RESEARCH PROPOSAL
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The Riemann hypothesis is a conjecture made by Riemann in the context of a hypergeometric function theory due to Euler. Applications of hypergeometric functions appear in examples of polynomials orthogonal on a line. The theory of positive linear functionals on polynomials due to Stieltjes is an axiomatic treatment of integration on a line.

The Riemann hypothesis is an innovation since it concerns the zeros of an entire function which is not a polynomial. This leads to the discovery of entire functions, not polynomials, which are essentially determined by their zeros. It is significant that Stieltjes worked in cooperation with Hermite who discovered such a class of entire functions.

The Stieltjes integration theory was reformulated by Hilbert as a structure theory for linear transformations with domain and range in a Hilbert space which are their own adjoints. The Hilbert-Schmidt class of linear transformations of a Hilbert space into itself contains quadratic estimates which are motivated by quadratic estimates of entire functions found by Hermite.

The research direction of Hilbert and students Hellinger and Schmidt departs from the interpretation of the hypergeometric series as a complex analytic function which is stimulated by the Cauchy formula. The Hardy spaces for the unit disk and for the upper half-plane restore complex analysis in the Hilbert space context.

The more fundamental Hardy space for the unit disk is the set of functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

of z which are represented in the disk by square summable power series

$$||f(z)||^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

The Hardy space for the upper half-plane is the set of functions

$$F(z) = \int_0^{\infty} exp(itz) f(t) dt$$

of z in the upper half-plane which are Fourier transforms of functions $f(t)$ of t which are square integrable on the positive half-line,

$$2\pi ||F(z)||^2 = \int_0^{\infty} |f(t)|^2 dt < \infty.$$

A function $F(z)$ of z which is analytic in the upper half-plane belongs to the Hardy space if, and only if, the integrals

$$||F(z)||^2 = \sup \int_{-\infty}^{+\infty} |F(x+iy)|^2 dx < \infty$$

are finite when y is positive and are bounded independently of y. The least upper bound is then a limit as y decreases to zero. Paley-Wiener spaces are Hilbert spaces of entire functions which were discovered by a student of Hardy in conjunction with Wiener, who applied them in prediction theory.

A Paley-Wiener space is defined for every positive number a as the set of entire functions

$$2\pi F(z) = \int_{-a}^{a} \exp(itz) f(t) dt$$

of z which are Fourier transforms of functions $f(t)$ of t which are square integrable

$$2\pi ||F(z)||^2 = \int_{-a}^{a} |f(t)|^2 dt < \infty$$

in the interval $(-a, a)$. An entire function $F(z)$ of z belongs to the Paley-Wiener space defined by a if, and only if, it is of exponential type at most a and is square integrable

$$||F(z)||^2 = \int_{-\infty}^{+\infty} |F(t)|^2 dt$$

on the real axis.

Prediction theory is an application of the Wiener operational calculus. A transformation $K(H)$ with domain and range in the Hilbert space of square integrable functions with respect to Lebesgue measure is defined by a Baire function $K(x)$ of real x. The transformation takes a function of $f(t)$ of t into a function $g(t)$ of t when their Fourier transforms

$$\int_{-\infty}^{\infty} \exp(itx) g(t) dt = K(x) \int_{-\infty}^{\infty} \exp(itx) f(t) dt$$

are related almost everywhere on multiplication by $K(x)$.

In applications to prediction theory the real variable is time, the negative half-line for past times, the positive half-line for future times. Prediction theory requires operators which map functions vanishing for past times into functions vanishing for past times. In nontrivial cases the function $K(x)$ of real x is the boundary value function of a function which is analytic and of bounded type in the upper half-plane.

The trivial case occurs when the domain of the operator contains no function which vanishes on the negative half-line and does not vanish almost everywhere. Levinson, a Wiener student, obtained a stronger conclusion under the hypothesis that the integral

$$\int_{-\infty}^{+\infty} (1+t^2)^{-1} \log_+ |K(t)| dt = \infty$$

diverges and the function $\log_+ |K(t)|$ of t is monotonic. The domain of the operator contains no nontrivial function which vanishes in the neighborhood of a point.

A problem of Beurling is to determine all local operators in the Wiener calculus. An operator is local if it takes functions vanishing in the neighborhood of a point into functions vanishing in a neighborhood of the same point.

The solution of the Beurling problem is given in the doctoral thesis of the present investigator. An operator is nontrivially local, if and only if, it is defined by a function $K(x)$ of real x which agrees almost everywhere with the restriction of an entire function of minimal exponential type. The integral

$$\int_{-\infty}^{+\infty} (1+t^2)^{-1} \log_+ |K(t)| dt < \infty$$

converges implying bounded type in the upper and lower half-planes.

