(\xi)^- \lambda(\xi + \xi^-)^{-1} g(\xi) d\xi$$

with integration with respect to the canonical measure for the adic half–plane whenever the Fourier transformation for the adic half–plane takes a square integrable analytic function $u(\xi)$ of ξ into a square integrable analytic function $v(\xi)$ of ξ whose product with $\lambda(\xi+\xi^-)^{-1}$ is square integrable.

The interchange of integrals is justified by absolute convergence.

The Radon transformation of harmonic ϕ for the ring of integral elements of the adic skew–plane is a restriction of the Radon transformation of harmonic ϕ for the adic skew–plane. The domain and range of the transformation are contained in the Hilbert space of functions $f(\xi)$ of ξ in the adic skew–plane which are square integrable with respect to the canonical measure for the adic skew–plane, which satisfy the identity

$$\phi(\xi) f(\omega\xi) = \phi(\omega\xi) f(\xi)$$

for every element ω of the adic skew–plane with conjugate as inverse, and which vanish at nonintegral elements of the adic skew–plane.

The ring of integral elements of the adic skew–plane is a compact Hausdorff space in the subspace topology inherited from the adic skew–plane. The canonical measure for the ring is the restriction to Baire subsets of the ring of the canonical measure for the adic skew–plane.

The spectral theory of the adjoint of the Radon transformation for the ring applies the quotient ring modulo r of the ring of integral elements of the adic skew–plane for a positive integer r. A function defined on the quotient ring is treated as a function defined on the ring which has equal values at elements which are congruent modulo r. The canonical measure for the ring of integral elements of the adic skew–plane modulo r is counting measure divided by the number of elements in the ring. The projection of the ring of integral elements of the adic skew–plane onto the quotient ring modulo r is a continuous open mapping which is a homomorphism of conjugated ring structure and which maps the canonical measure into the canonical measure.

The ring of integral elements of the adic plane is contained in the ring of integral elements of the adic skew–plane. The ring is a compact Hausdorff space in the subspace topology inherited from the adic plane, which is identical with the subspace topology inherited from the ring of integral elements of the adic skew–plane. The canonical measure for the ring is the restriction to Baire subsets of the ring of the canonical measure for the adic plane.

The ring of integral elements of the adic plane modulo r is the quotient ring of the ring of integral elements of the adic plane modulo the ideal generated by r. The ring of integral elements of the adic plane modulo r is isomorphic to the image of the ring of integral elements of the adic plane in the ring of integral elements of the adic skew–plane modulo r. The ring of integral elements of the adic plane modulo r is treated as a subring of the ring of integral elements of the adic skew–plane modulo r. The canonical measure for the quotient ring is counting measure divided by the number of elements in the ring. The projection of the ring of integral elements of the adic plane onto the quotient ring is a continuous open mapping which is a homomorphism of conjugated ring structure and which maps the canonical measure into the canonical measure.

The complementary space to the ring of integral elements of the adic plane in the ring of integral elements of the adic skew–plane is defined as the set of integral elements of the complementary space to the adic plane in the adic skew–plane. The set of integral elements of the complementary space is an additive group which is a compact Hausdorff space in the subspace topology inherited from the complementary space to the adic plane in the adic skew–plane. The canonical measure for the set of integral elements of the complementary space is the restriction to its Baire subsets of the canonical measure for the complementary space.

The complementary space to the ring of integral elements of the adic plane modulo r in the ring of integral elements of the adic skew–plane modulo r is defined as the image in the ring of integral elements of the adic skew–plane modulo r of the set of integral elements of the complementary space to the adic plane in the adic skew–plane.

The complementary space modulo r has a finite number of elements and is given the discrete topology. The set is an additive group whose canonical measure is counting measure divided by the number of elements of the set. The projection of the set of integral elements of the complementary space to the adic plane in the adic skew–plane onto the complementary space modulo r is a continuous open mapping which is a homomorphism of additive structure and which maps the canonical measure into the canonical measure.

The Radon transformation for the ring of integral elements of the adic skew–plane takes a function $f(\xi)$ of ξ in the ring into a function $g(\xi)$ of ξ in the ring when the identity

$$g(\omega\xi)/\phi(\omega\xi) = \int f(\omega\xi + \omega\eta)/\phi(\omega\xi + \omega\eta)d\eta$$

holds for every element ω of the adic skew–plane with conjugate as inverse when ξ is an integral element of the adic plane with integration with respect to the canonical measure for the complementary space to the ring of integral elements of the adic plane in the ring of integral elements of the adic skew–plane.

The Radon transformation for the ring of integral elements of the adic skew–plane modulo r is a restriction of the Radon transformation for the ring of integral elements of the adic skew–plane. The domain and range of the transformation are contained in the Hilbert space of functions $f(\xi)$ of ξ in the ring of integral elements of the r–adic skew–plane modulo r which are square integrable with respect to the canonical measure for the ring of integral elements of the adic skew–plane modulo r and which satisfy the identity

$$\phi(\xi)f(\omega\xi) = \phi(\omega\xi)f(\xi)$$

for every element ω of the adic skew–plane with conjugate as inverse.

The Radon transformation for the ring of integral elements of the adic skew–plane modulo r takes a function $f(\xi)$ of ξ in the ring into a function $g(\xi)$ of ξ in the ring when the identity

$$g(\omega\xi)/\phi(\omega\xi) = \int f(\omega\xi + \omega\eta)/\phi(\omega\xi + \omega\eta) d\eta$$

holds for every element ω of the adic skew–plane with conjugate as inverse when ξ is an element of the ring of integral elements of the adic plane modulo r with integration with respect to the canonical measure for the complementary space to the ring of integral elements of the adic plane modulo r in the ring of integral elements of the adic skew–plane modulo r.

A function

$$f(\xi) = \phi(\xi) h(\xi^- \xi)$$

of ξ in the ring of integral elements of the adic skew–plane, which is square integrable with respect to the canonical measure for the adic skew–plane, and which satisfies the identity

$$\phi(\xi) f(\omega\xi) = \phi(\omega\xi) f(\xi)$$

for every element ω of the adic skew–plane with conjugate as inverse, is parametrized by an analytic function $h(\xi)$ of ξ in the adic half–plane which is square integrable with respect to the canonical measure for the adic half–plane and which vanishes at nonintegral elements of the adic half–plane.

The set of integral elements of the adic half–plane is subring which is a compact Hausdorff space in the subspace topology inherited from the adic half–plane. The canonical measure for the subring is the restriction to Baire subsets of the subring of the canonical measure for the adic half–plane.

The identity

$$\int |f(\xi)|^2 d\xi = 2 \int |\mathrm{ext} h(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the ring of integral elements of the adic skew–plane and with integration on the right with respect to the canonical measure for the ring of integral elements of the adic half–plane.

