

RESEARCH PROPOSAL

The proof in 1984 of the Bieberbach conjecture ends the first half of a project whose aim is a proof of the Riemann hypothesis. The second half nears completion twenty-five years later since a proof of the Riemann hypothesis is offered. The second half is not over since the argument remains to be confirmed.

Publicity attached to the Riemann hypothesis makes it seem that a proof would attract immediate attention. The Riemann hypothesis is the only one of the seminal problems stated by Hilbert at the beginning of the twentieth century which were not solved at the end of the century. The prize for a solution offered by the Clay Foundation is comparable to a Nobel prize. Despite appearances a proof of the Riemann hypothesis is an unexpected event. No body of learned experts is competent for a verification of a proof. The Riemann hypothesis is seen as an unreachable goal whose very mention draws knowing smiles and glances of amusement.

Yet there was a time within living memory when the Riemann hypothesis was an accepted goal of research. The conjecture concerns the zeros of a function which is treated as resembling a polynomial. The required class of functions was identified by Hermite in the second half of the nineteenth century.

A polynomial is said to be of Hermite class if it has no zeros in a half-plane which is now chosen as the upper half-plane. (The lower half-plane is often chosen instead.) An entire function is said to be of Hermite class if it has no zeros in the upper half-plane and is a limit in the complex plane of polynomials which have no zeros in the upper half-plane. Entire functions of Hermite class admit a factorization in terms of zeros which generalizes the factorization of a nonconstant polynomial as a product of linear factors.

Properties of the Hermite class relevant to the Riemann hypothesis were found by Stieltjes in a representation of nonnegative linear functionals on polynomials. A linear functional on polynomials is defined to be nonnegative if it has nonnegative values on polynomials whose values are nonnegative on the real axis. A nonnegative linear functional on polynomials admits a representation as a Stieltjes integral with respect to a nondecreasing function of a real variable. The determination of the nondecreasing functions which represent a given nonnegative linear functional is a computation made by Stieltjes in the Hermite class of entire functions. The work of Stieltjes came to a premature end before the turn of the century because of his death from tuberculosis. Hilbert brought the Stieltjes integral representation to Göttingen after postdoctoral years in Paris and applied it as an integral representation of self-adjoint transformations in terms of invariant subspaces. Integral representations of transformations which are not self-adjoint appear in Hilbert spaces of analytic functions introduced by Hardy. The Riemann hypothesis is emphasized as an underlying problem in texts of Titchmarsh continuing the mathematical tradition founded by Hardy.

The Riemann hypothesis was a natural choice for serious research twenty-five years before the proof of the Bieberbach conjecture. The value of the Hermite class of entire functions was demonstrated by Stieltjes only in the case of polynomials. Entire functions of Hermite class which are not polynomials are essential to the Riemann hypothesis. A gap was left which was filled during the first five years of research on the Riemann hypothesis. A structural analysis was made of Hilbert spaces of entire functions which have the relationship to the Hermite class observed by Stieltjes in the case of polynomials. An

axiomatic treatment is given which applies the techniques of Hilbert. The construction of the spaces applies the Hardy space of analytic functions for the upper half-plane. The significance of the Riemann hypothesis is not easily understood without appeal to these Hilbert spaces of entire functions.

The classical motivation for the Riemann hypothesis lies in the existence of entire functions which resemble the function studied by Riemann and for which the analogue of the conjecture made by Riemann is true. The resemblance lies in relationship to the gamma function and does not extend to the zeta function. The functions were discovered in 1880 by Sonine in Fourier analysis on the complex plane. Fourier analysis is applied to functions which depend only on distance from the origin and which are square integrable with respect to Lebesgue measure for the plane. The Fourier transform of such a function is again such a function. Sonine constructed fundamental examples of such functions which vanish in a circular neighborhood of the origin and whose Fourier transform vanishes in the same neighborhood.

Scientific journalists have noticed the persistence with which the Riemann hypothesis was maintained as a research objective over half a century. An early success [1] is the explanation for tenacity. A construction was made of all square integrable functions of a radial variable which vanish in a given circular neighborhood of the origin and whose Fourier transform vanishes in the same neighborhood. A parametric solution is obtained as an application of related Hilbert spaces of entire functions. The entire functions of Hermite class which define these spaces exhibit the linear pattern of zeros which is characteristic of the Riemann hypothesis.

