PROJECT SUMMARY
THE RIEMANN HYPOTHESIS

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An application of applied mathematics is proposed for the solution of a problem in pure mathematics. The Riemann hypothesis has acquired a reputation as the most important unsolved problem in mathematics. A proof of the conjecture has not only an intrinsic value, which can only be properly estimated by specialists, but also a value as a means of communication between the mathematical community and a general public. Since this relationship is essential to funding, serious proposals for a solution of the problem are deserving of serious attention. The solid foundation in applied mathematics for the present proposal removes it from the category of the many unsound proposals for a solution of the problem. There are fifty years of research behind the present work.

The analysis of spectra is so pervasive in association with the Riemann hypothesis as to defy identification of its source. Spectral analysis of periodic motion in astronomy was applied by Legendre and later by Gauss to the distribution of prime numbers treated as random motion. These contributions were made at the beginning of the nineteenth century or before. The first systematic treatment was made toward the end of the century by Hermite and Stieltjes, who treat zeros of polynomials and their generalization after the conjecture of Riemann in the middle of the century. Spectral analysis is essentially the study of invariant subspaces of transformations.

At the end of the nineteenth century Stieltjes axiomatized integration as it applies to polynomials treated as functions of a real variable. A polynomial is seen to be nonnegative if it has nonnegative values on the real line. An integral is characterized as a linear functional on polynomials which has nonnegative values on nonnegative polynomials. An integral is shown to be computed as a Stieltjes integral with respect to some nondecreasing function of a real variable.

At the beginning of the twentieth century Hilbert adapted Stieltjes integration to an abstract context. A Hilbert space is a complex vector space which is complete in the uniform topology defined by a nonnegative quadratic form. A linear transformation with domain and range in a Hilbert space is defined as nonnegative if an element of the domain always has a nonnegative scalar product with the corresponding element of the range. Such a transformation is continuous if it is everywhere defined. A sufficient condition for spectral analysis is that the transformation be maximal: No proper linear extension exists which maintains positivity. The existence of invariant subspaces is shown for maximal nonnegative transformations. The transformation is a Stieltjes integral of invariant subspaces.

An elementary but essential step in adapting the Hilbert spectral theory to the Riemann hypothesis is to return to the context in which Stieltjes was working before by his death from tuberculosis. The axiomatization in 1959 of the required Hilbert spaces of entire functions marks the beginning of the present project on the Riemann hypothesis.

Hilbert spaces are introduced whose elements are entire functions and which have these properties:

(H1) Whenever an entire function $F(z)$ of z belongs to the space and has a nonreal zero w the entire function

$$F(z)(z - w^-)/(z - w)$$

of z belongs to the space and has the same norm as $F(z)$.

(H2) A continuous linear functional on the space is defined for every nonreal number w by taking an entire function $F(z)$ of z into its value $F(w)$ at w.

(H3) Whenever an entire function $F(z)$ of z belongs to the space, the conjugate entire function

$$F^*(z) = F(z^-)^-$$

of z belongs to the space and has the same norm as $F(z)$.

A Hilbert space $\mathcal{H}$ whose elements are entire functions, which satisfies the axioms (H1), (H2), and (H3), and which contains a nonzero element, has an elementary structure which was discovered by Stieltjes for spaces of finite dimension whose elements are polynomials. The space is determined by an entire function $E(z)$ of z which has no zeros above the real axis since the inequality

$$|E(x - iy)| < |E(x + iy)|$$

holds for all real x when y is positive. The elements of the space are the entire functions $F(z)$ of z which are smaller than $E(z)$ in the sense that the inequality

$$|F(z)|^2 \leq \|F\|^2 \; \frac{|E(z)|^2 - |E(z^-)|^2}{2\pi i(z^- - z)}$$

holds for all complex z where the integral

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt$$

converges.

Formulated as dynamics the structure of a Stieltjes space of polynomials is a vibrating string fastened at a finite number of points. The project on the Riemann hypothesis continues by showing that every Hilbert space of entire functions which satisfies the axioms has the structure of a vibrating string. The string satisfies constraints which permit analysis by entire functions instead of the more general Hilbert spectral theory. The formulation is advantageous for the Riemann hypothesis since spectral analysis is treated by the inverse problem: The generating differential operator is reconstructed from a knowledge of its spectral properties. The results are new since the inverse problem is shown to have a unique solution.