Nontriviality means that some nontrivial function in the domain of the operator vanishes in the neighborhood of a point. In the thesis it is shown that monotonicity in the Levinson theorem can be replaced by uniform continuity of the function

$$\log_+ |K(t)|$$

of t. The Beurling problem is a variant of the Bernstein problem in polynomial approximation. It is a step in the transition from polynomial approximation to approximation by entire functions. In the original context these problems are stated for approximation L^1. The mean square formulation was chosen for postdoctoral work. Hilbert spaces are studied whose elements are entire functions and which have these properties:

(H1) Whenever an entire function $F(z)$ of z belongs to the space and has a nonreal zero w, the entire function

$$F(z)(z - w^-)/(z - w)$$

of z belongs to the space and has the same scalar self-product as $F(z)$.

(H2) A continuous linear functional is defined on the space by taking a function $F(z)$ of z into its value $F(w)$ at w for every nonreal number w.

(H3) The conjugate function

$$F^*(z) = F(z^-)^-$$

of z belongs to the space whenever the function $F(z)$ of z belongs to the space, and it has the same scalar self-product as $F(z)$.

A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) is said to be a Stieltjes space since such spaces first appeared in the Stieltjes integral representation of positive linear functionals on polynomials.

An example of a Stieltjes space is defined by an entire function

$$E(z) = A(z) - iB(z)$$

of z which satisfies the inequality

$$|E(x - iy)| < |E(x + iy)|$$

for all real x when y is positive. The entire functions $A(z)$ and $B(z)$ of z are self-conjugate. The space is the set of entire functions $F(z)$ of z such that the integral

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt < \infty$$

converges and which satisfy the inequality

$$|F(z)|^2 \leq \|F\|^2 K(z, z)$$

for all complex numbers z, where

$$K(w, z) = [B(z)A(w^-) - A(z)B(w^-)]/[\pi(z - w^-)]$$

for all complex numbers z and w.

The function $K(w, z)$ of z belongs to the space for all complex numbers w and acts as reproducing kernel function for function values at w.

Every Stieltjes space which contains a nonzero element is isometrically equal to the Stieltjes space defined by some entire function $E(z)$. The function is not unique.

The function $E(z)$ and the elements of the defined space are polynomials in the application due to Stieltjes. In this case the Stieltjes space determines a nested family of Stieltjes spaces which are contained isometrically in the given space. The Stieltjes spaces determined by a positive linear functional on polynomials belong to a maximal totally ordered family of Stieltjes spaces. There may be no greatest member of the family. There always exists a nonnegative measure on the Baire subsets of real line such that the Stieltjes spaces are contained isometrically in the Hilbert space of square integrable functions with respect to the measure. The measure need not be unique. Nonuniqueness occurs when, and only when, the family of Stieltjes spaces has a greatest element.

The axiomatization of Stieltjes spaces permits generalization to spaces of infinite dimension whose elements are entire functions.

When a Stieltjes space is defined by an entire function $E(z)$, multiplication by z is the transformation which takes $F(z)$ into $zF(z)$ whenever the functions of z belong to the space.

The closure of the domain of multiplication by z is a Stieltjes space which is contained isometrically in the given space. Notation is required when a new Stieltjes space containing a nonzero element is created.

There is then a Stieltjes space with defining function

$$E(a, z) = A(a, z) - iB(a, z)$$

which is contained isometrically in a Stieltjes space with defining function

$$E(b, z) = A(b, z) - iB(b, z)$$

and whose orthogonal complement has dimension one.

The defining functions of the Stieltjes spaces can be chosen so that an element

$$A(a, z)u + B(a, z)v = A(b, z)u + B(b, z)v$$

of norm one in the orthogonal complement is defined by the some complex numbers u and v. The product

$$v^- u = u^- v$$

is then real.

Orthogonal polynomials define continued fractions, which are best treated as infinite products of two-by-two matrices.