Twenty years were required to formulate the Riemann hypothesis for Hilbert spaces of entire functions which includes the Sonine functions in the same category as the function studied by Riemann. The intervening years have a relevance to the proof of the Riemann hypothesis which is not evident but which needs to be known by a competent reader of the proof.

The Hardy space for the unit disk is identical with the Hilbert space of power series with complex coefficient for which the sum of the squares of absolute values of coefficients converges. Treated with equal facility are Hilbert spaces of power series with coefficients in a Hilbert space such that the sum of the squares of norms of coefficients converges.

A Hilbert space \mathcal{H} whose elements are power series with coefficients in a Hilbert space \mathcal{C} is said to satisfy the inequality for difference quotients if $[f(z) - f(0)]/z$ belongs to the space whenever $f(z)$ belongs to the space and if the inequality

$$\|[f(z) - f(0)]/z\|_{\mathcal{H}}^2 \leq \|f(z)\|_{\mathcal{H}}^2 - \|f(0)\|_{\mathcal{C}}^2$$

is satisfied. The space is said to satisfy the identity for difference quotients if equality always holds. The Hilbert space $\mathcal{C}(z)$ of all square summable power series with coefficients in \mathcal{C} satisfies the identity for difference quotients.

The structure of spaces which satisfy the inequality for difference quotients was explored in joint work with James Rovnyak in ten years prior to the proof of the Bieberbach conjecture. The methods introduced apply to the Riemann hypothesis as well as to the Bieberbach conjecture.

An innovative aspect of this work is the use of Hilbert spaces which are contained contractively in a given Hilbert space as opposed to Hilbert spaces which are contained

isometrically in the space. A Hilbert space of power series with coefficients in a Hilbert space \mathcal{C} which satisfies the inequality for difference quotients is contained contractively in the Hilbert space $\mathcal{C}(z)$ of all square summable power series with coefficients in \mathcal{C} . A convex structure and a compatible topology apply to Hilbert spaces which are contained contractively in a Hilbert space. Since the convex set is compact, it is the closed convex span of its extreme points.

An existence theorem for invariant subspaces of an isometric transformation of a Hilbert space into itself is an application of the integral representation of self-adjoint transformations. A nontrivial closed subspace exists which is an invariant subspace for every continuous transformation which commutes with the given transformation if the given transformation is not a scalar multiple of the identity. The same conclusion is expected to apply to contractive transformations of a Hilbert space into itself.

A proof of the invariant subspace conjecture is made possible by joint work with James Rovnyak [5] when the extreme points can be determined of the convex set of spaces of power series which satisfy the inequality for difference quotients [2]. A space which satisfies the identity for difference quotients is known to be an extreme point of the convex set. Every extreme point is conjectured to be of this form. Since the convex set is compact, it is the closed convex span of its extreme points.

The determination of extreme points of the convex set is already interesting when the coefficients of power series are complex numbers. A Schur function is a function which is analytic and bounded by one in the unit disk. A Schur function is represented by a power series which defines a contractive multiplication of the Hilbert space of square summable power series with complex coefficients into itself. A Hilbert space of power series which satisfies the inequality for difference quotients is associated with a Schur function. The space is the state space of a canonical conjugate-isometric linear system whose transfer function is the Schur function. Since an extensive interpolation theory and estimation theory of Schur functions is associated with the spaces, it is surprising that the spaces are not applied in the proof of the Bieberbach conjecture [3].

The Bieberbach conjecture is an estimate of coefficients of a power series which represents an injective function in the unit disk. The Löwner parametrization of such Riemann mapping functions and the properties of contractively contained Hilbert spaces of entire functions reduce the estimation problem to one for Schur functions which are Riemann mapping functions for special regions. The verification of the argument at the Steklov Mathematical Institute was confirmed at the University of San Diego. An unfortunate aspect of the verification procedure was the perception that readers had simplified the argument. What occurred was deletion of a part of the estimation theory which seemed irrelevant to the proof. The omission was significant since it concerned the relationship of the proof of the Bieberbach conjecture to the Riemann hypothesis for Hilbert spaces of entire functions.

The eagerness of readers of the proof of the Bieberbach conjecture prevented the research effort required for the simplest proof. The existing proof is unnecessarily complicated because it applies Hilbert spaces of analytic functions due to Helmut Grunsky rather than the spaces naturally associated with Schur functions. These are spaces which satisfy the inequality for difference functions. The inequalities due to Lebedev and Milin should be omitted from the proof of the Bieberbach conjecture. Their purpose is to transfer estimates in Grunsky spaces to estimates in spaces satisfying the inequality for difference quotients.