The treatment is interesting because of an example in which there is a computation of string structure. The resulting Hilbert space of entire functions satisfying the axioms is defined by an entire function $E(z)$ of z which has a zero–free half–plane larger than the upper half–plane. The zeros of the function lie on a horizontal line at distance one–half below the real axis. The relevance to the classical zeta function lies in its relationship to the Euler gamma function, a predecessor of the zeta function which appears in the functional identity discovered by Euler for the zeta function. The example applies the hypergeometric series, discovered by Euler and applied in his proof of the functional identity.

Publication of these results in 1965 completes preliminaries to research on the Riemann hypothesis. A search begins for other Hilbert spaces of entire functions whose defining functions have the pattern of zeros expected in the Riemann hypothesis. Examples collected over twenty years prepare an axiomatic treatment of spaces having the desired properties.

A linear transformation with domain and range in a Hilbert space is defined as dissipative if the real part of the scalar product of an element of the domain with a corresponding element of the range is always nonnegative. Such a transformation is continuous if it is everywhere defined. Effectiveness of the dissipative property requires that the transformation be maximal dissipative: No proper linear extension exists which maintains nonnegativity of the real part of the scalar product of an element of the domain with the corresponding element of the range.

A generalization of the Riemann hypothesis which applies to Hilbert spaces of entire functions satisfying the axioms (H1), (H2), and (H3) is the existence of a maximal dissipative transformation whose domain and range are contained in the space. The transformation takes an entire function $F(z)$ into an entire function $F(z+i)$ whenever the functions of z belong to the space.

The Riemann hypothesis for Hilbert spaces of entire functions is a generalization of the Riemann hypothesis in the sense of imposing a constraint on zeros of the defining entire function $E(z)$: The function has no pair of distinct zeros λ^- and $\lambda - i$. The hypothesis implies that the zeros of $E(z)$ lie on the line

$$iz^- - iz = -1$$

if the functions $E^*(z)$ and $E(z+i)$ are linearly dependent.

The Riemann hypothesis for Hilbert spaces of entire functions is not a conjecture to be verified but a hypothesis in a theorem whose elementary proof removes all possible doubt about its validity. The theorem supplies a procedure for verification of an expected pattern of zeros in a function whose resemblance to the classical zeta function creates such expectations. The theorem was published in 1986 as a research announcement in the Bulletin of the American Mathematical Society.

A symposium on Fourier analysis was held at Cornell University in the summer 1956 to which Arne Beurling was an invited speaker. He posed a problem in complex analysis which was taken as the topic of the thesis in preparation for the Riemann hypothesis. Its publication in 1958 was followed by a postdoctoral position at the Institute for Advanced Study and an invitation to the International Symposium on Functional Analysis held at Stanford University in the summer 1961. In his invited address Paul Malliavin presented joint work with Beurling on the problem which is the topic of the 1956 lecture and the thesis. The theorem of Beurling and Malliavin is a preliminary to the Riemann hypothesis for Hilbert spaces of entire functions.

The Hardy space for the upper half–plane is the Hilbert space of functions $F(z)$ of z, analytic in the upper half–plane, such that the integrals

$$\int_{-\infty}^{+\infty} |F(x+iy)|^2 dx$$

are bounded functions of positive y. The least upper bound

$$\int_{-\infty}^{+\infty} |F(x)|^2 dx = \inf \int_{-\infty}^{+\infty} |F(x+iy)|^2 dx$$

is attained in the limit as y decreases to zero as the integral of a boundary value function $F(x)$ of x defined almost everywhere on the real axis. An element of the space is characterized as a Fourier transform

$$F(z) = \int_0^{\infty} \exp(2\pi izt) f(t) dt$$

of a square integrable function $f(t)$ of real t which vanishes for negative arguments. The identity

$$\int_{-\infty}^{+\infty} |F(x)|^2 dx = \int_0^{\infty} |f(t)|^2 dt$$

states the isometric property of the Fourier transformation. Norbert Wiener treated the real variable t as time and applied the Hardy space in a prediction theory for future events applying data from past times.

A generalization of Fourier analysis arises when functions which do not have representations as Fourier integrals have a meaningful action on Fourier transforms according to the Wiener operational calculus. Hilbert spaces of functions analytic in the upper half–plane appear which generalize the classical Hardy space. The spaces are defined by functions which are analytic and without zeros in the upper half–plane. The defining functions are typically too large to be represented as Fourier transforms. Wiener created a Fourier analysis of unbounded functions.