A matrix

$$M(a,b,z) = \begin{pmatrix} A(a,b,z) & B(a,b,z) \\ C(a,b,z) & D(a,b,z) \end{pmatrix}$$

appears when the defining functions of the Stieljes spaces are chosen with value one at the origin. The identity

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))M(a,b,z)$$

is satisfied with

$$A(a,b,z) = 1 - \beta z, \quad B(a,b,z) = \alpha z$$

and

$$C(a,b,z) = \gamma z, \quad D(a,b,z) = 1 + \beta z$$

where

$$\alpha = \pi u^- u, \quad \beta = \pi u^- v = \pi v^- u, \quad \gamma = \pi v^- v.$$

Stieltjes spaces now appear which are contained in the Stieltjes space defined by $E(b,z)$ and which contain the Stieltjes space defined by $E(a,z)$. If the parameters a and b are positive numbers such that $a < b$, a Stieltjes space is defined by an entire function

$$E(t,z) = A(t,z) - iB(t,z)$$

when $a \leq t \leq b$. The functions $A(t,z)$ and $B(t,z)$ are defined by linearity in t to have the given values when $t = a$ and when $t = b$.

The recurrence relations for Stieltjes spaces of finite dimension are replaced by differential equations which apply in arbitrary dimensions. The differential equations are stated for a family of Stieltjes spaces with defining functions

$$E(t,z) = A(t,z) - iB(t,z)$$

which are parametrized by positive numbers t so that the space with parameter a is contained in the space with parameters b when $a < b$. The inclusion is contractive and is isometric on the closure of the domain of multiplication by z.

The functions $A(t,z)$ and $B(t,z)$ are absolutely continuous functions of t for every complex number z which satisfy the differential equations

$$B'(t,z) = zA(t,z)\alpha'(t) + zB(t,z)\beta'(t)$$

and

$$-A'(t,z) = zA(t,z)\beta'(t) + zB(t,z)\gamma'(t)$$

for absolutely continuous functions $\alpha(t), \beta(t)$ and $\gamma(t)$ of positive t with real values. The matrix

$$m'(t) = \begin{pmatrix} \alpha'(t) & \beta'(t) \\ \beta'(t) & \gamma'(t) \end{pmatrix}$$

is nonnegative for almost all t. The solution $B(t,z)$ has derivative zero at the origin as a function of z. The solution $A(t,z)$ value one at the origin.

Matrix notation is advantageous for the treatment of the differential equations. Two-by-two matrices with complex entries act on the right of row vectors with two complex entries and on the left of column vectors with two complex entries. A bar denotes the conjugate transpose of a row or column vector as well as of a matrix. The matrix

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is a generalization of the imaginary unit.

The differential equation reads

$$(A'(t,z), B'(t,z))I = z(A(t,z), B(t,z))m'(t).$$

The reproducing kernel function

$$[B(t,z)A(t,w)^- - A(t,z)B(t,w)^-]/[\pi(z - w^-)]$$

is an absolutely continuous function of t with derivative

$$\pi^{-1}(A(t,z), B(t,z))m'(t)(A(t,w), B(t,w))^-.$$

The increment

$$[B(b,z)A(b,w)^- - A(b,z)B(b,w)^-]/[\pi(z - w^-)]$$
$$-[B(a,z)A(a,w)^- - A(a,z)B(a,w)^-]/[\pi(z - w^-)]$$

in reproducing kernel functions is the integral

$$\int_a^b (A(t,z), B(t,z))dm(t)(A(t,w), B(t,w))^-.$$

A Hilbert space $L^2(m)$ is defined whose elements are equivalence classes of pairs $(f(t), g(t))$ of Baire functions of positive t such that the integral

$$\int_0^\infty (f(t), g(t))dm(t)(f(t), g(t))^- < \infty$$

converges. The integral defines the scalar self-product. Equivalence of pairs means that the scalar self-product of the difference vanishes.

The elements of the Stieltjes space with parameter a are entire functions

$$F(z) = \int_0^a (A(t,z)B(t,z))m'(t)(f(t)^-, g(t)^-)^- dt$$

of z which are represented by pairs $(f(t), g(t))$ of Baire functions of t such that the integral

$$\pi \int_0^a (f(t), g(t))m'(t)(f(t), g(t))^- dt < \infty$$

converges. The integral is equal to the scalar self-product of the entire function in the Stieltjes space defined by $E(a,z)$.

The representation of elements of a Stieltjes space as integrals is a generalization of the Fourier transformation for the real line. The Stieltjes space defined by

$$E(a,z) = \exp(-iaz)$$

is the Paley-Wiener space of entire functions $F(z)$ of exponential type at most a which are square integrable on the real axis. An element of the space is represented as the Fourier integral

$$F(z) = \int_{-a}^{a} \exp(itz) f(t) dt$$

of a square integrable functions $f(t)$ of real t which vanishes outside of the interval $(-a, a)$. The isometric property of the Fourier transformation reads

$$2\pi \int_{-\infty}^{\infty} |F(t)|^2 dt = \int_{-a}^{a} |f(t)|^2 dt.$$

An analytic weight function is a function $W(z)$ of z which is analytic and without zeros in the upper half-plane. The weighted Hardy space $\mathcal{F}(W)$ is the set of functions $F(z)$ of z analytic in the upper half-plane such that the integral

$$\int_{-\infty}^{\infty} |F(t+iy)/W(t+iy)|^2 dt < \infty$$

converges when y is positive and is a bounded function of y. As y decreases to zero, the integral increases to the scalar self-product of the function in the weighted Hardy space.