The Fourier transformation for the ring of integral elements of the adic half–plane applies a quotient space of the adic half–plane. Elements ξ and η of the adic half–plane are said to be congruent modulo 1 if they differ by an integral element $\eta - \xi$ of the adic half–plane.

The adic half–plane modulo 1 is the quotient space of the adic half–plane defined by the equivalence relation. The quotient space is an additive group which has the discrete topology and whose canonical measure is counting measure. The projection of the adic half–plane onto the adic half–plane modulo 1 is a continuous open mapping which is a homomorphism of additive structure and which takes the canonical measure into the canonical measure. A function defined on the adic half–plane modulo 1 is treated as a function defined on the adic half–plane which has equal values at elements which are congruent modulo 1.

An example

$$\exp(\pi i(\xi^- \eta + \eta^- \xi))$$

of a function of ξ in the adic half–plane modulo 1 is obtained when η is an integral element of the adic half–plane.

The Fourier transformation for the ring of integral elements of the adic half–plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the ring onto the Hilbert space of square integrable functions with respect to the canonical measure for the adic half–plane modulo 1 which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^- \eta + \eta^- \xi)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure for the group. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^- \eta + \eta^- \xi)) g(\eta) d\eta$$

applies with integration with respect to the canonical measure for the adic half–plane modulo 1 when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

An element of the adic half–plane modulo 1 is said to be stable if it is the product of an element of the adic line modulo 1 and a stable element of the adic half–plane with conjugate as inverse.

A function $g(\xi)$ of ξ in the adic half–plane modulo 1 is said to be analytic of signature σ if the identity

$$g(\omega\xi) = \sigma(\omega) g(\xi)$$

holds for every stable element ω of the adic half–plane with conjugate as inverse and if the function vanishes at unstable elements of the adic half–plane modulo 1. The σ–extension of the analytic function is the function $\text{ext}g(\xi)$ of ξ in the r–adic half–plane modulo 1 which agrees with $g(\xi)$ when ξ is a stable element of the adic half–plane modulo 1 and which satisfies the identity

$$\text{ext}g(\omega\xi) = \sigma(\omega)\text{ext}g(\xi)$$

for every element ω of the adic half–plane with conjugate as inverse.

A function $g(\xi)$ of ξ in the adic half–plane modulo 1 is said to be analytic if it is a linear combination of analytic functions of signatures σ. The signatures are chosen to agree on all elements of the adic half–plane with conjugate as inverse if they agree on elements of the adic line with self–inverse. An extension of the analytic function is a function $\text{ext}g(\xi)$ of ξ in the adic half–plane modulo 1 which is a sum of σ–extensions of σ–components of the function $g(\xi)$ of ξ.

The Fourier transform of an analytic function on the ring of integral elements of the adic half–plane is an analytic function on the adic half–plane modulo 1. The inverse Fourier transform of an analytic function on the adic half–plane modulo 1 is an analytic function on the ring of integral elements of the adic half–plane.

The Laplace transform of harmonic ϕ for the ring of integral elements of the adic skew–plane of a function

$$f(\xi) = \phi(\xi)h(\xi^- \xi)$$

of ξ in the ring which is square integrable with respect to the canonical measure for the ring and which is parametrized by a square integrable analytic function $h(\xi)$ of ξ in the ring of integral elements of the adic half–plane is the square integrable analytic function $g(\xi)$ of ξ in the adic half–plane modulo 1 which is the Fourier transform of the function $h(\xi)$ of ξ.

The identity

$$\int |f(\xi)|^2 d\xi = 2 \int |\text{ext} g(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the ring of integral elements of the adic skew–plane and with integration on the right with respect to the canonical measure for the adic half–plane modulo 1.

The adic modulus

$$\lambda(\xi) = \sup \lambda(\eta)$$

of an element ξ of the adic half–plane modulo 1 is defined as the maximum adic modulus of an element η of the adic half–plane which represent ξ.

The adjoint of the Radon transformation of harmonic ϕ takes a function

$$f(\xi) = \phi(\xi)h(\xi^- \xi)$$

of ξ in the ring of integral elements of the adic skew–plane whose Laplace transform of harmonic ϕ is a function $g(\xi)$ of ξ in the adic half–plane modulo 1 into an analytic function

$$f'(\xi) = \phi(\xi)h'(\xi^- \xi)$$

of ξ in the ring of integral elements of the adic skew–plane whose Laplace transform of harmonic ϕ is an analytic function $g'(\xi)$ of ξ in the adic half–plane modulo 1 if, and only if, the identity

$$\int v(\xi)^- g'(\xi) d\xi = \int v(\xi)^- \lambda(\xi + \xi^-)^{-1} g(\xi) d\xi$$

holds with integration with respect to the canonical measure for the adic half–plane modulo 1 for every square integrable analytic function $v(\xi)$ of ξ in the adic half–plane modulo 1 which vanishes when $\lambda(\xi + \xi^-)$ vanishes.

The r–annihilated subgroup of the adic half–plane modulo 1 is the set of elements ξ such that $r\xi$ vanishes. Elements of the r–annihilated subgroup are represented by elements ξ of the adic half–plane such that $r\xi$ is integral. Elements ξ and η of the adic half–plane such that $r\xi$ and $r\eta$ are integral represent the same element of the adic half–plane modulo 1 if, and only if, $r\xi$ and $r\eta$ represent the same integral element of the adic half–plane modulo r.

The r–annihilated subgroup of the adic half–plane modulo 1 is a finite set which has the discrete topology and whose canonical measure is counting measure.

The Fourier transformation for the ring of integral elements of the adic half–plane modulo r is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the ring onto the Hilbert space of square integrable functions with respect to the canonical measure for the r–annihilated subgroup of the adic half–plane modulo 1 which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^- \eta + \eta^- \xi)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure for the ring. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^- \eta + \eta^- \xi)) g(\eta) d\eta$$

holds with integration with respect to the canonical measure for the r–annihilated subgroup when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

An element ω of the adic half–plane with conjugate as inverse is said to be stable modulo r if any two signatures modulo divisors of r which agree on elements of the adic line with self–inverse agree on ω.

An element of the ring of integral elements of the adic half–plane modulo r is said to be stable if it is the product of an element of the ring of integral elements of the adic line modulo r and an element of the adic half–plane with conjugate as inverse which is stable modulo r.

A function $h(\xi)$ of ξ in the ring of integral elements of the adic half–plane modulo r is said to be analytic of signature σ for a signature σ modulo a divisor of r if the identity

$$h(\omega\xi) = \sigma(\omega) h(\xi)$$

holds for every element ω of the adic half–plane with conjugate as inverse which is stable modulo r and if the function vanishes at unstable elements of the ring of integral elements of the adic half–plane modulo r.

