The Riemann Hypothesis

Project summary

The Riemann hypothesis is a conjecture about the zeros of a particular entire function which was made at a time when polynomials were known to be essentially determined by their zeros, when information was available about the zeros of special polynomials expressed as hypergeometric functions, and when applications of these polynomials were known in Fourier analysis. The classical approach to the Riemann hypothesis searches for properties of entire functions which generalize those of polynomials in the expectation that they will apply to the entire function treated by Riemann. A progress report is given on this approach.

The first step was made by Hermite when he discovered a class of entire functions which are essentially determined by their zeros. The Riemann hypothesis is the conjecture that the entire function treated by Riemann belongs to the Hermite class.

Entire functions of Hermite class have a chosen half–plane free of zeros. The upper half–plane is best chosen for applications to the Riemann hypothesis. Polynomials which have no zeros in the upper half–plane belong to the Hermite class. The Riemann hypothesis presumes that it is interesting for an entire function to have the upper half–plane free of zeros. The interest of this property is clear in examples of polynomials computed as hypergeometric series. The interest of a zero–free upper half–plane was clarified for general polynomials by Stieltjes.

The contribution of Stieltjes is a structure theory for nonnegative linear functionals on polynomials. A linear functional on polynomials is said to be nonnegative if it has nonnegative values on polynomials whose values on the real axis are nonnegative. A nonnegative linear functional on polynomials is shown to be represented as a Stieltjes integral with respect to a nondecreasing function of a real variable. When the nonnegative linear functional is applied to polynomials of degree less than r, then the nonnegative linear functional is represented as an integral with respect to a nondecreasing step–function with at most r jumps. The computation of such step functions applies the properties of polynomials of degree r which have no zeros in the upper half–plane.

Stieltjes did not make a generalization to entire functions of Hermite class since he died from tuberculosis. An extensive generalization of the representation of nonnegative linear functionals on polynomials is made by Hilbert.

Hilbert introduces the study of invariant subspaces for contractive transformations of a Hilbert space into itself. Polynomials are applied by Gauss in the construction of invariant subspaces when the space has finite dimension. Hilbert shows that an isometric transformation admits nontrivial invariant subspaces if it is nontrivial. Subspaces are constructed which are invariant subspaces for all continuous transformations which commute with the given isometric transformation. When the transformation has an isometric inverse, it has an integral representation as a Stieltjes integral of projections into invariant subspaces.

Hilbert and his students were frustrated in their attempts to prove the Riemann hypothesis because they overlooked an application of complex analysis which was found by Hardy.

The Hardy space for the unit disk is the Hilbert space of analytic functions which are represented in the unit disk by square summable power series. The Hardy space for the upper half–plane is fundamental to an understanding of the Riemann hypothesis.

The application of square summable power series to the construction of invariant subspace is due to Beurling. Invariant subspaces for some contractive transformations are constructed from the factorization theory of functions which are analytic and bounded by one in the unit disk. The generalization of the construction to all contractive transformations is preparation for research on the Riemann hypothesis.

In 1965 an announcement was made with inadequate proof and in joint work with James Rovnyak that a nontrivial contractive transformation of a Hilbert space into itself admits a nontrivial invariant subspace. The completed proof appears in lecture notes on square summable power series [5] which serve as an introduction to the methods applied to the Riemann hypothesis. The subspace constructed is an invariant subspace for every continuous transformation which commutes with the given contractive transformation.

The relationship between factorization and invariant subspaces as formulated by M.S. Livšic is the foundation of a theory of linear systems which has a major impact on society in its applications to engineering. Valuable applications are found even when the state space has finite dimension and is not equipped with a scalar product. The lecture notes on square summable power series construct a natural scalar product in all cases of a satisfactory relationship between factorization and invariant subspaces. The scalar products need not be positive. The generalization is relevant to the Riemann hypothesis when a hidden singularity appears, as it does for the entire function treated by Riemann. The scalar product is nearly positive since the Pontryagin index of negativity is one.

Weighted Hardy spaces of functions analytic in the upper half–plane are applied to the Riemann hypothesis. An analytic weight function is a function $W(z)$ of z which is analytic and without zeros in the upper half–plane. The weighted Hardy space $\mathcal{F}(W)$ is the set of functions $F(z)$ of z which are analytic in the upper half–plane such that the least upper bound

$$\|F\|^2_{\mathcal{F}(W)} = \sup \int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 dx$$

taken over all positive y is finite. The weighted Hardy space is a Hilbert space of functions analytic in the upper half–plane which is characterized by two properties: If w is in the upper half–plane, multiplication by

$$(z-w)/(z-w^-)$$

is an isometric transformation of the space onto the subspace of functions which vanish at w. A continuous linear functional on the space is defined when w is in the upper half–plane by taking a function $F(z)$ of z into its value $F(w)$ at w.