The Askey–Gasper inequalities should be omitted from the proof of the Bieberbach conjecture. They apply to polynomials which appear in Grunsky spaces. The corresponding polynomials which appear in spaces satisfying the inequality for difference quotients are those introduced by Hardy as elementary functions for which an analogue of the Riemann hypothesis is true. These polynomials resemble the Sonine functions in producing examples of the Riemann hypothesis for Hilbert spaces of entire functions. Motivation for the proof of the Riemann hypothesis was supplied by David Trutt when he constructed plane measures with respect to which the Newton interpolation polynomials are orthogonal. These measures are constructed by an iterated application of the Riemann hypothesis for Hilbert spaces of entire functions as it applies to the polynomials introduced by Hardy [6].

A status report on the proof of the Bieberbach conjecture was presented in a seminar at the Ben Gurion University of the Negev in July and August 1997. The offer of joint work by Daniel Alpay was declined in view of the incomplete status of the proof of the Riemann hypothesis.

The Riemann hypothesis for Hilbert spaces of entire functions was presented at the retirement celebration of Wolfgang Fuchs at Cornell University in March 1985. Yang Lo of Academia Sinica Beijing was encouraged by the talk to send a talented graduate student for a doctoral thesis on the Riemann hypothesis. Xian–Jin Li arrived at Purdue University in the fall semester 1986 and was the principal student in graduate courses on the Riemann hypothesis until his graduation in the spring semester 1993. The thesis prepares a new proof of a known case of the Riemann hypothesis obtained by André Weil.

The Riemann hypothesis proved by Weil resembles the classical Riemann hypothesis in its relationship to Fourier analysis but differs in the context to which Fourier analysis is applied. Although the Fourier analysis applied by Weil is more elementary than the Fourier analysis applied by Riemann, it is equally obscure from the perspective of the Riemann hypothesis for Hilbert spaces of entire functions. The methods applied by Li have yet to give a new proof of the Weil theorem.

Doubts about the Riemann hypothesis for Hilbert spaces of entire functions appeared when expected proofs in Fourier analysis failed to materialize. An acknowledgement is here given to Yashowanto Ghosh who replaced Li for six years as the principal graduate student in courses on the Riemann hypothesis. Yet he declined doctoral research on the Riemann hypothesis in favor of a thesis in mathematical statistics. In the fall of 2004 he returned to Purdue University for a postdoctoral year in which an alternative to the Riemann hypothesis for Hilbert spaces of entire functions was discussed and a preliminary manuscript was produced.

The seminar on the Riemann hypothesis continued with Victor Katznelson, a visitor from the Weizmann Institute of Science, in the fall semester 2005. Here was someone who could understand the proposed proof of the Riemann hypothesis if anyone could. When he failed to understand the argument despite repeated efforts, it became clear that the manuscript is unreadable.

Seminar discussions showed that alternatives to the Riemann hypothesis for Hilbert spaces of entire functions were not convincing as a proof of the Riemann hypothesis. They are technically too difficult for a skeptical audience. A renewed effort to apply the Riemann hypothesis for Hilbert spaces of entire functions began when the semester was over.

The Riemann hypothesis for Hilbert spaces of entire functions was conjectured to apply

to suitable zeta functions. The desired zeta functions were expected to appear in Fourier analysis on quaternions. These zeta functions were conjectured to be identical with those constructed from modular forms.

These conjectures fail their purpose without a relationship between zeta functions to which the Riemann hypothesis for Hilbert spaces of entire functions applies and zeta functions to which it does not. This makes an additional conjecture. More than three years were required to verify the conjectures which together give a proof of the Riemann hypothesis.

The manuscript is difficult to read because of the large class of zeta functions treated. A proof of the Riemann hypothesis is proposed for special zeta functions which are sufficient for application to the Euler zeta function and to Dirichlet zeta functions. The proof of the Riemann hypothesis originates in examples of Hilbert spaces of entire functions derived from the work of Sonine. The argument which applies in this case can be generalized by changing the context in which Fourier analysis is applied. A proof of the Riemann hypothesis is expected which is readable to James and Virginia Rovnyak [8] as well as to Victor Katznelson, who are acquainted with the work of Sonine.

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<http://www.math.purdue.edu/~branges/site//Papers> Riemann Zeta functions