The σ–extension of a function $h(\xi)$ of ξ in the ring of integral elements of the adic half–plane modulo r which is analytic of signature σ is the function $\text{ext}h(\xi)$ of ξ in the ring of integral elements of the adic half–plane modulo r which agrees with $h(\xi)$ when ξ is stable and which satisfies the identity

$$\text{ext}h(\omega\xi) = \sigma(\omega)\text{ext}h(\xi)$$

for every element ω of the adic half–plane with conjugate as inverse.

A function $h(\xi)$ of ξ in the ring of integral elements of the adic half–plane modulo r is said to be analytic if it is a linear combination of functions analytic of signatures σ for signatures modulo divisors of r. The signatures are chosen to agree on elements of the adic half–plane with conjugate as inverse if they agree on elements of the adic line with self–inverse. An extension of the analytic function $h(\xi)$ of ξ is a function $\text{ext}h(\xi)$ of ξ which is a sum of the σ–extensions of the components which are analytic of signatures σ.

A function $g(\xi)$ of ξ in the r–annihilated subgroup of the adic half–plane modulo 1 is determined by a function $h(\xi)$ of ξ in the ring of integral elements of the adic half–plane modulo r such that

$$g(\xi) = h(r\xi)$$

for every element ξ of the adic half–plane such that $r\xi$ is integral. The function $g(\xi)$ of ξ is defined as analytic if, and only if, the function $h(\xi)$ of ξ is analytic.

The Fourier transformation takes analytic functions on the ring of integral elements of the adic half–plane modulo r into analytic functions on the r–annihilated subgroup of the adic half–plane modulo 1. The inverse Fourier transformation takes analytic functions on the r–annihilated subgroup of the adic half–plane modulo 1 into analytic functions on the ring of integral elements of the adic half–plane modulo r.

The Laplace transform of harmonic ϕ of a function

$$f(\xi) = \phi(\xi)h(\xi^-\xi)$$

of ξ in the ring of integral elements of the adic skew–plane modulo r which is parametrized by an analytic function $h(\xi)$ of ξ in the ring of integral elements of the adic half–plane modulo r is the analytic function $g(\xi)$ of ξ in the r–annihilated subgroup of the adic half–plane modulo 1 which is the Fourier transform of the function $h(\xi)$ of ξ.

The identity

$$\int |f(\xi)|^2 d\xi = 2\int |\text{ext}g(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the ring of integral elements of the adic skew–plane modulo r and with integration on the right with respect to the canonical measure for the r–annihilated subgroup of the adic half–plane modulo 1.

The Radon transformation of harmonic ϕ takes a function

$$f(\xi) = \phi(\xi)h(\xi^-\xi)$$

of ξ in the ring of integral elements of the adic skew–plane modulo r whose Laplace transform is an analytic function $g(\xi)$ of ξ in the r–annihilated subgroup of the adic half–plane modulo 1 into a function

$$f'(\xi) = \phi(\xi)h'(\xi^-\xi)$$

of ξ in the ring of integral elements of the adic skew–plane modulo r whose Laplace transform is an analytic function $g'(\xi)$ of ξ in the r–annihilated subgroup of the adic half–plane modulo 1 if, and only if, the identity

$$\int v(\xi)^- g'(\xi)d\xi = \int v(\xi)^- \lambda(\xi + \xi^-)^{-1} g(\xi)d\xi$$

holds with integration with respect to the canonical measure for the r–annihilated subgroup of the adic half–plane modulo 1 for every analytic function $v(\xi)$ of ξ in the r–annihilated subgroup of the adic half–plane modulo 1 which vanishes when $\lambda(\xi + \xi^-)$ vanishes.

5. Harmonic Analysis on an Adelic Skew–Plane

The adelic skew–plane is the Cartesian product of the complex skew–plane and the adic skew–plane. An element ξ of the adelic skew–plane has a component ξ_+ in the complex skew–plane and a component ξ_- in the adic skew–plane. The adelic skew–plane is a conjugated ring with coordinate addition and multiplication.

The sum $\xi + \eta$ of elements ξ and η of the adelic skew–plane is the element of the adelic skew–plane whose component in the complex skew–plane is the sum

$$\xi_+ + \eta_+$$

of components in the complex skew–plane and whose component in the adic skew–plane is the sum

$$\xi_- + \eta_-$$

of components in the r–adic skew–plane.

The product $\xi\eta$ of elements ξ and η of the adelic skew–plane is the element of the adelic skew–plane whose component in the complex skew–plane is the product

$$\xi_+\eta_+$$

of components in the complex skew–plane and whose component in the r–adic skew–plane is the product

$$\xi_-\eta_-$$

of components in the adic skew–plane.

The conjugate of an element ξ of the adelic skew–plane is the element ξ^- of the adelic skew–plane whose component in the complex skew–plane is the conjugate

$$\xi_+^-$$

of the component in the complex skew–plane and whose component in the adic skew–plane is the conjugate

$$\xi_-^-$$

of the component in the r–adic skew–plane.

The adelic skew–plane is a locally compact Hausdorff space in the Cartesian product topology of the topology of the complex skew–plane and the topology of the adic skew–plane. Addition is continuous as a transformation of the Cartesian product of the adelic skew–plane with itself into the adelic skew–plane. Multiplication by an element of the adelic skew–plane is a continuous transformation of the r–adelic skew–plane into itself. Conjugation is a continuous transformation of the adelic skew–plane into itself.

The canonical measure for the adelic skew–plane is the Cartesian product measure of the canonical measure for the complex skew–plane and the canonical measure for the adic skew–plane. The measure is defined on Baire subsets of the adelic skew–plane. A measure preserving transformation of the adelic skew–plane into itself is defined by taking ξ into $\xi + \eta$ for every element η of the adelic skew–plane. Measure preserving transformations of the adelic skew–plane into itself are defined by taking ξ into $\omega\xi$ and into $\xi\omega$ for every element ω of the adelic skew–plane with has conjugate as inverse.

The Fourier transformation for the adelic skew–plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the adelic skew–plane into itself which takes an integrable function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the adelic skew–plane into the continuous function

$$g(\xi_+, \xi_-) = \int \exp(\pi i(\xi_+^- \eta_+ + \eta_+^- \xi_+)) \exp(\pi i(\xi_-^- \eta_- + \eta_-^- \xi_-)) f(\eta_+, \eta_-) d\eta$$

of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi_+, \xi_-) = \int \exp(-\pi i(\xi_+^- \eta_+ + \eta_+^- \xi_+)) \exp(-\pi i(\xi_-^- \eta_- + \eta_-^- \xi_-)) g(\eta_+, \eta_-) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ is integrable and the function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ is continuous.