Notation is required for precision. An analytic weight function is a function which is analytic and without zeros in the upper half–plane. The weighted Hardy space $\mathcal{F}(W)$ defined by an analytic weight function $W(z)$ is the set of function $F(z)$ of z which are analytic in the upper half–plane, such that the integrals

$$\int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 dx$$

are bounded functions of positive y. Multiplication by $W(z)$ acts as an isometric transformation of the classical Hardy space onto the weighted Hardy space.

An analytic weight function acts as a symbol in the Wiener operational calculus. The properties of the function are reflected in properties of the associated space of functions. The relationship between space and defining function is an essential feature in the Riemann hypothesis for Hilbert spaces of entire functions.

Prediction requires a good relationship between past and future. Time reversal interchanging past into future is represented by the conjugation which takes a function $F(z)$ of z into the function

$$F^*(z) = F(z^-)^-.$$

Nontrivial functions $F(z)$ are needed such that the functions $F(z)$ and $F^*(z)$ of z both belong to the weighted Hardy space. Such functions must be defined and analytic in the upper and lower half–planes. Consistency of boundary value functions on the real axis implies that the functions are analytic in the complex plane. Prediction requires the existence of nontrivial entire functions $F(z)$ of z such that the restrictions to the upper half–plane of the functions $F(z)$ and $F^*(z)$ of z belong to the weighted Hardy space.

When conditions on the weight function $W(z)$ due to Norman Levinson are satisfied, the set of entire functions $F(z)$ such that the functions $F(z)$ and $F^*(z)$ of z belong to the space $\mathcal{F}(W)$ is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) in the scalar product of the space $\mathcal{F}(W)$.

Beurling and Malliavin give conditions on the weight function $W(z)$ for the existence of a nontrivial entire function $F(z)$ such that the functions $F(z)$ and $F^*(z)$ of z belong to the space $\mathcal{F}(W)$.

The Riemann hypothesis for Hilbert spaces of entire functions admits a formulation as a condition on analytic weight functions. Since these weight functions are generalizations of the weight function

$$W(z) = \Gamma(\tfrac{1}{2} - iz)$$

associated with the Euler gamma function, they are called Euler weight functions.

An analytic weight function $W(z)$ is said to be an Euler weight function if for every h in the interval $[0, 1]$ a maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space. The function $W(z)$ of z then admits an extension to the half–plane $iz^- - iz > -1$ which is analytic and without zeros. For h in the interval $[0, 1]$ the function

$$W(z + \tfrac{1}{2}ih)/W(z - \tfrac{1}{2}ih)$$

of z is analytic and has nonnegative real part in the upper half–plane. The definition of an Euler weight function replaces the functional identity for the gamma function by an inequality.

The theorem of Beurling and Malliavin is not known to apply generally to the space $\mathcal{F}(W)$ when $W(z)$ is an Euler weight function. When it does, the set of entire functions

$F(z)$ of z such that the functions $F(z)$ and $F^*(z)$ of z belong to the space $\mathcal{F}(W)$ is a nontrivial Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) in the scalar product of the space into the space $\mathcal{F}(W)$. For h in the interval $[0,1]$ a maximal dissipative transformation is defined in the space by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space.

A theorem of von Neumann is implicit in the definition of an Euler weight function. A maximal dissipative transformation T belongs to a semi–group of transformations T^h whose elements are maximal dissipative transformations when h is in the interval $[0,1]$. The transformations constructed by the von Neumann operational calculus are required to be shifts taking $F(z)$ into $F(z+ih)$. These conditions are verified when

$$W(z) = \Gamma(\tfrac{1}{2} - iz)$$

by properties of the gamma function discovered by Euler.

The canonical measure for the complex plane is the Cartesian product measure of Lebesgue measure for two coordinate lines. The Fourier transformation for the complex plane is an isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure which takes an integrable function $f(z)$ into the continuous function

$$g(z) = \int \exp(\pi i(z^- w + w^- z)) f(w) dw$$

of z defined by integration with respect to the canonical measure. Fourier inversion

$$f(z) = \int \exp(-\pi i(z^- w + w^- z)) g(w) dw$$

applies with integration with respect to the canonical measure when the function $g(z)$ of z is integrable and the function $f(z)$ of z is continuous.