When an analytic weight function $W(z)$ is given, there may exist a nontrivial entire function $F(z)$ of z such that for some real number τ the functions

$$\exp(i\tau z) F(z)$$

and

$$\exp(i\tau z) F^*(z)$$

of z in the upper half-plane belong to the weighted Hardy space. The set of all such functions is then a Stieltjes space which is mapped isometrically into the weighted Hardy space on multiplication by $\exp(i\tau z)$.

A parametrized family of Stieltjes spaces is defined by entire functions

$$E(t, z) = A(t, z) - iB(t, z)$$

of z such that for every positive number t a least real number $\tau(t)$ exists such that multiplication by $\exp(i\tau(t)z)$ is a contractive transformation of the Stieltjes space defined by $E(t, z)$ into the weighted Hardy space and is isometric on the closure of the domain of multiplication by z in the space. The image of the Stieltjes space defined by $E(a, z)$ is contained in the image of the Stieltjes space defined by $E(b, z)$ when $a < b$. The function $\tau(t)$ of t, which is nondecreasing and absolutely continuous, satisfies the differential equation

$$\tau'(t)^2 = \alpha'(t)\gamma'(t) - \beta'(t)^2.$$

An analytic weight function

$$W_\infty(z) = \lim \exp(i\tau(t)z) E(t, z)$$

is obtained as a limit uniformly on compact subsets of the upper half-plane as t increases to infinity. Multiplication by

$$\exp(i\tau(t)z)$$

is a contractive transformation of the Stieltjes space defined by $E(t,z)$ into the weighted Hardy space defined by $W_\infty(z)$ and is isometric on the closure of the domain of multiplication by z. The weighted Hardy space $\mathcal{F}(W_\infty)$ is contained isometrically in the weighted Hardy space $\mathcal{F}(W)$ but can be a proper subspace of $\mathcal{F}(W)$. The analytic weight function $W_\infty(z)$ is said to be the scattering function of the parametrized family of Stieltjes spaces defined by the analytic weight function $W(z)$.

Two fundamental problems arise in the construction of Stieltjes spaces by analytic weight functions: 1) When does an analytic weight function define a parametrized family of Stieltjes spaces? 2) When is an analytic weight function the scattering function of the parametrized family of Stieltjes spaces which it defines?

Both problems are solved under a hypothesis which is relevant to the Riemann hypothesis. An analytic weight function $W(z)$ is said to be an Euler weight function if for every h in the interval $-1 \leq h \leq 1$ a maximal accretive transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space.

A linear relation with domain and range in a Hilbert space is said to be accretive if the sum

$$< a, b > + < b, a > \ \geq \ 0$$

of conjugate scalar products is nonnegative for every element (a, b) of its graph.

An accretive linear relation is said to be maximal accretive if its graph is not a proper vector subspace of the graph of an accretive linear relation with domain and range in the same Hilbert space.

A linear transformation with domain and range in a Hilbert space is said to be maximal accretive if it is maximal accretive as a linear relation.

Every Euler weight function defines a parametrized family of Stieltjes spaces and is the scattering function of the spaces which it defines.

The defined Stieltjes spaces inherit maximal accretive transformations. When $-1 \leq h \leq 1$, a maximal accretive transformation in the Stieltjes space defined by $E(t,z)$ takes $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space.

An Euler weight function $W(z)$ has an analytic extension to the half-plane $iz^- - iz > -1$ which satisfies the recurrence relation

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z)$$

for a function $\phi(z)$ of z which is analytic and has nonnegative real part in the upper half-plane. The recurrence relation denies the existence of zeros in the half-plane.

If a nontrivial function $\phi(z)$ of z is given which is analytic and has nonnegative real part in the upper half-plane, an Euler weight function exists which satisfies the recurrence relation for the given function $\phi(z)$. The Euler weight function is unique within a constant factor.