The adelic half–plane is the set of elements (z, ξ) of the adelic skew–plane whose component z in the complex skew–plane belongs to the upper half–plane and whose component ξ in the adic skew–plane belongs to the adic half–plane. The topology of the adelic half–plane is the Cartesian product topology of the topology of the upper half–plane and the topology of the adic half–plane. The canonical measure for the adelic half–plane is the Cartesian product measure of the canonical measure for the upper half–plane and the canonical measure for the adic half–plane.

Harmonic functions of order $\rho = (\rho_+, \rho_-)$ for the adelic skew–plane are defined when harmonic functions of order ρ_+ are defined for the complex skew–plane and harmonic functions of order ρ_- are defined for the adic skew–plane.

A harmonic function of order ρ for the adelic skew–plane is a continuous function $\phi(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the adelic skew–plane such that for every element ξ_- of the adic skew–plane the function of ξ_+ in the complex skew–plane is a harmonic function of order ρ_+ and for every element ξ_+ of the complex skew–plane the function of ξ_- in the adic skew–plane is a harmonic function of order ρ_-.

The set of harmonic functions of order ρ for the adelic skew–plane is a Hilbert space whose scalar product is defined from the scalar product of the Hilbert space of harmonic functions of order ρ_+ for the complex skew–plane and the scalar product for the Hilbert space of harmonic functions of order ρ_- for the adic skew–plane.

The Hilbert space of harmonic functions of order ρ_+ for the complex skew–plane admits an orthogonal basis whose elements are monomials. The Hilbert space of harmonic functions of order ρ for the adelic skew–plane is the orthogonal sum of subspaces determined by the monomials for the complex skew–plane. A subspace contains the product of the monomial with a harmonic function of order ρ_- for the adic skew–plane. The scalar product of two elements of the subspace is the scalar self–product of the monomials for the complex skew–plane multiplied by the scalar product of the harmonic functions for the adic skew–plane.

Multiplication on left by an element of the adelic skew–plane with conjugate as inverse on the argument of a harmonic function of order ρ is an isometric transformation of the Hilbert space of harmonic functions of order ρ into itself. The dimension of the Hilbert space of harmonic functions of order ρ for the adelic skew–plane is the product of the dimension of the Hilbert space of harmonic functions of order ρ_+ for the complex skew–plane and the dimension of the Hilbert space of harmonic functions of order ρ_- for the adic skew–plane.

Hecke operators are self–adjoint transformations of the Hilbert space of harmonic functions of order ρ for the adelic skew–plane into itself. An isometric transformation of the Hilbert space into itself is defined by every nonzero element ω of the discrete skew–field by taking a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the adelic skew–plane into the function $f(\xi_+\omega, \xi_-\omega)$ of $\xi = (\xi_+, \xi_-)$ in the adelic skew–plane.

Define N as the number of elements of the discrete skew–plane such that $\langle \omega, \omega \rangle = 1$.

A Hecke operator $\Delta(n)$ is defined for every positive integer n. The transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the adelic skew–plane into the function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the adelic skew–plane defined by the sum

$$Ng(\xi_+, \xi_-) = \sum f(\xi_+\omega, \xi_-\omega)$$

over the integral elements ω of the discrete skew–plane such that

$$n = \omega^- \omega.$$

The identity

$$\Delta(m)\Delta(n) = \sum k\Delta(mn/k^2)$$

holds for all positive integers m and n with summation over the common odd divisors k of m and n.

The Hecke operator $\Delta(1)$ is the orthogonal projection of the Hilbert space of harmonic functions of order ρ onto the subspace of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the adelic skew–plane which satisfy the identity

$$f(\xi_+, \xi_-) = f(\xi_+\omega, \xi_-\omega)$$

for every integral element ω of the discrete skew–plane such that $\omega^-\omega = 1$.

The kernel of $\Delta(1)$ is contained in the kernel of $\Delta(n)$, and the range of $\Delta(n)$ is contained in the range of $\Delta(1)$, for every positive integer n.

Hecke operators act as self–adjoint transformations in the range of $\Delta(1)$. The range of $\Delta(1)$ is the orthogonal sum of invariant subspaces whose elements are characterized as eigenfunctions of $\Delta(n)$ for a real eigenvalue $\tau(n)$ for every positive integer n.

Eigenvalues defining invariant subspaces satisfy the identity

$$\tau(m)\tau(n) = \sum k \ \tau(mn/k^2)$$

for all positive integers m and n with summation over the common odd divisors k of m and n. The eigenvalue $\tau(n)$ is one when n is one.

A Radon transformation of harmonic ϕ is defined for the adelic skew–plane when a nontrivial harmonic function ϕ of order ν defines an eigenfunction for Hecke operators. The transformation is defined by integration with respect to the canonical measure for the complementary space to the adelic plane in the adelic skew–plane.

The adelic plane is the set of elements $\xi = (\xi_+, \xi_-)$ of the adelic skew–plane whose component ξ_+ in the complex skew–plane belongs to the complex plane and whose component ξ_- in the adic skew–plane belongs to the adic plane. The adelic plane is isomorphic to the Cartesian product of the complex plane and the adic plane. The subspace topology of the adelic plane inherited from the adelic skew–plane is identical with the Cartesian product topology of the topology of the complex plane and the topology of the adic plane. The canonical measure for the adelic plane is the Cartesian product measure of the canonical measure for the complex plane and the canonical measure for the adic plane.

The complementary space to the adelic plane in the adelic skew–plane is the set of elements $\xi = (\xi_+, \xi_-)$ of the adelic skew–plane whose component ξ_+ in the complex skew–plane belongs to the complementary space to the complex plane in the complex skew–plane and whose component ξ_- in the adic skew–plane belongs to the complementary space to the adic plane in the adic skew–plane. The complementary space to the adelic plane in the adelic skew–plane is isomorphic to the Cartesian product of the complementary space of the complex plane in the complex skew–plane and the complementary space to the adic plane in the adic skew–plane.

The topology which the adelic plane inherits from the adelic skew–plane is identical with the Cartesian product topology of the topology of the complex plane and the topology of the adic plane. The canonical measure for the adelic plane is the Cartesian product measure of the canonical measure for the complex plane and the canonical measure for the adic plane.

The topology which the complementary space to the adelic plane in the adelic skew–plane inherits from the adelic skew–plane is identical with the Cartesian product topology of the topology of the complementary space of the complex plane in the complex skew–plane and the topology of the complementary space to the adic plane in the adic skew–plane.

The canonical measure for the complementary space to the adelic plane in the adelic skew–plane is the Cartesian product measure of the canonical measure for the complementary space to the complex plane in the complex skew–plane and the canonical measure for the complementary space to the adic plane in the adic skew–plane.

