The Riemann Hypothesis

Project summary

The spectral theory of the vibrating string is applied to a proof of the Riemann hypothesis for the Hecke zeta functions in the theory of modular forms. A proof of the Riemann hypothesis for the Euler zeta function is an application.

The Riemann hypothesis is a conjecture about the zeros of a particular entire function which presumes a significance of zeros known for polynomials. A clarification of the conjecture was made by Hermite by introducing a class of entire functions essentially determined by their zeros. The upper half–plane is chosen when the spectral theory of the vibrating string is applied.

An entire function $E(z)$ of z is said to be of Hermite class if it has no zeros in the upper half–plane, if it satisfies the inequality

$$|E(x - iy)| \leq |E(x + iy)|$$

for all real x when y is positive, and if the modulus $|E(x + iy)|$ of $E(x + iy)$ is a nondecreasing function of positive y for every real x. The Riemann hypothesis is strengthened as the conjecture that a given entire function belongs to the Hermite class.

A polynomial belongs to the Hermite class if it has no zeros in the upper half–plane. An entire function of Hermite class is a limit uniformly on compact subsets of the complex plane of polynomials having no zeros in the upper half–plane.

The interest of polynomials having no zeros in the upper half–plane was discovered in applications to celestial mechanics by Legendre.

A clarification of the conjectured estimate of the number of primes less than a given positive number, due to Legendre, is the declared purpose of the Riemann hypothesis.

The significance of zeros of polynomials was demonstrated by Gauss in a construction of invariant subspaces for linear transformations of a complex vector space into itself.

A polynomial whose zeros are real is an example of a polynomial which has no zeros in the upper half–plane. If a polynomial

$$E(z) = A(z) - iB(z)$$

has no zeros in the upper half–plane, then it decomposes into polynomials $A(z)$ and $B(z)$ which take real values on the real axis. The zeros of these polynomials are real and are intersticed.

Gauss was only the first of those who followed Legendre in the construction of examples of polynomials which have no zeros in the upper half–plane. The Riemann hypothesis is the expectation that the properties of such polynomials extend to more general entire functions.

The Riemann hypothesis is the stimulus for a coherent treatment by Stieltjes of polynomials which have no zeros in the upper half–plane.

The contribution of Stieltjes is a structure theory for nonnegative linear functionals on polynomials. A linear functional on polynomials is said to be nonnegative if it has nonnegative values on polynomials whose values on the real axis are nonnegative. A nonnegative linear functional on polynomials is shown to be represented as a Stieltjes integral with respect to a nondecreasing function of a real variable. When the nonnegative linear functional is applied to polynomials of degree less than r, then the nonnegative linear functional is represented as an integral with respect to a nondecreasing step–function with at most r jumps. The computation of such step functions applies the properties of polynomials of degree r which have no zeros in the upper half–plane.

Stieltjes made no generalization to entire functions of Hermite class since he died from tuberculosis. An extensive generalization of the representation of nonnegative linear functionals on polynomials is made by Hilbert.

Hilbert introduces the study of invariant subspaces for contractive transformations of a Hilbert space into itself. Polynomials are applied by Gauss in the construction of invariant subspaces when the space has finite dimension. Hilbert shows that an isometric transformation admits nontrivial invariant subspaces if it is nontrivial. Subspaces are constructed which are invariant subspaces for all continuous transformations which commute with the given isometric transformation. When the transformation has an isometric inverse, it has an integral representation as a Stieltjes integral of projections into invariant subspaces.

Hilbert and his students were frustrated in their attempts to prove the Riemann hypothesis because they overlooked an application of complex analysis which was found by Hardy.

The Hardy space for the unit disk is the Hilbert space of analytic functions which are represented in the unit disk by square summable power series. The Hardy space for the upper half–plane is fundamental to an understanding of the Riemann hypothesis.