The unweighted Hardy space for the upper half–plane is obtained when $W(z)$ is identically one. If $W(z)$ is an analytic weight function, multiplication by $W(z)$

is an isometric transformation of the unweighted Hardy space onto the weighted Hardy space $\mathcal{F}(W)$. The function

$$W(z)W(w)^-/[2\pi i(w^- - z)]$$

of z belongs to the weighted Hardy space when w is in the upper half–plane and acts as reproducing kernel function for function values at w.

The Stieltjes representation of positive linear functionals on polynomials is clarified by the use of weighted Hardy spaces. The conjugate

$$F^*(z) = F(z^-)^-$$

of an entire function $F(z)$ of z is an entire function whose values on the real axis are complex conjugates of the values of the given entire function. The conjugate of a polynomial is a polynomial of the same degree. A polynomial has nonnegative values on the real axis if, and only if, it is a product

$$F^*(z)F(z)$$

of a polynomial $F(z)$ and its conjugate. If a nonnegative linear functional on polynomials annihilates the product for no nontrivial polynomial $F(z)$ of degree less than r, then the polynomials of degree less than r become a Hilbert space with scalar product $\langle F, G \rangle$ for polynomial $F(z)$ and $G(z)$ defined by the action of the nonnegative linear functional on the polynomial

$$G^*(z)F(z).$$

In substance the Stieltjes theorem states that the Hilbert space is contained isometrically in the weighted Hardy space $\mathcal{F}(W)$ for some polynomial $W(z)$ of degree r which has no zeros in the upper half–plane.

The generalization of the Stieltjes spaces of polynomials to Hilbert spaces of entire functions permits another formulation of the Riemann hypothesis. The spaces are characterized by these properties:

(H1) Whenever an entire function $F(z)$ of z belongs to the space and has a nonreal zero w, the entire function

$$F(z)(z - w^-)/(z - w)$$

of z belongs to the space and has the same norm as $F(z)$.

(H2) A continuous linear functional is defined on the space for every nonreal number w by taking an entire function $F(z)$ of z into its value $F(w)$ at w.

(H3) The entire function $F^*(z)$ of z belongs to the space whenever an entire function $F(z)$ of z belongs to the space, and it always has the same norm as $F(z)$.

A Hilbert space $\mathcal{H}(E)$ of entire functions which satisfies the axioms is defined by an entire function $E(z)$ of z which satisfies the inequality

$$|E(x - iy)| < |E(x + iy)|$$

for all real x when y is positive. The condition implies that the function has no zeros in the upper half–plane. The space $\mathcal{H}(E)$ is contained isometrically in the weighted Hardy space $\mathcal{F}(W)$ with analytic weight function

$$W(z) = E(z).$$

The elements of the space $\mathcal{H}(E)$ are the entire functions $F(z)$ of z which belong to the weighted Hardy space and whose conjugate $F^*(z)$ belongs to the space.

The entire function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

of z belongs to the space $\mathcal{H}(E)$ for every complex number w and acts as reproducing kernel function for function values at w. A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains a nonzero element is isometrically equal to a space $\mathcal{H}(E)$ for some entire function $E(z)$ of z which satisfies the inequality

$$|E(x - iy)| < |E(x + iy)|$$

for all real x when y is positive. The entire function $E(z)$ can be chosen to have a given zero in the lower half–plane. The function is then unique within a constant factor of absolute value one [1].

The Riemann hypothesis is the conjecture that a given entire function is the defining function of a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). The application of the conjecture to the estimated number of primes less than a given positive number is an application of the properties of the resulting Hilbert space of entire functions.

Riemann did not state his motivation for the conjecture. The classical motivation for the Riemann hypothesis lies in properties of the Fourier transformation which were observed in 1880 by N. Sonine. These aspects of Fourier analysis are clarified by the construction of Hilbert spaces of entire functions.