The Radon transformation of harmonic ϕ has domain and range in the Hilbert space of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the adelic skew–plane which satisfy the identity

$$\phi(\xi_+, \xi_-)f(\omega_+\xi_+, \omega_-\xi_-) = \phi(\omega_+\xi_+, \omega_-\xi_-)f(\xi_+, \xi_-)$$

for every element $\omega = (\omega_+, \omega_-)$ of the adelic skew–plane with conjugate as inverse.

The transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the adelic skew–plane into the function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the adelic skew–plane when the identity

$$g(\omega_+\xi_+, \omega_-\xi_-)/\phi(\omega_+\xi_+, \omega_-\xi_-)$$
$$= \int f(\omega_+\xi_+ + \omega_+\eta_+, \omega_-\xi_- + \omega_-\eta_-)/\phi(\omega_+\xi_+ + \omega_+\eta_+, \omega_-\xi_- + \omega_-\eta_-)d\eta$$

holds with integration with respect to the canonical measure for the complementary space to the adelic plane in the adelic skew–plane for every element $\xi = (\xi_+, \xi_-)$ of the adelic plane and every element $\omega = (\omega_+, \omega_-)$ of the adelic skew–plane with conjugate as inverse.

The integral is taken in the weak topology of the Hilbert space of square integrable functions with respect to the canonical measure for the adelic skew–plane of integrals over compact subsets of the complementary space to the adelic plane in the adelic skew–plane which contain $\omega\xi = (\omega_+\xi_+, \omega_-\xi_-)$ for every element $\omega = (\omega_+, \omega_-)$ of the adelic skew–plane with conjugate as inverse whenever they contain $\xi = (\xi_+, \xi_-)$.

The Laplace transformation of harmonic ϕ for the adelic skew–plane is a spectral analysis of the adjoint of the Radon transformation of harmonic ϕ for the adelic skew–plane in an invariant subspace. A Laplace transformation of harmonic ϕ is defined for a harmonic function ϕ of order ρ which has norm one in the Hilbert space of harmonic functions of order ρ.

The domain of the Laplace transformation of harmonic ϕ is contained in the Hilbert space of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the adelic skew–plane which are square integrable with respect to the canonical measure for the adelic skew–plane, which satisfy the identity

$$\phi(\xi_+, \xi_-)f(\omega_+\xi_+, \omega_-\xi_-) = \phi(\omega_+\xi_+, \omega_-\xi_-)f(\xi_+, \xi_-)$$

for every element $\omega = (\omega_+, \omega_-)$ of the adelic skew–plane with conjugate as inverse, and which vanish at noninvertible elements of the adelic skew–plane.

The adelic half–plane is a locally compact group which is the Cartesian product of the upper half–plane and the adic half–plane. The topology of the adelic half–plane is the Cartesian product topology of the topology of the upper half–plane and the topology of the adic half–plane. The canonical measure for the adelic half–plane is the Cartesian product measure of the canonical measure for the upper half–plane and the canonical measure for the adic half–plane.

A function

$$f(\xi_+, \xi_-) = \phi(\xi_+, \xi_-) h(\xi_+^- \xi_+, \xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the adelic skew–plane which belongs to the domain of the Laplace transformation of harmonic ϕ is parametrized by a function $h(z, \xi)$ of (z, ξ) in the adelic half–plane.

For every element ξ of the adic half–plane the function of z in the upper half–plane admits an extension to the complex plane which satisfies the identity

$$h(\omega z, \xi) = h(z, \xi)$$

for every element ω of the complex plane with conjugate as inverse. The function of ξ in the adic half–plane is analytic for every element z of the upper half–plane.

The identity

$$\int |f(\xi_+, \xi_-)|^2 d\xi = 2\pi \int \ |\xi_+|^\nu |h(\xi_+, \xi_-)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the r–adelic skew–plane and with integration on the right with respect to the canonical measure for the r–adelic half–plane.

The Laplace transform of harmonic ϕ of the function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the adelic skew–plane is a function

$$g(z, \xi) = \int \exp(\pi i |\eta_+| z) \exp(\pi i (\xi^- \eta_- + \eta_-^- \xi)) |\eta_+|^\nu h(\eta_+, \eta_-) d\eta$$

of (z, ξ) in the adelic half–plane which is analytic as a function of z in the upper half–plane for every element ξ of the adic half–plane and analytic as a function of ξ in the adic half–plane for every element z of the upper half–plane. The integral representation applies when the integral is absolutely convergent.

The identity

$$\int \int_0^\infty \int_{-\infty}^{+\infty} |g(x + iy, \xi)|^2 y^\nu dx dy d\xi = (2\pi)^{-\nu} \Gamma(1 + \nu) \int |\eta_+|^\nu |\mathrm{ext} h(\eta_+, \eta_-)|^2 d\eta$$

holds with outer integration on the left with respect to the canonical measure for the adic half–plane and with integration on the right with respect to the canonical measure for the adelic half–plane.

The adjoint of the Radon transformation of harmonic ϕ takes a function

$$f(\xi_+, \xi_-) = \phi(\xi_+, \xi_-) h(\langle \xi_+, \xi_+ \rangle, \xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the adelic skew–plane whose Laplace transform of harmonic ϕ is a function $g(z, \xi)$ of (z, ξ) in the adelic half–plane into a function

$$f'(\xi_+, \xi_-) = \phi(\xi_+, \xi_-) h'(\langle \xi_+, \xi_+ \rangle, \xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the adelic skew–plane whose Laplace transform of harmonic ϕ is a function $g'(z, \xi)$ of (z, ξ) in the adelic half–plane if, and only if, the identity

$$\int v(\xi)^- g'(z,\xi) d\xi = (i/z) \int v(\xi)^- \lambda(\xi + \xi^-)^{-1} g(z,\xi) d\xi$$

holds with integration with respect to the canonical measure for the adic half–plane for every square integrable analytic function $v(\xi)$ of ξ in the adic half–plane whose product with $\lambda(\xi + \xi^-)^{-1}$ is square integrable.

An element $\xi = (\xi_+, \xi_-)$ of the adelic skew–plane is treated as integral if its adic component ξ_- is integral. The set of integral elements of the adelic skew–plane is a ring which is a locally compact Hausdorff space in the topology inherited from the adelic skew–plane. The canonical measure for the ring is the restriction to Baire subsets of the ring of the canonical measure for the adelic skew–plane.

The ring of integral elements of the adelic skew–plane is isomorphic to the Cartesian product of the complex skew–plane and the ring of integral elements of the adic skew–plane. The topology of the ring of integral elements of the adelic skew–plane is the Cartesian product topology of the topology of the complex skew–plane and the topology of the ring of integral elements of the adic skew–plane. The canonical measure for the ring of integral elements of the adelic skew–plane is the Cartesian product measure of the canonical measure for the complex skew–plane and the canonical measure for the ring of integral elements of the adic skew–plane.