The application of square summable power series to the construction of invariant subspace is due to Beurling. Invariant subspaces for some contractive transformations are constructed from the factorization theory of functions which are analytic and bounded by one in the unit disk. The generalization of the construction to all contractive transformations is preparation for research on the Riemann hypothesis.

In 1965 an announcement was made with inadequate proof in joint work with James Rovnyak that a nontrivial contractive transformation of a Hilbert space into itself admits a nontrivial invariant subspace. The completed proof appears in lecture notes on complex analysis [5] which serve as an introduction to the methods applied to the Riemann hypothesis. The subspace constructed is an invariant subspace for every continuous transformation which commutes with the given contractive transformation.

Weighted Hardy spaces of functions analytic in the upper half–plane are applied to the Riemann hypothesis. An analytic weight function is a function $W(z)$ of z which is analytic and without zeros in the upper half–plane. The weighted Hardy space $\mathcal{F}(W)$ is the set of functions $F(z)$ of z which are analytic in the upper half–

plane such that the least upper bound

$$\|F\|^2_{\mathcal{F}(W)} = \sup \int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 dx$$

taken over all positive y is finite. The weighted Hardy space is a Hilbert space of functions analytic in the upper half–plane which is characterized by two properties: If w is in the upper half–plane, multiplication by

$$(z-w)/(z-w^-)$$

is an isometric transformation of the space onto the subspace of functions which vanish at w. A continuous linear functional on the space is defined when w is in the upper half–plane by taking a function $F(z)$ of z into its value $F(w)$ at w.

The unweighted Hardy space for the upper half–plane is obtained when $W(z)$ is identically one. If $W(z)$ is an analytic weight function, multiplication by $W(z)$ is an isometric transformation of the unweighted Hardy space onto the weighted Hardy space $\mathcal{F}(W)$. The function

$$W(z)W(w)^-/[2\pi i(w^- - z)]$$

of z belongs to the weighted Hardy space when w is in the upper half–plane and acts as reproducing kernel function for function values at w.

The Stieltjes representation of positive linear functionals on polynomials is clarified by the use of weighted Hardy spaces. The conjugate

$$F^*(z) = F(z^-)^-$$

of an entire function $F(z)$ of z is an entire function whose values on the real axis are complex conjugates of the values of the given entire function. The conjugate of a polynomial is a polynomial of the same degree. A polynomial has nonnegative values on the real axis if, and only if, it is a product

$$F^*(z)F(z)$$

of a polynomial $F(z)$ and its conjugate. If a nonnegative linear functional on polynomials annihilates the product for no nontrivial polynomial $F(z)$ of degree less than r, then the polynomials of degree less than r become a Hilbert space with scalar product $\langle F, G\rangle$ for polynomial $F(z)$ and $G(z)$ defined by the action of the nonnegative linear functional on the polynomial

$$G^*(z)F(z).$$

In substance the Stieltjes theorem states that the Hilbert space is contained isometrically in the weighted Hardy space $\mathcal{F}(W)$ for some polynomial $W(z)$ of degree r which has no zeros in the upper half–plane.

The Stieltjes spaces of polynomials are generalized as Hilbert spaces of entire functions. The spaces are characterized by these properties:

(H1) Whenever an entire function $F(z)$ of z belongs to the space and has a nonreal zero w, the entire function

$$F(z)(z - w^-)/(z - w)$$

of z belongs to the space and has the same norm as $F(z)$.

(H2) A continuous linear functional is defined on the space for every nonreal number w by taking an entire function $F(z)$ of z into its value $F(w)$ at w.

(H3) The entire function $F^*(z)$ of z belongs to the space whenever an entire function $F(z)$ of z belongs to the space, and if always has the same norm as $F(z)$.

