Title

Project Summary

The aim of the project is an estimation theory of those special functions of analysis called zeta functions after the zeta function of Euler (1730). The desired estimates generalize the estimates for the Euler zeta function conjectured by Riemann (1859), which are stated as properties of zeros of the functions: Nontrivial zeros are conjectured to lie on a line of symmetry for a functional identity satisfied by the zeta function. After an investment of fifty years the project is achieving its goals.

Intellectual Merit

The Riemann hypothesis is generally recognized to be the most important unsolved problem in mathematics. A solution is certain to attract readers internationally. The intellectual merit of the solution lies in the stimulus to mathematical activity which it gives not only at the highest levels of competence. A doctorate in mathematics is not needed to understand the significance of the conjecture for prime numbers.

Broader Impact

The Riemann hypothesis is not just another special problem of mathematics like the Fermat problem or the Poincaré conjecture to which it compares in stature. The solution of the problem gives an understanding of the symmetries of the three–dimensional space in which all physical theories are necessarily formulated. These properties of space are explored using Fourier analysis. The known formulation of quantum mechanics is an application of Fourier analysis which does not apply the information supplied by the Riemann hypothesis. Present theories assume that a particle orbiting about a nucleus has spherical symmetry. There is another possibility: the particle might have the symmetries of a cube. The symmetries of a cube generate another Fourier analysis, which is that of the Riemann hypothesis. If this is correct, then the present formulation of quantum mechanics is an approximation which needs to be improved. Atomic structure needs the best theory to become useful for meeting the energy needs of a crowded planet.

The Riemann Hypothesis

Project Summary

The Riemann hypothesis is unique as a research project since it is universally presumed not to exist. An application for funds must summarize the results of fifty years of nonexistence. The project began with the postdoctoral construction of Hilbert spaces whose elements are entire functions. After twenty–five years it continues with the introduction of special Hilbert spaces of entire functions to which an analogue of the Riemann hypothesis applies. And it reaches completion after another twenty–five years with the construction of spaces having the desired relationship to the Riemann hypothesis.

Hilbert spaces of entire functions supply a spectral theory of the vibrating string treated as a canonical model of an oscillator whose motion is constrained to one dimension. The model applies to the orbital motion of a particle about a nucleus. A fundamental problem is to determine the structure of the oscillator from scattering data. Hilbert spaces of entire functions give a proof that the inverse problem admits a unique solution when properly formulated.

Scattering data is stored in a function which is analytic and without zeros in the upper half–plane. Such scattering functions are called analytic weight functions in the theory of Hilbert spaces of entire functions. Associated with an analytic weight function $W(z)$ is a weighted Hardy space $\mathcal{F}(W)$ whose elements are the analytic function $F(z)$ of z in the upper half–plane such that the integrals

$$\int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 dx$$

converge when y is positive and are a bounded function of y. The weighted Hardy space is a Hilbert space whose scalar self–product is a limit of the integrals as y decreases to zero.

Related Hilbert spaces of entire functions are contained isometrically in the space $\mathcal{F}(W)$. A fundamental example of such a space is the set of all entire functions $F(z)$ of z such that both $F(z)$ and

$$F^*(z) = F(z^-)^-$$

are functions of z in the upper half–plane which belong to the space $\mathcal{F}(W)$. When the space contains a nonzero element, then a nested family of related Hilbert spaces of entire functions exist. The spaces are parametrized in such a way that a differential equation is satisfied. A vibrating string results.

Computable examples of analytic weight functions are derived from the gamma function. When

$$W(z) = \Gamma(\tfrac{1}{2} - iz),$$

a vibrating string is obtained which is related to Fourier analysis in the complex plane. The vibrating string describes the radial flow of heat from a source located at the origin. Hilbert spaces of entire functions are a construction of radially symmetric functions which are square integrable with respect to plane measure, which vanish in a disk about the origin, and whose Fourier transform vanishes in the same disk.

A generalization is motivated by examples taken from the gamma function. An Euler weight function is an analytic weight function $W(z)$ such that a maximal accretive transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space, $-1 \leq h \leq 1$.