The set of integral elements of the adelic plane is a ring which is a locally compact Hausdorff space in the topology inherited from the adelic plane. The canonical measure for the ring is the restriction to Baire subsets of the ring of the canonical measure for the adelic plane.

The ring of integral elements of the adelic plane is isomorphic to the Cartesian product of the complex plane and the ring of integral elements of the adic plane. The topology for the ring of integral elements of the adelic plane is the Cartesian product topology of the topology of the complex plane and the topology of the ring of integral elements of the adic plane. The canonical measure for the ring of integral elements of the adelic plane is the Cartesian product measure of the canonical measure for the complex plane and the canonical measure for the ring of integral elements of the adic plane.

The complementary space to the ring of integral elements of the adelic plane in the ring of integral elements of the adelic skew–plane is the set of integral elements of the complementary space to the adelic plane in the adelic skew–plane. The complementary space to the ring of integral elements of the adelic plane in the ring of integral elements of the adelic skew–plane is a locally compact Hausdorff space in the subspace topology inherited from the complementary space to the adelic plane in the adelic skew–plane. The canonical measure for the complementary space to the ring of integral elements of the adelic plane in the ring of integral elements of the adelic skew–plane is the restriction to its Baire subsets of the canonical measure for the complementary space to the adelic plane in the adelic skew–plane.

The complementary space to the ring of integral elements of the adelic plane in the ring of integral elements of the adelic skew–plane is isomorphic to the Cartesian product of the complementary space to the complex plane in the complex skew–plane and the complementary space to the ring of integral elements of the adic plane in the ring of integral elements of the adic skew–plane. The topology of the complementary space to the ring of integral elements of the adelic plane in the ring of integral elements of the adelic skew–plane is the Cartesian product topology of the topology of the complementary space to the complex plane in the complex skew–plane and the topology of the complementary space to the ring of integral elements of the adic plane in the ring of integral elements of the adic skew–plane. The canonical measure for the complementary space to the ring of integral elements of the adelic plane in the ring of integral elements of the adelic skew–plane is the Cartesian product measure of the canonical measure for the complementary space to the complex plane in the complex skew–plane and the canonical measure for the complementary space to the ring of integral elements of the adic plane in the ring of integral elements of the adic skew–plane.

The Radon transformation of harmonic ϕ for the ring of integral elements of the adelic skew–plane is a transformation whose domain and range are contained in a Hilbert space which is the domain of the Laplace transformation of harmonic ϕ for the ring of integral elements of the adelic skew–plane. The elements of the domain of the Laplace transformation are the functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the adelic skew–plane which are square integrable with respect to the canonical measure for the ring and which satisfy the identity

$$\phi(\xi_+, \xi_-) f(\omega_+ \xi_+, \omega_- \xi_-) = \phi(\omega_+ \xi_+, \omega_- \xi_-) f(\xi_+, \xi_-)$$

for every element $\omega = (\omega_+, \omega_-)$ of the adelic skew–plane with conjugate as inverse.

The Radon transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring into a function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring when the identity

$$g(\omega_+ \xi_+, \omega_- \xi_-) / \phi(\omega_+ \xi_+, \omega_- \xi_-)$$
$$= \int f(\omega_+ \xi_+ + \omega_+ \eta_+, \omega_- \xi_- + \omega_- \eta_-) / \phi(\omega_+ \xi_+ + \omega_+ \eta_+, \omega_- \xi_- + \omega_- \eta_-) d\eta$$

holds for every element $\omega = (\omega_+, \omega_-)$ of the adelic skew–plane with conjugate as inverse for every element $\xi = (\xi_+, \xi_-)$ of the ring of integral elements of the adelic plane with integration with respect to the canonical measure for the complementary space to the ring of integral elements of the adelic plane in the ring of integral elements of the adelic skew–plane. The integral is interpreted as a limit in the weak topology of the domain of the Laplace transformation of integrals over compact subsets of the complementary space which contain $\omega \xi = (\omega_+ \xi_+, \omega_- \xi_-)$ for every element $\omega = (\omega_+, \omega_-)$ of the adelic skew–plane with conjugate as inverse whenever they contain $\xi = (\xi_+, \xi_-)$.

An element $\xi = (\xi_+, \xi_-)$ of the adelic half–plane is treated as integral if its adic component ξ_- is an integral element of the adic half–plane. The set of integral elements of the adelic half–plane is an additive subgroup which is a locally compact Hausdorff space in the subspace topology inherited from the adelic half–plane. The canonical measure for the subgroup is the restriction to Baire subsets of the subgroup of the canonical measure for the adelic half–plane.

The group of integral elements of the adelic half–plane is isomorphic to the Cartesian product of the upper half–plane and the group of integral elements of the adic half–plane. The topology of the group of integral elements of the adelic half–plane is the Cartesian product topology of the topology of the upper half–plane and the topology of the group of integral elements of the adic half–plane. The canonical measure for the group of integral elements of the adelic half–plane is the Cartesian product measure of the canonical measure for the upper half–plane and the canonical measure for the group of integral elements of the adic half–plane.

A function

$$f(\xi_+, \xi_-) = \phi(\xi_+, \xi_-) h(\xi_+^- \xi_+, \xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the adelic skew–plane which belongs to the domain of the Laplace transformation of harmonic ϕ for the ring is parametrized by a function $h(z, \xi)$ of (z, ξ) in the group of integral elements of the adelic half–plane whose product with $|z|^{\frac{1}{2}\nu}$ is square integrable with respect to the canonical measure for the group. For every integral element ξ of the adic half–plane the function $h(z, \xi)$ of z in the upper half–plane admits an extension to the complex field which satisfies the identity

$$h(\omega z, \xi) = h(z, \xi)$$

for every element ω of the complex field with conjugate as inverse. The function $h(z, \xi)$ of ξ in the ring of integral elements of the adic half–plane is analytic for every element z of the upper half–plane.

The identity

$$\int |f(\xi_+, \xi_-)|^2 d\xi = 2\pi \int |\xi_+|^\nu |h(\xi_+, \xi_-)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the ring of integral elements of the adelic skew–plane and with integration on the right with respect to the canonical measure for the group of integral elements of the adelic half–plane.

Elements $\xi = (\xi_+, \xi_-)$ and $\eta = (\eta_+, \eta_-)$ of the adelic half–plane are treated as congruent modulo 1 if the components

$$\xi_+ = \eta_+$$

in the upper half–plane are equal and the components ξ_- and η_- in the adic half–plane are congruent modulo 1. The quotient space modulo the equivalence relation is the adelic half–plane modulo 1.

