

# RESEARCH PROPOSAL

## RIEMANN HYPOTHESIS

The original conjecture known as the Riemann hypothesis applies to the zeta function whose functional identity was obtained by Euler as an application of hypergeometric series. Fourier analysis gives another proof of the functional identity which generalizes to analogous zeta functions for which there is a conjecture also known as the Riemann hypothesis.

Fourier analysis is formulated for any locally compact Abelian group. The groups which produce a zeta function are completions of an initial discrete group in topologies which are compatible with additive structure. The discrete field of rational numbers is commonly used for its additive group to create zeta functions. The discrete skew-field of quaternions with rational numbers as coordinates is now used because a proof of the Riemann hypothesis is offered for these zeta functions.

The construction of zeta functions is motivated by the Euclidean algorithm. An element of the discrete skew-field is said to be integral if its coordinates are all integers or all halves of odd integers. Sums and products of integral elements are integral. The conjugate  $\xi^-$  of an integral element  $\xi$  is integral.

The Euclidean algorithm applies to pairs of integral elements of the discrete skew-field. The Cartesian product of the discrete skew-field with itself is treated as a left vector space over the discrete skew-field with indefinite scalar product

$$\langle (\alpha, \beta), (\gamma, \delta) \rangle = \delta^- \alpha + \gamma^- \beta$$

whose values are elements of the discrete skew-field.

A linear transformation of the vector space into itself is defined by a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

whose entries are elements of the discrete skew-field. The transformation takes an element  $(\alpha, \beta)$  of the vector space into the element

$$(\gamma, \delta) = (\alpha, \beta) \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The matrix is said to be symplectic if the transformation is isometric for the indefinite scalar product. The identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A^- & C^- \\ B^- & D^- \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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characterizes a symplectic matrix.

The transformation is said to be integral if it takes pairs of integral elements of the discrete skew-field into pairs of integral elements of the discrete skew-field. The transformation is integral if, and only if, the entries of its matrix are integral elements of the discrete skew-field.

The integral symplectic matrices are a group used to define an equivalence relation on the vector space. Equivalence of  $(\alpha, \beta)$  and  $(\gamma, \delta)$  means that

$$(\gamma, \delta) = (\alpha, \beta) \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for a matrix belonging to the group.

A formulation of the Euclidean algorithm applies in the presence of an indefinite scalar product. A pair  $(\alpha, \beta)$  of elements of the discrete skew-field whose scalar self-product

$$\beta^- \alpha + \alpha^- \beta = 0$$

vanishes is equivalent to a pair  $(\gamma, \delta)$  for which

$$\delta = 0$$

vanishes.

Since  $\alpha$  and  $\beta$  can be multiplied by a positive integer, it is sufficient to give a verification when  $\alpha$  and  $\beta$  are integral. If  $\alpha$  vanishes, the required integral symplectic matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

When neither  $\alpha$  nor  $\beta$  vanishes, it can be assumed by the use of the same matrix that

$$\beta^- \beta \leq \alpha^- \alpha.$$

By hypothesis the integral element

$$\beta^- \alpha$$

of the discrete skew-field is skew-conjugate.

A skew-conjugate integral element  $\gamma$  of the discrete skew-field exists such that the inequality

$$(\beta^- \alpha - \beta^- \beta \gamma)^- (\beta^- \alpha - \beta^- \beta \gamma) < (\beta^- \beta)^2$$

is satisfied. The inequality

$$(\alpha - \beta \gamma)^- (\alpha - \beta \gamma) < \beta^- \beta$$

is then satisfied by an equivalent pair

$$(\alpha - \beta \gamma, \beta) = (\alpha, \beta) \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

since the integral matrix

$$\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

is symplectic.

The desired pair of elements of the discrete skew-field is found by iteration.

Hecke operators are linear transformations on homogeneous functions  $f(\xi, \eta)$  of pairs of elements  $\xi$  and  $\eta$  of the discrete skew-field which have equal values at equivalent pairs: The identity

$$f(\alpha, \beta) = f(\gamma, \delta)$$

holds whenever

$$(\gamma, \delta) = (\alpha, \beta) \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for an integral symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Homogeneous means that the identity

$$f(\xi, \eta) = f(\omega\xi, \omega\eta)$$

holds for every positive rational number  $\omega$ .

A Hecke operator  $\Delta(r)$  is defined for every positive integer  $r$ . The definition of the transformation applies a representation of  $r$  by integral matrices.