Sonine observes that a nontrivial function which vanishes in a neighborhood of the origin can have a Fourier transform which vanishes in a neighborhood of the origin. This property applies in Fourier analysis on a line but is more relevant in Fourier analysis on a plane. A fundamental problem is to determine all square integrable functions in the plane which vanish in a given circular neighborhood of the origin and whose Fourier transform vanishes in the same neighborhood. The set of all such functions is invariant under the transformation which takes a function $f(z)$ of z in the plane into the function $f(\omega z)$ of z in the plane for every element ω of the plane of absolute value one. The set of all such functions is a Hilbert space which is the orthogonal sum of invariant subspaces. An invariant subspace is defined for every integer n as the set of functions which satisfy the identity

$$f(\omega z) = \omega^n f(z)$$

for every complex number ω of absolute value one. Attention is drawn to spaces parametrized by nonnegative integers n since other spaces are obtained by conjugation. A closer relationship to analytic function theory results when n is nonnegative.

The examples given by Sonine apply for every nonnegative integer n. For every positive number a nontrivial functions exist which vanish in the disk $|z| < a$ and whose Fourier transform vanishes in the disk. A determination of all such functions is made by the construction of a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). The space is isometrically equal to a space $\mathcal{H}(E(a))$ for an entire function $E(a, z)$ which is computed as a confluent hypergeometric function when n is zero [2].

A resemblance to the entire function treated by Riemann exists since the entire functions

$$E^*(a, z)$$

and

$$E(a, z - i)$$

of z are linearly dependent. The expectation created by the Riemann hypothesis is that the zeros of $E(a, z)$ lie on a horizontal line

$$z - z^- = -i$$

at distance one–half below the real axis. Motivation for the Riemann hypothesis results from the computation of zeros showing that they lie on the expected line.

The Sonine spaces of entire functions are constructed from the analytic weight function

$$W(z) = \Gamma(\tfrac{1}{2} - iz)$$

which is defined from the Euler gamma function. Multiplication by a^{-iz} is an isometric transformation of the space $\mathcal{H}(E(a))$ into the weighted Hardy space $\mathcal{F}(W)$. The space $\mathcal{H}(E(a))$ contains every entire function $F(z)$ of z such that the functions

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

of z belong to the space $\mathcal{F}(W)$.

The hypergeometric series is a generalization of the geometric series which is motivated by the binomial expansion and which was discovered by Euler subsequent to his discovery in 1730 of the gamma function. These special functions are an outgrowth of the infinitessimal calculus when it is treated as a limit of the calculus of finite differences, as it was by Newton. The gamma function is a limiting case of the Newton interpolation polynomials.

In the notation of Weierstrass the gamma function is a function $\Gamma(s)$ of a complex variable s which is analytic in the complex plane with the exception of singularities at the nonpositive integers and which satisfies the recurrence relation

$$s\Gamma(s) = \Gamma(s + 1).$$

Since

$$\Gamma(1) = 1.$$

$\Gamma(1+s)$ is a generalization of $s!$. The gamma function is a good solution of the recurrence relation is a sense essential to the Riemann hypothesis.

The entire function treated by Riemann is a product of two factors, of which one is constructed from the gamma function and the other from the classical zeta function. The classical zeta function was discovered by Euler in 1737 as a product which resembles the product for the gamma function. Euler also discovered the functional identity for the product of gamma functions and zeta functions as an application of the theory of hypergeometric series. A proof of the functional identity in Fourier analysis was later found as an application of the Poisson summation formula. Fourier analysis confirms the classical zeta function as an analogue of the gamma function.

The history of the Riemann hypothesis is fragmented by political upheavals which began with the French Revolution and did not end with the Russian Revolution. Facility in complex analysis resulting from the Cauchy formula is not achieved without diminished appreciation of the merits of hypergeometric functions. The consolidation of Fourier analysis made possible by Lebesgue integration conceals the applications of Stieltjes integration. Fourier analysis is enhanced by its extension to commutative groups whose topology sustains integration, but a problem of purpose for Fourier analysis appears in applications. An accumulation of obstacles hinders applications to the Riemann hypothesis.

The Fourier analysis of the Riemann hypothesis is made on fields or their products, with the exception that fields can be advantageously replaced by skew–fields. The Fourier transformation is an isometric transformation of the Hilbert space of square integrable functions with respect to the invariant measure onto itself.

A fundamental problem remains of determining all square integrable functions which vanish in a given neighborhood of the origin and whose Fourier transform vanishes in the same neighborhood. The neighborhoods must be invariant under multiplication by elements ω of the product ring for which multiplication by ω preserves the invariant measure. When the group of such elements is commutative, the Hilbert space of square integrable functions decomposes into subspaces of functions $f(\xi)$ of ξ which satisfy the identity

$$f(\omega\xi) = \chi(\omega) f(\xi)$$

for some character χ of the group. The group is noncommutative when skew–fields are used. Hecke operators then decompose the representation into irreducible representations. Zeta functions appear in the decomposition in both cases.