The Laplace transform of harmonic ϕ of a function

$$f(\xi_+, \xi_-) = \phi(\xi_+, \xi_-) h(\xi_+^-, \xi_+, \xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the adelic skew–plane is a function

$$g(z, \xi) = \int \exp(\pi i |\eta_+| z) \exp(\pi i (\xi^- \eta_- + \eta_-^- \xi)) |\eta_+|^\nu h(\eta_+, \eta_-) d\eta$$

of (z, ξ) in the adelic half–plane modulo 1 which is analytic as a function of z in the upper half–plane for every element ξ of the adic half–plane modulo 1 and which is analytic as a function of ξ in the adic half–plane modulo 1 for every element z of the upper half–plane. Integration is with respect to the canonical measure for the ring of integral elements of the adic half–plane.

The identity

$$\int \int_0^\infty \int_{-\infty}^{+\infty} |g(x + iy, \xi)|^2 y^\nu dx dy d\xi = (2\pi)^{-\nu} \Gamma(1 + \nu) \int |\eta_+|^\nu |h(\eta_+, \eta_-)|^2 d\eta$$

holds with outer integration on the left with respect to the canonical measure for the adic half–plane modulo 1 and with integration on the right with respect to the canonical measure for the group of integral elements of the adelic half–plane.

The adjoint of the Laplace transformation of harmonic ϕ takes a function

$$f(\xi_+, \xi_-) = \phi(\xi_+, \xi_-) h(\xi_+^- \xi_+ \xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the adelic skew–plane whose Laplace transform of harmonic ϕ is a function $g(z, \xi)$ of (z, ξ) in the adelic half–plane modulo 1 into a function

$$f'(\xi_+, \xi_-) = \phi(\xi_+, \xi_-) h'(\langle \xi_+, \xi_+ \rangle, \xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the adelic skew–plane whose Laplace transform of harmonic ϕ is a function $g'(z, \xi)$ in the adelic half–plane modulo 1 if, and only if, the identity

$$\int v(\xi)^- g'(z, \xi) d\xi = (i/z) \int v(\xi)^- \lambda(\xi + \xi^-)^{-1} g(z, \xi) d\xi$$

holds with integration with respect to the canonical measure for the adic half–plane modulo 1 for every square integrable analytic function $v(\xi)$ of ξ in the adic half–plane modulo 1 whose product with $\lambda_(\xi + \xi^-)^{-1}$ is square integrable.

The projection of the ring of integral elements of the adelic skew–plane onto the ring of integral elements of the adelic skew–plane modulo r is a homomorphism of conjugated ring structure which is a continuous open mapping and which takes the canonical measure into the canonical measure.

A function defined on the ring of integral elements of the adelic skew–plane modulo r is treated as a function defined on the ring of integral elements of the adelic skew–plane which has equal values at elements which are congruent modulo r.

The ring of integral elements of the adelic plane modulo r is the quotient ring of the ring of integral elements of the adelic plane defined by congruence modulo r. The ring of integral elements of the adelic plane modulo r is isomorphic to the image of the ring of integral elements of the adelic plane in the ring of integral elements of the adelic skew–plane modulo r.

The ring of integral elements of the adelic plane modulo r is isomorphic to the Cartesian product of the complex plane and the ring of integral elements of the adic plane modulo r. The topology of the ring of integral elements of the r–adelic plane modulo r is the Cartesian product topology of the topology of the complex plane and the topology of the ring of integral elements of the adic plane modulo r. The canonical measure for the ring of integral elements of the adelic plane modulo r is the Cartesian product measure of the canonical measure for the complex plane and the canonical measure for the ring of integral elements of the adic plane modulo r.

The projection of the ring of integral elements of the adelic plane onto the ring of integral elements of the adelic plane modulo r is a homomorphism of conjugated ring structure which is a continuous open mapping and which takes the canonical measure into the canonical measure.

A function defined on the ring of integral elements of the adelic plane modulo r is treated as a function defined on the ring of integral elements of the adelic plane which has equal values at elements which are congruent modulo r.

The complementary space to the ring of integral elements of the adelic plane modulo r in the ring of integral elements of the adelic skew–plane modulo r is defined as the image in the ring of integral elements of the adelic skew–plane modulo r of the complementary space to the ring of integral elements of the adelic plane in the ring of integral elements of the adelic skew–plane. The complementary space to the ring of integral elements of the adelic plane modulo r in the ring of integral elements of the adelic skew–plane modulo r is isomorphic to the quotient space modulo r of the complementary space to the ring of integral elements of the adelic plane in the ring of integral elements of the adelic skew–plane.

The complementary space to the ring of integral elements of the adelic plane modulo r in the ring of integral elements of the adelic skew–plane modulo r is isomorphic to the Cartesian product of the complementary space to the complex plane in the complex skew–plane and the complementary space to the ring of integral elements of the adic plane modulo r in the ring of integral elements of the adic skew–plane modulo r.

The topology of the complementary space to the ring of integral elements of the adelic plane modulo r in the ring of integral elements of the adelic skew–plane modulo r is the Cartesian product topology of the topology of the complementary space to the complex plane in the complex skew–plane and the topology for the complementary space to the ring of integral elements of the adic plane modulo r in the ring of integral elements of the adic skew–plane modulo r.

The canonical measure for the complementary space to the ring of integral elements of the adelic plane modulo r in the ring of integral elements of the adelic skew–plane modulo r is the Cartesian product measure of the canonical measure for the complementary space to the complex plane in the complex skew–plane and the canonical measure for the complementary space to the ring of integral elements of the adic plane modulo r in the ring of integral elements of the adic skew–plane modulo r.

The projection of the complementary space to the ring of integral elements of the adelic plane in the ring of integral elements of the adelic skew–plane into the complementary space to the ring of integral elements of the adelic plane modulo r in the ring of integral elements of the adelic skew–plane modulo r is a homomorphism of additive structure which is a continuous open mapping and which maps the canonical measure into the canonical measure.

A function defined on the complementary space to the ring of integral elements of the adelic plane modulo r in the ring of integral elements of the adelic skew–plane modulo r is treated as a function defined in the complementary space to the ring of integral elements of the adelic plane in the ring of integral elements of the adelic skew–plane which has equal values at elements which are congruent modulo r.

The Radon transformation of harmonic ϕ for the ring of integral elements of the adelic skew–plane modulo r is a transformation with domain and range in the domain of the Laplace transformation of harmonic ϕ for the ring of integral elements of the adelic skew–plane modulo r.

The domain of the Laplace transformation of harmonic ϕ for the ring of integral elements of the adelic skew–plane modulo r is the Hilbert space of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the adelic skew–plane modulo r which are square integrable with respect to the canonical measure for the ring and which satisfy the identity

$$\phi(\xi_+, \xi_-)f(\omega_+\xi_+, \omega_-\xi_-) = \phi(\omega_+\xi_+, \omega_-\xi_-)f(\xi_+, \xi_-)$$

for every element $\omega = (\omega_+, \omega_-)$ of the adelic skew–plane with conjugate as inverse.