An integral matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is said to represent a positive integer  $r$  if

$$\begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A^- & C^- \\ B^- & D^- \end{pmatrix}.$$

Two such matrices are considered equivalent for the definition of the Hecke operator if they are obtained from each other on multiplication on the left by an integral symplectic matrix. Every equivalence class contains a diagonal matrix

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$$

such that

$$r = \omega^- \omega.$$

Two diagonal matrices are equivalent if, and only if, they are obtained from each other on multiplication on the left by a diagonal matrix with equal entries on the diagonal which are integral elements of the discrete skew-field with integral inverse.

The Hecke operator  $\Delta(r)$  takes a function  $f(\xi, \eta)$  of pairs of elements of the discrete skew-field into the function

$$g(\xi, \eta) = \sum f(\xi\omega, \eta\omega)$$

of pairs of elements of the discrete skew-field defined as a sum over the equivalence classes of integral elements  $\omega$  of the discrete skew-field which represent

$$r = \omega^{-}\omega.$$

Equivalent elements are obtained from each other on multiplication on the left by an integral element of the discrete skew-field with integral inverse.

The identity

$$\Delta(m)\Delta(n) = \sum \Delta(mn/k^2)$$

holds for all positive integers  $m$  and  $n$  with summation over the common odd divisors  $k$  of  $m$  and  $n$ .

Hecke operators are applied to functions belonging to a Hilbert space of finite dimension such that an isometric transformation of the space into itself is defined by taking a function  $f(\xi, \eta)$  of pairs of integral elements of the discrete skew-field into the function  $f(\xi\omega, \eta\omega)$  of pairs of integral elements of the discrete skew-field for every nonzero element  $\omega$  of the discrete skew-field.

Hecke operators are commuting self-adjoint transformations. The Hilbert space is the orthogonal sum of invariant subspaces which are determined by eigenvalues of Hecke operators. The eigenvalue  $\tau(r)$  of  $\Delta(r)$  in an invariant subspace is a real number. The identity

$$\tau(m)\tau(n) = \sum \tau(mn/k^2)$$

holds for all positive integers  $m$  and  $n$  with summation over the common odd divisors  $k$  of  $m$  and  $n$ .

The zeta function defined by an invariant subspace is the Dirichlet series

$$Z(s) = \sum \tau(n)n^{-s}$$

whose coefficients are the eigenfunctions of Hecke operators. The Euler product

$$1/Z(s) = [1 - \tau(2)2^{-s}] \prod [1 - \tau(p)p^{-s} + p^{-2s}]$$

is taken over the odd primes  $p$  with an exceptional factor for the even prime.

A preliminary to the Riemann hypothesis is the Ramanujan hypothesis

$$-2 \leq \tau(p) \leq 2$$

for every odd prime  $p$  and the analogous hypothesis

$$-1 \leq \tau(2) \leq 1$$

for the even prime. When these inequalities are satisfied, the Dirichlet series and its Euler product converge in the half-plane  $\Re s > 1$  and define an analytic function of  $s$  which has no zeros in the half-plane.

Another preliminary to the Riemann hypothesis is the analytic extension of the function to the half-plane  $\Re s > \frac{1}{2}$  with the possible exception of a simple pole at  $s = 1$ . When these preliminaries are completed, the Riemann hypothesis states that the analytic extension has no zeros in the half-plane.

The Hilbert spaces of functions on which Hecke operators act are constructed in Fourier analysis on completions of the vector space of pairs of elements of the discrete skew-field in topologies which are compatible with vector space structure. The elements of a completion are pairs of elements of the completion of the discrete skew-field in topologies which are compatible with additive and multiplicative structure. A completion of the discrete skew-field is a space of quaternions whose coordinates are completions of the field of rational numbers which are compatible with additive and multiplicative structure.

It is sufficient to treat two completions of the field of rational numbers. The completion in the Dedekind topology is the field of real numbers. The canonical measure for the additive group of real numbers is Lebesgue measure on the Baire subsets of real numbers. The complex skew-field is the algebra of quaternions whose coordinates are real numbers. The canonical measure for the complex skew-field is the Cartesian product measure of the canonical measures for four coordinate lines. The canonical measure for the space of pairs of elements of the complex skew-field is the Cartesian product measure of the canonical measure of two components in the complex skew-field.

For Hecke operators the functions  $f(\xi, \eta)$  of pairs of elements  $\xi$  and  $\eta$  of the complex skew-field are homogeneous in the sense that the identity

$$f(\xi, \eta) = f(\omega\xi, \omega\eta)$$

holds for every positive real number  $\omega$ . The condition of square integrability with respect to the canonical measure is applied to the function

$$f(\xi, \eta)/(\xi^-\eta + \eta^-\xi)$$

over a fundamental domain.