The Fourier transformation for the complex plane commutes with the isometric transformation which takes a function $f(z)$ of z into the function $f(\omega z)$ of z for every element ω of the complex plane which has its conjugate as inverse. The Hilbert space of square integrable functions with respect to the canonical measure decomposes into the orthogonal sum of invariant subspaces for the commuting transformations. An invariant subspace is parametrized by an integer ν and consists of the functions $f(z)$ of z which satisfy the identity

$$f(\omega z) = \omega^\nu f(z)$$

for every element ω of the complex plane with conjugate as inverse. Attention is restricted to the case ν nonnegative since other cases are obtained under the isometric transformation which takes a function $f(z)$ of z into the function $f(z^-)$ of z.

The case ν equal to zero is of special interest since these methods produce the Hilbert spaces of entire functions constructed from the analytic weight function

$$W(z) = \Gamma(\tfrac{1}{2} - iz)$$

for which the analogue of the Riemann hypothesis true.

The classical zeta function is defined by Euler as a product taken over the primes whose factors are functions analogous to the gamma function. The introduction of Fourier analysis clarifies the transition from the gamma function to its analogue for a prime p. The definition of the gamma function applies the additive and multiplicative structure of the field of real numbers. A field of p–adic numbers is defined for every prime p as the completion of the rational numbers in a topology which resembles the topology with respect to which the field of real numbers is obtained. The definition of the p–adic analogue mimics the definition of the gamma function in a p–adic field.

The complex plane admits a p–adic analogue for every prime p as a field which is a quadratic extension of the field of p–adic numbers. The unique nontrivial automorphism of the field is called a conjugation by analogy with the conjugation of the complex plane. The prime p is required to factor as the product of an element of the field and its conjugate. The field is called the p–adic plane since it is uniquely determined by these properties within an isomorphism.

The constructions made in Fourier analysis on the complex plane generalize to constructions on the p–adic plane since they apply only the additive and multiplicative properties of a conjugated field. The analytic weight function

$$W(z) = \frac{1}{1 - p^{-\frac{1}{2} + iz}} \, \frac{1}{1 - p^{-\frac{3}{2} + iz}}$$

which replaces the analytic weight function

$$W(z) = \Gamma(\tfrac{1}{2} - iz)$$

applies a doubling of the p–adic analogue of the gamma function which is analogous to the Euler duplication formula for the gamma function:

$$2^{s-1} \Gamma(\tfrac{1}{2}s) \Gamma(\tfrac{1}{2}s + \tfrac{1}{2}) = \Gamma(\tfrac{1}{2}) \Gamma(s).$$

The classical zeta function is an Euler product of factors contributed by primes. Fourier analysis on the real line produces an additional factor of a gamma function in the functional identity. The analogous product in Fourier analysis on planes requires a reformulation of previous results due to the absence of a conjugated field which is contained as a dense subset of the complex plane and of the p–adic plane for every prime p.

A skew–field associated with a conjugated field contains the given field and extends the conjugation of the field as an anti–automorphism of the skew–field. When the conjugated field is called a plane, it is natural to call the associated skew–field a skew–plane. The complex skew–plane is then the skew–field constructed from the complex plane. The elements of the complex skew–plane are quaternions

$$t + ix + jy + kz$$

with real numbers as coordinates. The dense subset of the complex skew–plane whose elements have rational numbers as coordinates is a conjugated skew–field. The p–adic skew–plane is obtained by completion in a p–adic topology. The elements of the p–adic skew–plane are the quaternions with p–adic numbers as coordinates.

Fourier analysis on a skew–plane is related to Fourier analysis on a plane by a Radon transformation. The complementary space to a plane in a skew–plane is defined as the set of elements η which satisfy the identity

$$\eta\xi = \xi^- \eta$$

for every element ξ of the plane. Elements of the complementary space are skew–conjugate. The product of an element of the plane and an element of the complementary space is an element of the complementary space. The product of two elements of the complementary space is an element of the plane. An element of the skew–plane is the unique sum of an element of the plane and an element of the complementary space. The canonical measure for the skew–plane is the Cartesian product measure of the canonical measure for the plane and a canonical measure for the complementary space.

The Radon transformation for a skew–plane is a maximal dissipative transformation in the Hilbert space of functions which are square integrable with respect to the canonical measure. The transformation is defined as an integral on those elements of its domain which are integrable with respect to the canonical measure. The transformation takes a function $f(\xi)$ of ξ in the skew–plane which is integrable with respect to the canonical measure into the function $g(\xi)$ of ξ in the skew–plane defined almost everywhere by the equation

$$g(\omega\xi) = \int f(\omega\xi + \omega\eta) d\eta$$

for every element ω of the skew–plane with conjugate as inverse with integration with respect to the canonical measure for the complementary space. The inequality

$$\int |g(\omega\xi)| d\xi \leq \int |f(\xi)| d\xi$$

holds for every element ω of the skew–plane with conjugate as inverse with integration on the left with respect to the canonical measure for the plane and with integration on the right with respect to the canonical measure for the skew–plane.