The Radon transformation of harmonic ϕ takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the adelic skew–plane modulo r into a function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the adelic skew–plane modulo r when the identity

$$g(\omega_+\xi_+, \omega_-\xi_-)/\phi(\omega_+\xi_+, \omega_-\xi_-)$$
$$= \int f(\omega_+\xi_+ + \omega_+\eta_+, \omega_-\xi_- + \omega_-\eta_-)/\phi(\omega_+\xi_+ + \omega_+\eta_+, \omega_-\xi_- + \omega_-\eta_-)d\eta$$

holds for every element $\omega = (\omega_+, \omega_-)$ of the adelic skew–plane with conjugate as inverse when $\xi = (\xi_+, \xi_-)$ is an element of the ring of integral elements of the adelic plane modulo r and when integration is with respect to the canonical measure for the complementary space to the ring of integral elements of the adelic plane modulo r in the ring of integral elements of the adelic skew–plane modulo r.

The integral is interpreted as a limit of integrals over compact subsets of the complementary space to the ring of integral elements of the adelic plane modulo r in the ring of integral elements of the adelic skew–plane modulo r. which contain $\omega\xi = (\omega_+\xi_+, \omega_-\xi_-)$ for every integral element $\omega = (\omega_+, \xi_-)$ of the adelic skew–plane with integral inverse whenever they contain $\xi = (\xi_+, \xi_-)$. The limit is taken in the metric topology of the Hilbert space of square integrable functions with respect to the canonical measure for the ring of integral elements of the adelic skew–plane modulo r.

Integral elements $\xi = (\xi_+, \xi_-)$ and $\eta = (\eta_+, \eta_-)$ of the adelic half–plane are treated as congruent modulo r if their components

$$\xi_+ = \eta_+$$

in the upper half–plane are equal and if their components ξ_- and η_- in the adic half–plane differ by an element $\eta_- - \xi_-$ of the adic half–plane which is the product of r and an integral element of the adic half–plane.

The quotient group of the group of integral elements of the adelic half–plane is the group of integral elements of the adelic half–plane modulo r. The group is isomorphic to the Cartesian product of the upper half–plane and the group of integral elements of the adic half–plane modulo r.

The topology of the group of integral elements of the adelic half–plane modulo r is the Cartesian product topology of the topology of the upper plane and the topology of the group of integral elements of the adic half–plane modulo r. The canonical measure for the group of integral elements of the adelic half–plane modulo r is the Cartesian product measure of the canonical measure for the upper half–plane and the canonical measure for the ring of integral elements of the adic half–plane modulo r.

The projection of the group of integral elements of the adelic half–plane onto the group of integral elements of the adelic half–plane modulo r is a homomorphism of additive structure which is a continuous open mapping and which maps the canonical measure into the canonical measure.

A function defined on the group of integral elements of the adelic half–plane modulo r is treated as a function defined on the group of integral elements of the adelic half–plane which has equal values at elements which are congruent modulo r.

A function

$$f(\xi_+, \xi_-) = \phi(\xi_+, \xi_-) h(\xi_+^- \xi_+, \xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the adelic skew–plane modulo r which belongs to the domain of the Laplace transformation of harmonic ϕ for the ring is parametrized by a function $h(z, \xi)$ of (z, ξ) in the group of integral elements of the adelic half–plane modulo r whose product with $|z|^{\frac{1}{2}\nu}$ is square integrable with respect to the canonical measure for the group.

For every element ξ of the group of integral elements of the adic half–plane modulo r the function $h(z, \xi)$ of z in the upper half–plane admits an extension to the complex field which satisfies the identity

$$h(\omega z, \xi) = h(z, \xi)$$

for every element ω of the complex field with conjugate as inverse. For every element z of the upper half–plane the function $h(z, \xi)$ of ξ in the ring of integral elements of the adic half–plane modulo r is analytic.

The identity

$$\int |f(\xi_+, \xi_-)|^2 d\xi = 2\pi \int |\xi_+|^\nu |\mathrm{ext} h(\xi_+, \xi_-)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the ring of integral elements of the adelic skew–plane modulo r and with integration on the right with respect to the canonical measure for the group of integral elements of the adelic half–plane modulo r.

The Laplace transform of harmonic ϕ of the function

$$f(\xi_+, \xi_-) = \phi(\xi_+, \xi_-) h(\xi_+^- \xi_+, \xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the adelic skew–plane modulo r is defined as the function

$$g(z, \xi) = \int \exp(\pi i |\eta_+| z) \exp(\pi i (\xi^- \eta_- + \eta_-^- \xi)) |\eta_+|^\nu h(\eta_+, \eta_-) d\eta$$

of (z, ξ) in the adelic half–plane modulo such that $r\xi$ vanishes defined by integration with respect to the canonical measure for the group of half–integral elements of the adelic half–plane modulo r.

The identity

$$\int \int_0^\infty \int_{-\infty}^{+\infty} |\mathrm{ext} g(x + iy, \xi)|^2 y^\nu dx dy d\xi = (2\pi)^{-\nu} \Gamma(1 + \nu) \int \int_0^\infty r^\nu |\mathrm{ext} h(r, \eta)|^2 dr d\eta$$

holds with outer integration on the left with respect to the canonical measure for the adic half–plane modulo 1 over the set of elements η such that $r\eta$ vanishes and with outer integration in the right with respect to the canonical measure for the ring of integral elements of the adic half–plane modulo r.

The adjoint of the Radon transformation of harmonic ϕ takes a function

$$f(\xi_+, \xi_-) = \phi(\xi_+, \xi_-) h(\xi_+^- \xi_+, \xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the adelic skew–plane modulo r whose Laplace transform of harmonic ϕ is a function $g(z, \xi)$ of (z, ξ) in the group of elements (z, ξ) of the adelic half–plane modulo 1 such that $r\xi$ vanishes into a function

$$f'(\xi_+, \xi_-) = \phi(\xi_+, \xi_-) h'(\xi_+^- \xi_+, \xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the adelic skew–plane modulo r whose Laplace transform of harmonic ϕ is a function $g'(z, \xi)$ of (z, ξ) in the group of elements (z, ξ) of the adelic half–plane modulo 1 such that $r\xi$ vanishes if, and only if, the identity

$$\int v(\xi)^- g'(z, \xi) d\xi = (i/z) \int v(\xi)^- \lambda(\xi + \xi^-)^{-1} g(z, \xi) d\xi$$

holds with integration with respect to the canonical measure for the r–annihilated subgroup of the adic half–plane modulo 1 for every analytic function $v(\xi)$ of ξ in the r–annihilated subgroup of the adic half–plane modulo 1 which vanishes when $\lambda(\xi + \xi^-)$ vanishes.

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