Accretive means that the scalar product

$$\int_{-\infty}^{+\infty} \frac{F(t)^- F(t+ih)}{W(t)^- W(t)}\, dt$$

has nonnegative real part. Maximal accretive means that the accretive property is preserved in no proper linear extension.

An analytic weight function $W(z)$ is an Euler weight function if, and only if, the function

$$W(z + \tfrac{1}{2}ih)/W(z - \tfrac{1}{2}ih)$$

has an extension which is analytic and has nonnegative real part in the upper half–plane when $-1 \leq h \leq 1$.

An Euler weight function admits an analytic extension without zeros to the half–plane

$$-1 < iz^{-} - iz.$$

The function

$$W(z) = \Gamma(\tfrac{1}{2} - iz)$$

of z is an Euler weight function. A relationship to the Riemann hypothesis is indicated.

The concept of an Euler weight function is well–related to the vibrating string since the maximal accretive property is inherited in associated Hilbert spaces of entire functions. It is therefore a property of the vibrating string. A fundamental problem is to characterize vibrating strings which generate Euler weight functions.

The concept of an Euler weight function clarifies the nature of heat flow in the complex plane and guides generalization to other contexts in which Fourier analysis applies. The flow of heat in the plane is generated by a differential operator whose inverse is a Radon transformation relating Fourier analysis on a plane to Fourier analysis on a line. Radon transformations define the flow of heat on locally compact abelian groups. Groups with good multiplicative structure are applied to the Riemann hypothesis.

The underlying locally compact abelian groups of the Riemann hypothesis are skew–fields. They originate in the theorem, attributed to Diophantus and proved by Lagrange, that every positive integer is the sum of four squares of integers. The proof makes an implicit use of quaternions. Quaternions with rational numbers as coordinates are applied in the proof given by Hurwitz.

A Hurwitz integer is a quaternion

$$\xi = t + ix + jy + kz$$

whose coordinates are all integers or all halves of odd integers. The conjugated ring of Hurwitz integers admits a Euclidean algorithm. Nontrivial left and right ideals are generated by elements of the ring. The theorem of Diophantus states that every positive integer

$$n = \omega^{-}\omega$$

is the product of a Hurwitz integer ω and its conjugate ω^{-}. The number of representations is divisible by twenty–four since there are twenty–four representations of one. The number of representations of a positive integer n is twenty–four times the sum of the odd divisors of n.

A homogeneous harmonic polynomial φ of degree ν is a linear combination of monomials

$$t^{d}x^{a}y^{b}z^{c}$$

with nonnegative integer exponents whose sum is

$$\nu = a + b + c + d$$

and which is annihilated by the Laplacian

$$\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The homogeneous harmonic polynomials of degree ν inherit a scalar product from the Hilbert space of homogeneous polynomials of degree ν: The monomials form an orthogonal set with

$$\frac{a!b!c!d!}{\nu!}$$

as the scalar self–product of the monomial with exponents a, b, c, and d.

Zeta functions of Fourier analysis on skew–fields are Dirichlet series whose coefficients are defined as eigenfunctions of Hecke operators. Hecke operators are self–adjoint transformations acting on the Hilbert space of homogeneous harmonic polynomials of degree ν for an even nonnegative integer ν. A Hecke operator $\Delta(n)$ is defined for every positive integer n. The transformation takes $\phi(\xi)$ into $\psi(\xi)$ when they are functions of $\xi = t + ix + jy + kz$ such that

$$24n^{\frac{1}{2}+\frac{1}{2}\nu}\psi(\xi) = \sum \phi(\omega\xi)$$

is a sum over the Hurwitz integers ω which represent

$$n = \omega^{-}\omega.$$

The identity

$$\Delta(m)\Delta(n) = \sum \Delta(mn/k^2)$$

holds for all positive integers m and n with summation over the common odd divisors k of m and n.