A solution of the problem is given in a general context by the construction of a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). The entire function treated by Riemann appears in the defining function of such a space.

In this formulation the Riemann hypothesis is treated as an issue in the theory of Hilbert spaces of entire functions as they appear in Fourier analysis. A hypothesis on Hilbert spaces of entire functions is required for the desired location of zeros of defining functions. Properties of Fourier analysis prepare the needed hypothesis.

In the introduction to Fourier analysis Fourier gave convincing applications to the flow of heat. The treatment of heat flow is an original contribution to dynamics since energy is lost whereas motion is otherwise conceived as preserving energy.

In the language of invariant subspaces Fourier introduces contractive transformations in a context where isometric transformations were previously conceived as relevant. In the language of dynamics Fourier applies dissipative transformations in a context where skew–adjoint transformations were seen as relevant.

A dissipative transformation is a linear transformation T with domain and range in a Hilbert space such that the real part of the scalar product

$$\langle Tf, f\rangle$$

is nonnegative for every element f of the domain of the transformation. A skew–adjoint transformations is a dissipative transformation such that the real part of the scalar product always vanishes. Since the definition admits transformations with small domain, it needs to be supplemented by the requirement that the dissipative transformation is maximal: The transformation cannot be properly extended as a dissipative transformation.

The infinitessimal generator for the flow of heat is a maximal dissipative transformation. Since the infinitessimal generator is a differential operator, heat flow would seem to be restricted to Fourier analysis on commutative groups which admit a differentiable structure. Fortunately for the Riemann hypothesis this is not the case. Other groups appear.

The inverse of the infinitessimal generator for the flow of heat is an integral transformation which is maximal dissipative. When the flow of heat is treated on a plane rather than on a line, the integral transformation has properties which were observed by Radon.

The Radon transformation permits the Fourier transformation for the plane to be treated formally as a composition with the Fourier transformation for a line. The transformation averages a function of two variables with respect to one variable so as to produce a function of the other variable. Rotations about the origin recover functions of two variables. The Radon transformation is a tool of applied mathematics which may seem to lack the rigor required for pure mathematics. But it has a rigorous meaning in Fourier analysis on any group which is the Cartesian product of two commutative groups.

An example of a context in which the flow of heat is meaningful is given by a quadratic extension of the p–adic numbers. In applications to the Riemann hypothesis a choice needs to be made since the field of p–adic numbers has two quadratic extensions. An unramified extension is obtained by adjoining the square root of a unit. A ramified extension is obtained by adjoining a square root of the prime p. Ramified extensions are required for application to the Euler zeta function.

The Laplace transformation was introduced by Fourier for a spectral theory of the infinitessimal generator in the flow of heat. An analogous treatment of heat flow applies in any context in which the Radon transformation is meaningful. A Laplace transformation applies on any quadratic extension of the p–adic numbers. And it applies in product spaces constructed from the complex plane and the p–adic plane for every prime p. The image of the infinitessimal generator for the flow of heat under the Laplace transformation is a multiplication operator in the space of square integrable functions with respect to a nonnegative measure. The infinitessimal generator for the flow of heat is a subnormal operator which is diagonalized by the Laplace transformation. The maximal dissipative property of the infinitessimal generator implies that multiplication is by a function whose values have nonnegative real part.

The Hilbert spaces of entire functions which appear in Fourier analysis have a property resulting from the maximal dissipative property of the infinitessimal generator for the flow of heat. It has been seen that such a space satisfies the axioms (H1), (H2), and (H3) and contains a nonzero element. The space is then isometrically equal to a space $\mathcal{H}(E)$ for an entire function $E(z)$ of z which satisfies the inequality

$$|E(x - iy)| < |E(x + iy)|$$

for all real x when y is positive. A maximal dissipative transformation is defined in the space by taking $F(z)$ into $F(z + i)$ whenever the functions of z belong to the space.

The special property of a Hilbert space of entire functions implies that the defining function $E(z)$ of the space admits no pair of distinct zeros w and $w^- - i$ which are symmetric about a horizontal line at distance one–half below the real axis. All zeros lie on the line when the functions $E(z + i)$ and $E^*(z)$ are linearly dependent.