There are two regions to be treated according to the sign of the scalar self-product

$$\xi^-\eta + \eta^-\xi,$$

one in which the scalar self-product is positive and one in which the scalar self-product is negative. The set of which the scalar self-product vanishes is ignored since it has zero measure. The set on which

$$\eta + \eta^-$$

vanishes is ignored for the same reason. Again there are two regions depending on a choice of sign. When positive signs are chosen, a fundamental region is defined by

$$\eta + \eta^- = 2.$$

The adic topology of the rational numbers applies integrality for its definition. The ring of integral adic numbers is isomorphic to the Cartesian product of the rings of integral  $p$ -adic numbers taken over all primes  $p$ . The ring of adic numbers is the ring of quotients of the ring of integral adic numbers with positive integers as denominators. An invertible adic number  $\omega$  is the unique product of a positive rational number  $\lambda(\omega)$  and an integral adic number with integral inverse. The ring of integral adic numbers is compact and has measure one with respect to the canonical measure for the ring of adic numbers.

The adic algebra is the algebra of quaternions whose coordinates are adic numbers. An element of the adic algebra is integral if its coordinates are all integral or all nonintegral halves of integral adic numbers. The ring of integral elements of the adic algebra is compact and has measure one with respect to the canonical measure for the adic algebra. The canonical measure for the space of pairs of elements of the adic algebra is the Cartesian product measure of the canonical measures of its two components in the adic algebra.

For Hecke operators the function  $f(\xi, \eta)$  of pairs of elements  $\xi$  and  $\eta$  in the adic algebra are homogeneous in the sense that the identity

$$f(\xi, \eta) = f(\omega\xi, \omega\eta)$$

holds for every positive rational number  $\omega$ . Square integrability with respect to the canonical measure is applied to the function

$$f(\xi, \eta)/\lambda(\xi^-\eta + \eta^-\xi)$$

over a fundamental domain. The set on which

$$\xi^-\eta + \eta^-\xi$$

is noninvertible is ignored since it has zero measure. The set on which

$$\eta + \eta^-$$

is noninvertible is ignored for the same reason. A fundamental domain is defined by

$$\lambda(\eta + \eta^-) = 2.$$

The adelic algebra is the Cartesian product of the complex skew-field and the adic algebra. An element  $\xi$  of the adelic algebra has a component  $\xi_+$  in the complex skew-field and a component  $\xi_-$  in the adic algebra. The canonical measure for the adelic algebra is the Cartesian product measure of the canonical measure for the complex skew-field and the canonical measure for the adic algebra.

Since an element  $\omega$  of the discrete skew-field is an element of the complex skew-field and an element of the adic algebra, the product

$$\eta = \xi\omega$$

of an element  $\xi$  of the adelic algebra and an element  $\omega$  of the discrete skew-field is defined by

$$\eta_+ = \xi_+ \omega$$

and

$$\eta_- = \xi_- \omega.$$

Multiplication by  $\omega$  is a measure preserving transformation with respect to the canonical measure for the adelic algebra.

Hecke operators are applied in Hilbert spaces whose elements are homogeneous functions  $f(\xi, \eta)$  of pairs of elements  $\xi$  and  $\eta$  of the adelic algebra such that the identity

$$f(\gamma, \delta) = f(\alpha, \beta)$$

holds whenever

$$(\gamma, \delta) = (\alpha, \beta) \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for an integral symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The Hecke operator  $\Delta(r)$  takes a function  $f(\xi, \eta)$  of pairs of elements of the adelic algebra into the function

$$g(\xi, \eta) = \sum f(\xi\omega, \eta\omega)$$

of pairs of elements of the adelic algebra defined by summation over the equivalence classes of elements  $\omega$  of the discrete skew-field which represent

$$r = \omega^- \omega.$$

equivalence of nonzero elements of the discrete skew-field means that they are obtained from each other on multiplication on the left by an integral element of the discrete skew-field with integral inverse.

Hilbert spaces of finite dimension on which Hecke operators act are obtained as invariant subspaces for a compact group of commuting isometric transformations. The transformations are defined by elements  $\omega$  of the adelic algebra with conjugate as inverse which take a function  $f(\xi, \eta)$  of pairs of elements of the adelic algebra into the functions  $f(\omega\xi, \eta)$  and  $f(\xi, \omega\eta)$  of pairs of elements of the adelic algebra.