A Hilbert space $\mathcal{H}(E)$ of entire functions which satisfies the axioms is defined by an entire function $E(z)$ of z which satisfies the inequality

$$|E(x - iy)| < |E(x + iy)|$$

for all real x when y is positive. The condition implies that the function has no zeros in the upper half–plane. The space $\mathcal{H}(E)$ is contained isometrically in the weighted Hardy space $\mathcal{F}(W)$ with analytic weight function

$$W(z) = E(z).$$

The elements of the space $\mathcal{H}(E)$ are the entire functions $F(z)$ of z which belong to the weighted Hardy space and whose conjugate $F^*(z)$ belongs to the space.

The entire function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

of z belongs to the space $\mathcal{H}(E)$ for every complex number w and acts as reproducing kernel function for function values at w. A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains a nonzero element is isometrically equal to a space $\mathcal{H}(E)$ for some entire function $E(z)$ of z which satisfies the inequality

$$|E(x - iy)| < |E(x + iy)|$$

for all real x when y is positive. The entire function $E(z)$ can be chosen to have a given zero in the lower half–plane. The function is then unique within a constant factor of absolute value one [1].

The Stieltjes spaces of polynomials appear in the spectral analysis of a vibrating string which is tied at a finite number of points on the positive half–line. The treatment of an arbitrary string requires the Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3).

Every Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) appears in the spectral theory of some string. The string is a canonical model of a one–dimensional dynamical system such as appears in the scattering of particles in a central force field. The string measures distance from the origin. A Hilbert space of entire functions is created when constraining forces are removed outside of some spherical surface. The spectral analysis of the string describes the scattering of particles by the reflecting surface. The interior of the sphere is a black box capable of arbitrary scattering. The mechanical system described by the strong is uniquely determined inside the spherical surface by the Hilbert space of entire functions.

The classical motivation for the Riemann hypothesis lies in properties of the Fourier transformation which were observed in 1880 by N. Sonine. Sonine observes that a nontrivial function which vanishes in a neighborhood of the origin can have a Fourier transform which vanishes in a neighborhood of the origin. This property applies in Fourier analysis on a line but is more relevant in Fourier analysis on a plane. A fundamental problem is to determine all square integrable functions in the plane which vanish in a given circular neighborhood of the origin and whose Fourier transform vanishes in the same neighborhood. The set of all such functions is invariant under the transformation which takes a function $f(z)$ of z in the plane into the function $f(\omega z)$ of z in the plane for every element ω of the plane of absolute value one. The set of all such functions is a Hilbert space which is the orthogonal sum of invariant subspaces. An invariant subspace is defined for every integer n as the set of functions which satisfy the identity

$$f(\omega z) = \omega^n f(z)$$

for every complex number ω of absolute value one. Attention is drawn to spaces parametrized by nonnegative integers n since other spaces are obtained by conjugation. A closer relationship to analytic function theory results when n is nonnegative.

The examples given by Sonine apply for every nonnegative integer n. For every positive number a nontrivial functions exist which vanish in the disk $|z| < a$ and whose Fourier transform vanishes in the disk. A determination of all such functions is made by the construction of a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). The space is isometrically equal to a space $\mathcal{H}(E(a))$ for an entire function $E(a, z)$ which is computed as a confluent hypergeometric function when n is zero [2].

A resemblance to the entire function treated by Riemann exists since the entire functions

$$E^*(a, z)$$

and

$$E(a, z - i)$$

of z are linearly dependent. The expectation created by the Riemann hypothesis is that the zeros of $E(a, z)$ lie on a horizontal line

$$z - z^- = -i$$

at distance one–half below the real axis. Motivation for the Riemann hypothesis results from the computation of zeros showing that they lie on the expected line.

The Sonine spaces of entire functions are constructed from the analytic weight function

$$W(z) = \Gamma(\tfrac{1}{2} - iz)$$

which is defined from the Euler gamma function. Multiplication by a^{-iz} is an isometric transformation of the space $\mathcal{H}(E(a))$ into the weighted Hardy space $\mathcal{F}(W)$. The space $\mathcal{H}(E(a))$ contains every entire function $F(z)$ of z such that the functions

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

of z belong to the space $\mathcal{F}(W)$.