The Riemann hypothesis is proved for a nonsingular zeta function by constructing an associated Euler weight function. In application to a singular zeta function the analytic weight function $W(z)$ has positive values on the upper half of the imaginary axis. The resulting symmetry

$$W^*(z) = W(-z)$$

implies symmetry of zeros about the imaginary axis. The shift defined when $-1 \leq h \leq 1$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the weighted Hardy space has a one-dimensional extension which is a maximal accretive relation. The function $W(z)$ of z has an analytic extension without zeros to the half-plane $iz^- - iz > -1$ with the exception of a simple pole at the origin.

The analytic weight functions which apply to zeta functions are products of Euler weight functions constructed from the hypergeometric series and its Heine generalization. An example of an Euler weight function

$$W(z) = \Gamma(h - iz)/\Gamma(k - iz)$$

is defined when $k \geq h \geq \frac{1}{2}$. The Euler spaces construct unitary representations of the group of matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with complex entries such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A^- & C^- \\ B^- & D^- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrices define conformal mappings z into

$$(Az + B)/(Cz + D)$$

of the unit disk into itself, and also of the complement of the unit disk into itself, which preserve hyperbolic structure.

Each prime p contributes a component Euler weight function which is periodic of period

$$\pi^{-1} log(p).$$

An analytic weight function which is periodic of period λ for a positive number λ satisfies the recurrence relation

$$W(z + \lambda) = W(z).$$

An isometric transformation of the weighted Hardy space onto itself is defined by taking a function $F(z)$ of z into the function $F(z + \lambda)$ of z. An isometric transformation of every associated Stieltjes space onto itself is defined by taking a function $F(z)$ of z into the function $F(z + \lambda)$ of z. The structure of a periodic space reduces by a change of variable to the case $\lambda = 2\pi$. Under the change of variable the Stieltjes spaces contributed by primes are identical.

The Euler weight function $W(z)$ satisfies the recurrence relation

$$W(z + i) = W(z) \sin(z + ih)/\sin(z + ik)$$

for numbers h and k, $k \geq h \geq \frac{1}{2}$.

Paley-Wiener spaces are fundamental examples of periodic spaces. The structure of an arbitrary periodic Stieltjes space is reduced to the structure of Paley-Wiener spaces by an inductive construction related to the construction of polynomials orthogonal with respect to a nonnegative measure on the Baire subsets of the unit circle.

Maximal accretive transformations are given by the Laplace transformation. Hilbert spaces are constructed for positive integers ν whose elements are analytic functions $F(z)$ of z in the upper half-plane. A function $F(z)$ of z belongs to the space of order ν if the integral

$$\int_0^\infty \int_{-\infty}^{+\infty} |F(x + iy)|^2 y^{\nu - 1} dx dy < \infty$$

converges. A space is also defined when $\nu = 0$ as the Hardy space for the upper half-plane. A maximal accretive transformation is defined by taking $F(z)$ into $G(z)$ when

$$G(z) = (i/z)F(z)$$

for functions $F(z)$ and $G(z)$ of z which belong to the space.

An analogue of the Laplace transformation exist for every prime p. It is found by treating the Laplace transformation as the spectral analysis of a Radon transformation. In the original application the Radon transformation relates the Fourier transformation for the plane to the Fourier transformation for the line by an integral. The Radon transformation is new in p-adic analysis.

Since a product of Euler weight functions need not be an Euler weight function, it needs to be explained why it is, or nearly is, in the context of the Riemann hypothesis. A composite Laplace transformation is constructed from the Laplace transformations for each factor. A partial construction appears in the construction of Hecke zeta functions from modular forms.

The elements of the modular group are matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

which integer entries and determinant one defining conformal mappings z into

$$(Az + B)/(Cz + D)$$

of the upper half-plane onto itself. For every positive integer ν a Hilbert space of finite dimension is defined as the set of functions $F(z)$ of z analytic in the upper half-plane such that

$$F(z) = F((Az + B)/(Cz + D))(Cz + D)^{-1-\nu}$$

for every element of the modular group and such that the integral

$$||F(z)||^2 = \int \int |F(x + iy)|^2 y^{\nu-1} dxdy < \infty$$

over a fundamental domain converges.

The Hilbert space is the orthogonal sum of invariant subspaces of dimension one under the action of Hecke operators. Eigenvalues of Hecke operators define the coefficients of zeta functions as Dirichlet series.

The construction of zeta functions is applied in modified form since the subgroup of the modular group is used whose elements are matrices with two even entries. A Hecke operator is not defined for the even prime.

An eigenfunction of Hecke operators defines an analytic weight function which is a product of Euler weight functions. A composite Laplace transformation is constructed from component Laplace transformations. The product analytic weight function is an Euler weight function if it has no singularity at the origin. In the case of a singularity there remains enough information for a proof of the Riemann hypothesis.

References

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