The property is satisfied by the spaces which appear in Fourier analysis for the complex plane and which supply the motivation for the Riemann hypothesis. It gives a new proof that zeros lie on the expected line. For this reason the property has been called the Riemann hypothesis for Hilbert spaces of entire functions [3].

Despite its name the Riemann hypothesis for Hilbert spaces of entire functions is not applicable in the original context of Riemann. The condition is incompatible with the singularity of the Euler zeta function. A variant of the condition is required to circumvent the obstacle created by the singularity. Those who are familiar with indefinite scalar products will have no difficulty in reading the modification.

The singularity of the Euler zeta function is not a substantial obstacle to a proof of the Riemann hypothesis. There is however an obstacle which is substantial: The Euler zeta function is not a Hecke zeta function. The Riemann hypothesis for Hilbert spaces of entire functions applies in Fourier analysis on a plane, not in Fourier analysis on a line. This obstacle is circumvented by the relationship between Fourier analysis on a plane and Fourier analysis on a line.

The relationship is seen in the Euler duplication formula

$$2^{s-1}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s + \tfrac{1}{2}) = \Gamma(\tfrac{1}{2})\Gamma(s)$$

for the gamma function. An analogous construction is made of a Hecke zeta function from the Euler zeta function. The Riemann hypothesis for the Hecke zeta function implies the Riemann hypothesis for the Euler zeta function.

A proof of the Riemann hypothesis for Hecke zeta functions appears in a preprint [4]. The argument is not in final form since it contains expository material which is not publishable. Some exposition needs to be relegated to lecture notes. The argument needs to be presented in seminar in order to find the presentation which is both sufficiently terse and sufficiently detailed for a mature reader. Who such readers are and what needs they have is difficult to foresee since the argument spans the fields of complex analysis, Fourier analysis, and functional analysis. Some exposition may be desirable in publication.

An alternative to the Riemann hypothesis for Hilbert spaces of entire functions is formulated in weighted Hardy spaces. A maximal dissipative transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space if, and only if, the analytic weight function $W(z)$ admits an extension as an analytic function of z in the half–plane

$$iz^{-} - iz > -1$$

such that the recurrence relation

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z)$$

holds for a function $\phi(z)$ of z which is analytic and has nonnegative real part in the upper half–plane.

The recurrence relation is satisfied with

$$\phi(z) = -iz$$

when

$$W(z) = \Gamma(\tfrac{1}{2} - iz).$$

A good solution of the recurrence relation is given by the gamma function since a maximal dissipative transformation is defined when h belongs to the interval $(0, 1)$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space.

A good solution of the recurrence relation is available generally [6]. If an analytic weight function $W(z)$ is a solution of the recurrence relation for some function $\phi(z)$ which is analytic and has nonnegative real part in the upper half–plane, then an entire function $S(z)$ which is periodic of period i and has no zeros exists such that the analytic weight function

$$S(z)W(z)$$

is a good solution of the recurrence relation.

Good solutions of the recurrence relation determine generalizations of the gamma function. The Riemann hypothesis can be viewed as the conjecture that the entire function treated by Riemann is a generalization of the gamma function.

The Riemann hypothesis for Hilbert spaces of entire functions was presented at the retirement celebration of Wolfgang Fuchs from Cornell University in 1985. As thesis advisor he guided the first project on the Riemann hypothesis. Yang Lo of Academia Sinica Beijing was inspired by the lecture to send a talented graduate student to Purdue University for research on the Riemann hypothesis. In 1992 Xian-Jin Li completed a thesis on an analogue of the Riemann hypothesis which was proved by André Weil.

Since Li was unable to apply the Riemann hypothesis for Hilbert spaces of entire functions to a new proof of the Weil theorem, he concluded that he had been sent on a wild goose chase. Instead he killed the goose that laid the golden egg. He made computer calculations at the American Institute of Mathematics which are generally accepted by reviewers as invalidating the Riemann hypothesis for Hilbert spaces of entire functions. Since these calculations are made in joint work with an expert in computing, they may be correct. If they are, they confirm the belief that the Riemann hypothesis for Hilbert spaces of entire functions is not applicable in Fourier analysis on a line.

Those who make computer calculations need to guard themselves against misreading of their work. They must also make certain that they do not themselves draw false conclusions from it.

A proof of the Weil theorem as an application of the Riemann hypothesis for Hilbert spaces of entire functions remains as an interesting project. Zeta functions constructed in Fourier analysis on a line need to be related to zeta functions constructed in Fourier analysis on a plane for the application. Comparison with the Weil argument should be made.

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