The Radon transformation is the source of the maximal dissipative transformations required for Euler weight functions.

Consistent use of Fourier analysis clarifies the nature of the functions to which the Riemann hypothesis is expected to apply. These generalizations of the gamma function are products of a duplicated gamma function and its analogues in p–adic Fourier analysis. Further clarification results when the Riemann hypothesis as a conjecture about zeros is replaced by the conjectured presence of an Euler weight function which has the desired implication for zeros. The Riemann hypothesis is interesting only in a limit taken over an infinite number of primes. But the presence of Euler weight functions is interesting when only a finite number of primes are taken since the properties of Euler weight functions are preserved in a limit.

Fourier analysis is applied on the Cartesian product of the complex skew–plane and the p–adic skew–plane for a finite number of primes p. The Hilbert space of square integrable functions with respect to the Cartesian product measure of the product space is acted upon by a group of isometric transformations determined by the chosen primes.

A theorem attributed to Diophantus and confirmed by Lagrange states that every positive integer is the sum of four squares of integers. Equivalently a positive integer

$$n = \omega^- \omega$$

is the product of an integral element ω of the complex skew–plane and its conjugate ω^-. An integral element

$$\omega = t + ix + jy + kz$$

is according to Hurwitz not only an element whose coordinates x, y, z, and t are all integers but also an element whose coordinates are all halves of odd integers. A generalization of the Euclidean algorithm to integral elements of the complex skew–plane gives an elementary proof of the representation of a positive integer as a sum of four squares. The essential case of the representation occurs when the positive integer is a prime. Since there are twenty–four representations of one, the number of representations of a positive integer is always divisible by twenty–four. The number of representations of a positive integer n is shown by Jacobi to be equal to twenty–four times the sum of the odd positive divisors of n.

The representation is applied to positive integers n whose prime divisors are restricted to a given finite set of primes and to positive rational numbers whose numerator and denominator are such positive integers. These positive rational numbers form a group under multiplication. A noncommutative group of elements of the complex skew–plane is generated by the integral elements of the complex skew–plane which represent such positive integers n.

Nonzero elements of the complex skew–plane act as isometric operators on the Hilbert space of square integrable functions with respect to the Cartesian product measure. They act as multipliers on the independent variable. Such action is familiar in applications of operator theory to complex analysis, but now has a new aspect since the independent variable lies in a skew–field. Multiplication is noncommutative.

The Hilbert space decomposes into irreducible invariant subspaces under the group action producing special functions which are generalizations of theta functions. Jacobi introduced theta functions in the first application of Fourier analysis to the Euler zeta function. The zeta function appears multiplied by a gamma function when the Mellin transformation is applied to the theta function. A proof of the functional identity for the zeta function is obtained as an application of the Poisson summation in Fourier analysis.

The Jacobi construction of theta functions is an application of doubly periodic functions. A more structured construction is needed to gather information from different components of Fourier analysis and assemble it in a quotient space. The desired maximal dissipative property of a transformation is proved by a spectral decomposition using the theta functions. The transformation is subnormal: It is unitarily equivalent to multiplication by an analytic function in a Hilbert space whose elements are functions analytic in the upper half–plane. The transformation is maximal dissipative because the multiplying analytic function $-iz$ has nonnegative real part in the half–plane. These properties generalize familiar properties of the Laplace transformation. The theta function defines a generalization of the Laplace transformation.

The theta functions which generalize the Jacobi theta function produce a generalization of the Euler zeta function on application of the Mellin transformation. The Mellin transform of a theta function is an Euler weight function which factors as the product of a gamma function and a zeta function which satisfies a generalization of the Riemann hypothesis. The zeta functions constructed are not new. They are identical with zeta functions constructed by Erich Hecke from modular forms.

A proof of the classical Riemann hypothesis does not follow since an exceptional situation arises in which the underlying transformation is not maximal dissipative. There is a problem of convergence in mixing the components of Fourier analysis. The transformation is however nearly maximal dissipative: It has a one–dimensional extension which is maximal dissipative. The zeta function has an exceptional symmetry which permits the desired information concerning zeros to be obtained from a weaker hypothesis.

The production and acceptance of a manuscript for publication is a challenge not only for its author but also for its readers and depends on the support of organizations which fund both activities.

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