The hypergeometric series is a generalization of the geometric series which is motivated by the binomial expansion and which was discovered by Euler subsequent to his discovery in 1730 of the gamma function. These special functions are an outgrowth of the infinitessimal calculus when it is treated as a limit of the calculus of finite differences, as it was by Newton. The gamma function is a limiting case of the Newton interpolation polynomials.

In the notation of Weierstrass the gamma function is a function $\Gamma(s)$ of a complex variable s which is analytic in the complex plane with the exception of singularities at the nonpositive integers and which satisfies the recurrence relation

$$s\Gamma(s) = \Gamma(s+1).$$

Since

$$\Gamma(1) = 1.$$

$\Gamma(1+s)$ is a generalization of $s!$

The properties of the gamma function as they appear in the Sonine spaces of entire functions identify a special class of vibrating strings.

An Euler weight function is an analytic weight function $W(z)$ for which a maximal dissipative transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ for every h in the interval $[0,1]$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space. An equivalent condition is that the function

$$W(z - \tfrac{1}{2}ih)/W(z + \tfrac{1}{2}ih)$$

of z admits an extension which is analytic and has nonnegative real part in the upper half–plane.

An example of an Euler weight function is

$$W(z) = \Gamma(\tfrac{1}{2} - iz).$$

Associated with Euler weight functions are vibrating strings whose Hilbert spaces of entire functions resemble the Sonine spaces of entire functions.

An Euler space of entire functions is a nontrivial Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and for which a maximal dissipative transformation is defined in the space for every h in the interval $[0,1]$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space.

The defining function $E(z)$ of an Euler space of entire functions admits no pair of distinct zeros $\lambda - ih$ and λ^- with h in the interval $[0, 1]$. This observation indicates a proof of the Riemann hypothesis by the construction of an Euler space of entire functions [5].

The entire function treated by Riemann is a product of two factors, of which one is constructed from the gamma function and the other from the classical zeta function. The classical zeta function was discovered by Euler in 1737 as a product which resembles the product for the gamma function. Euler also discovered the functional identity for the product of gamma functions and zeta functions as an application of the theory of hypergeometric series. A proof of the functional identity in Fourier analysis was later found as an application of the Poisson summation formula. Fourier analysis confirms the classical zeta function as an analogue of the gamma function.

The Fourier analysis of the Riemann hypothesis is made on fields or their products, with the exception that fields can be advantageously replaced by skew–fields. The Fourier transformation is an isometric transformation of the Hilbert space of square integrable functions with respect to the invariant measure onto itself.

A fundamental problem is the determination of all square integrable functions which vanish in a given neighborhood of the origin and whose Fourier transform vanishes in the same neighborhood. The neighborhoods must be invariant under multiplication by elements ω of the product ring for which multiplication by ω preserves the invariant measure. When the group of such elements is commutative, the Hilbert space of square integrable functions decomposes into subspaces of functions $f(\xi)$ of ξ which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for some character χ of the group. The group is noncommutative when skew–fields are used. Hecke operators then decompose the representation into irreducible representations. Zeta functions appear in the decomposition in all cases.

A solution of the problem is given in a general context by the construction of a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3).

In the creation of Fourier analysis Fourier gave convincing applications to the flow of heat. The treatment of heat flow is an original contribution to dynamics since energy is lost whereas motion is otherwise conceived as preserving energy.

In the language of invariant subspaces Fourier introduces contractive transformations in a context where isometric transformations were seen as relevant. In the language of dynamics Fourier applies dissipative transformations in a context where skew–adjoint transformations were seen as relevant.

A dissipative transformation is a linear transformation T with domain and range in a Hilbert space such that the real part of the scalar product

$$\langle Tf, f\rangle$$

is nonnegative for every element f of the domain of the transformation. A skew–adjoint transformations is a dissipative transformation such that the real part of the scalar product always vanishes. Since the definition admits transformations with small domain, it needs to be supplemented by the requirement that the dissipative transformation is maximal: The transformation cannot be properly extended as a dissipative transformation.