The Riemann hypothesis is a consequence of a phenomenon in Fourier analysis which Fourier called the flow of heat. Since the infinitesimal generator of heat flow is a differential operator, an inverse integral operator is needed for application to locally compact Abelian groups which do not have a differential structure.

The generalization of heat flow is made by a Radon transformation. The relevant features of the transformation are that it is maximal accretive and has a subnormal adjoint.

Subnormal means that a transformation is the restriction of a normal transformation to an invariant subspace. A spectral expansion of a subnormal transformation is expected from the spectral expansion of a normal transformation. A spectral expansion of a Radon transformation is indeed obtained in this way and is called a Laplace transformation. The Radon transformation is then unitarily equivalent to multiplication by some function in a Hilbert space of square integrable functions with respect to a nonnegative measure.

A linear relation with domain and range in a Hilbert space is said to be accretive if the sum

$$\langle a, b \rangle + \langle b, a \rangle \geq 0$$

of scalar products is nonnegative for all elements  $(a, b)$  of its graph. A maximal accretive linear relation is an accretive linear relation which has no proper extension as an accretive linear relation with domain and range in the same Hilbert space. The adjoint of a maximal accretive linear relation is a maximal accretive linear relation. A linear transformation is said to be maximal accretive if it is a maximal accretive linear relation.

A Laplace transformation for the complex skew-field applies a Hilbert space of functions analytic in the upper half-plane which are square integrable with respect to a nonnegative measure. The adjoint of the Radon transformation is unitarily equivalent to multiplication by

$$i/z$$

where  $z$  is the independent variable for the complex plane. The accretive property is immediate since the multiplier has positive real part in the upper half-plane. The maximal accretive property is verified by showing that the adjoint is accretive. The adjoint is computed on the elements of the Hilbert space whose scalar product with a function assigns the function value at an element of the upper half-plane.

A Laplace transformation for the adic algebra imitates the spectral expansion for the Radon transformation for the complex skew-field. All computations for the adic algebra are reduced to computations in Hilbert spaces of finite dimension. When the maximal accretive property is verified in Hilbert spaces of arbitrary finite dimension, it follows immediately in Hilbert spaces of infinite dimension.

A Laplace transformation for the adelic algebra combines a Laplace transformation for the complex skew field and a Laplace transformation for the adic algebra. The range of the Laplace transformation for the adelic algebra is a tensor product of the range of a Laplace transformation for the complex skew-field and the range of a Laplace transformation for the adic algebra. The functions in the adelic case are defined on a space which is the Cartesian product of the space on which functions are defined in the complex case and the space on which the functions are defined in the adic case. The multiplier in the adelic case has values which are products of the values of the multiplier in the complex case and the values of the multiplier in the adic case. Since the values of the multiplier for the complex skew-field have positive real part and since the values of the multiplier for the adic algebra are positive, the values of the multiplier for the adelic algebra have positive real part. Verification of the maximal accretive property for the adelic algebra is made in a finite calculation.



There is no obstacle to the proof of the Riemann hypothesis when the zeta function has no singularity at the unit of the complex plane. The argument fails in the singular case because the Radon transformation for the adic algebra fails to be maximal accretive. In this case the Radon transformation commutes with an isometric transformation which is its own inverse. The Hilbert space has an orthogonal decomposition into a subspace of functions of even parity and a subspace of functions of odd parity. The isometric transformation multiplies a function of even parity by one and a function of odd parity by minus one. The graph of the Radon transformation decomposes into the graph of the Radon transformation in functions of even parity and the graph of the Radon transformation on functions of odd parity. The Radon transformation acts as a maximal accretive transformation on functions of even parity. This information is sufficient for a proof of the Riemann hypothesis in the singular case.

The proof of the Riemann hypothesis creates a canonical mechanical system to whose spectral function the zeta function contributes a factor. A fundamental problem is to determine whether a mechanical system is determined by its spectral function. The issue is treated in certain Hilbert spaces of entire functions which originate with Stieltjes. He treated mechanical systems which are finite. An axiomatic treatment of the Stieltjes spaces removes finiteness conditions. A fundamental theorem [1] states that the mechanical system is indeed determined by its spectral properties.

Applications of the Riemann hypothesis require the mechanical system determined by a zeta function together with gamma function factors in a spectral function. The proof of the Riemann hypothesis is treated as a computation of the mechanical system. The expected applications are to the quantum mechanics of orbital systems such as electrons in an atom or molecule. The present structure is a model for a one-electron atom. An extensive generalization in process. The initial discrete Abelian group is a finite Cartesian product of discrete skew-fields.