A Hecke zeta function is associated with a homogeneous harmonic polynomial of even degree ν which is an eigenfunction of $\Delta(n)$ for an eigenvalue $\tau(n)$ for every positive integer n and with a primitive character χ modulo ρ for a positive integer ρ. The eigenvalues are real numbers which satisfy the identity

$$\tau(m)\tau(n) = \sum \tau(mn/k^2)$$

for all positive integers m and n with summation over the common odd divisors k of m and n.

The Hecke zeta function

$$Z(\chi, s) = \sum n^{-\frac{1}{2}}\tau(n)\chi(n)n^{-s}$$

of harmonic ϕ and character χ is defined as a sum over the positive integers n. The series converges in the half–plane $\mathcal{R}s > 1$ and represents a function of s which has an analytic extension to the complex plane with the exception of a simple pole at one when ν is zero and ρ is one. The Euler product

$$Z(\chi, s)^{-1} = (1 - 2^{-\frac{1}{2}}\tau(2)\chi(2)2^{-s})$$
$$\times \prod (1 - p^{-\frac{1}{2}}\tau(p)\chi(p)p^{-s} + p^{-1}\chi(p)^2 p^{-2s})$$

is taken over the odd primes p with an exceptional factor for the even prime. The product converges in the half–plane $\mathcal{R}s > 1$ and denies zeros in the half–plane. The functional identity

$$(2\pi)^{-\frac{1}{2}\nu - s}\Gamma(\tfrac{1}{2}\nu + s)Z(\chi, s)$$
$$= \epsilon(\chi)(2\pi)^{-\frac{1}{2}\nu - 1 + s}\Gamma(\tfrac{1}{2}\nu + 1 - s)Z(\chi^-, 1 - s)$$

denies zeros in the half–plane $\mathcal{R}s < 0$ except for trivial zeros due to poles of the gamma function. The Riemann hypothesis is generalized as the conjecture that nontrivial zeros lie on the line of symmetry $\mathcal{R}s = \frac{1}{2}$. The original conjecture of Riemann concerns only the exceptional zeta function which has a singularity.

An Euler weight function which is related to a Hecke zeta function appears when the flow of heat is treated on the skew–fields of quaternions with real coordinates and with p–adic numbers as coordinates for every prime p. The locally compact abelian group which is a composite of these skew–fields is a conjugated ring on which multiplication by a nonzero Hurwitz integer is a measure preserving transformation. Poisson summation is treated as a summation over induced isometric transformations to produce functions which are left fixed by action on the independent variable. The flow of heat is treated in the space of invariant functions. A generalization of the gamma function appears. This is the Euler weight function associated with a Hecke zeta function which has no singularity.

The Euler weight function associated with $Z(\chi, s)$ is

$$W(z) = \Gamma(\tfrac{1}{2}\nu + 1 - iz)Z(\chi, 1 - iz).$$

An Euler weight function is obtained when for some positive number r the Hecke zeta function, treated as an Euler product, is replaced by the partial product over the prime divisors p of r. The partial products converge to the full product uniformly on compact subsets of the half–plane

$$-1 < iz^- - iz.$$

This gives a proof of the Riemann hypothesis for a Hecke zeta function which has no singularity. A modification of the argument is expected for the conjecture of Riemann, which concerns the exceptional Hecke zeta function with a singularity.

Funding of research on the Riemann hypothesis was obtained by applying funds given after the proof in 1984 of the Bieberbach conjecture. Funding ceased when it became clear that funds were applied to other purposes. There remains a backlog of research related to the Bieberbach conjecture. A noncommutative generalization of complex analysis aims to prove generalizations of the Riemann mapping theorem, for example in the skew–field of quaternions with real coordinates.

Bibliography

1. L. de Branges, *Hilbert Spaces of Entire Functions*, Prentice Hall, New York, 1968.

2. ———, *The Riemann hypothesis for Hilbert spaces of entire functions*, Bull. Amer. Math. Soc. **15** (1986), 1-17.

3. ———, *A proof of the Riemann hypothesis*, (in preparation), preprint, Purdue University (2013), http://math.purdue.edu/∼branges.