The infinitessimal generator for the flow of heat is a maximal dissipative transformation. Since the infinitessimal generator is a differential operator, heat flow seems to be restricted to Fourier analysis on commutative groups which admit a differentiable structure. This is not the case. Other groups appear.

The inverse of the infinitessimal generator for the flow of heat is an integral transformation which is maximal dissipative. When the flow of heat is treated on a plane rather than on a line, the integral transformation has properties which were observed by Radon.

The Radon transformation permits the Fourier transformation for the plane to be treated formally as a composition with the Fourier transformation for a line. The transformation averages a function of two variables with respect to one variable so as to produce a function of the other variable. Rotations about the origin recover functions of two variables. The Radon transformation has a meaning in Fourier analysis on any group which is the Cartesian product of two commutative groups.

An example of a context in which the flow of heat is meaningful is given by a quadratic extension of the p–adic numbers. In applications to the Riemann hypothesis a choice needs to be made since the field of p–adic numbers has two quadratic extensions. An unramified extension is obtained by adjoining the square root of a unit. A ramified extension is obtained by adjoining a square root of the prime p. Ramified extensions are required for application to the Euler zeta function.

The Laplace transformation was introduced by Fourier for a spectral analysis of the flow of heat. An analogous treatment of heat flow applies in any context in which the Radon transformation is meaningful. A Laplace transformation applies on any quadratic extension of the p–adic numbers. And it applies in product spaces constructed from the complex plane and the p–adic plane for every prime p. The image of the infinitessimal generator for the flow of heat under the Laplace transformation is a multiplication operator in the space of square integrable functions with respect to a nonnegative measure. The infinitessimal generator for the flow of heat is a subnormal operator which is diagonalized by the Laplace transformation. The maximal dissipative property of the infinitessimal generator implies that multiplication is by a function whose values have nonnegative real part.

The flow of heat appears in a context of Fourier analysis which has the structure of a vibrating string. The Hilbert spaces of entire functions associated with the string are Euler spaces of entire functions. The spaces are constructed from an Euler weight function.

A generalization of the Riemann hypothesis is thereby formulated in the spectral analysis of the vibrating string. Euler weight functions are the defining functions of Euler spaces of entire functions whose defining functions admit the pattern of zeros expected for the Euler zeta function by the Riemann hypothesis.

The results are seen as a generalization of the Riemann hypothesis in their application to the generalization of zeta functions introduced by Hecke in the theory of modular forms. The coefficients of these zeta functions are eigenvalues of self–adjoint operators discovered by Hecke. A computation of coefficients is possible in the simplest cases.

A character χ modulo ρ is defined for a positive integer ρ as a function $\chi(n)$ of integers n which is periodic of period ρ and which vanishes at integers not relatively prime to ρ, which satisfies the identity

$$\chi(mn) = \chi(m)\chi(n)$$

for all integers m and n, and which does not vanish identically. A character χ modulo ρ is said to be primitive modulo ρ if it does not agree on integers relatively prime to ρ with a character modulo a proper divisor of ρ.

The Dirichlet function

$$\sum \chi(n)n^{-s} = \prod (1 - \chi(p)p^{-s})^{-1}$$

of s is a generalization of the Euler zeta function which is defined by summation over the positive integers n and which is equal to a product over the primes which are not divisors of ρ. The Dirichlet function is the Euler zeta function when ρ is one.

A Dirichlet function admits an analytic extension to the complex plane when ρ is not one. A Hecke function

$$\left(\sum \rho_n n^{-s}\right)\left(\sum \rho_n n^{-s-1}\right)$$

is then obtained from the Dirichlet function. The Dirichlet function has no zeros in the half–plane $\mathcal{R}s > \frac{1}{2}$ since the Hecke function has no zeros in the half–plane.

These results apply in modified form when ρ is one. The Euler zeta function has no zeros in the half–plane as conjectured by Riemann.

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