A PROOF OF THE RIEMANN HYPOTHESIS II

Louis de Branges*

Abstract. The Riemann hypothesis originates as a conjecture concerning the zeros of the classical zeta function which permits an estimate of the error in the Gauss and Legendre approximations to the number of primes with a given bound. The proof of the Riemann hypothesis applies to a large class of zeta functions which resemble the classical zeta function in Euler product and functional identity. The meaning of the Riemann hypothesis is clarified by the proof. The proof applies properties of a maximal dissipative transformation which originates in Fourier analysis on Cartesian space. Properties of Cartesian space are applied which are related to the orbital motion of planets about the sun and of electrons about a nucleus. Prime numbers appear in Fourier analysis on Cartesian space when it is applied to the motion of a particle having the symmetries of a cube.

Fourer analysis on Cartesian space applies properties of skew–fields whose elements are hypercomplex numbers. The initial skew–field contains linear combinations

$$\xi = \delta + i\alpha + j\beta + k\gamma$$

of basic hypercomplex numbers i, j, k, and 1 with rational numbers α, β, γ, and δ as coordinates. The conjugate hypercomplex number is

$$\xi^- = \delta - i\alpha - j\beta - k\gamma.$$

The multiplication of hypercomplex numbers is determined by the identities

$$ij = k, \ \ jk = i, \ \ ki = j$$

and the requirement

$$(\xi\eta)^- = \eta^-\xi^-$$

making conjugation an anti–automorphism. The hypercomplex numbers with rational coordinates form an associative algebra with 1 as unit. Since a self–conjugate hypercomplex number is a rational multiple of the unit, it commutes with every hypercomplex number. Every nonzero hypercomplex number has an inverse.

* Research supported by the National Science Foundation

Polynomials are applied as finite linear combinations

$$\xi_0 t^r + \xi_1 t^{r-1} + \ldots + \xi_r$$

of nonnegative integral powers of t whose coefficients are hypercomplex numbers $\xi_0, \ldots, \xi_r$ with rational coordinates. Since the indeterminate t is self–conjugate, it commutes with every hypercomplex number. A self–conjugate polynomial commutes with every polynomial since it has rational coefficients. A polynomial with hypercomplex coefficients is a linear combination

$$d(t) + ia(t) + jb(t) + kc(t)$$

of polynomials $a(t), b(t), c(t)$, and $d(t)$ with rational coefficients. The polynomials form an associative algebra with one as unit. Conjugation is an anti–automorphism of the algebra. The degree of a nonzero polynomial is the largest exponent on t which appears with a nonzero coefficient. The product of two nonzero polynomial is a nonzero polynomial whose degree is the sum of the degrees of the factors.

The Euclidean algorithm applies to polynomials with hypercomplex coefficients. If $a(t)$ is a polynomial and if $b(t)$ is a nonzero polynomial, then polynomials $c(t)$ and $d(t)$ exist such that

$$a(t) = b(t)c(t) + d(t)$$

and such that the degree of $d(t)$ is less than the degree of $b(t)$ if $d(t)$ is nonzero. The proof of the Euclidean algorithm for polynomials with hypercomplex coefficients is an application of the Euclidean algorithm for polynomials with rational coefficients. Since the nonzero polynomial

$$b(t)^- b(t)$$

has rational coefficients and since the polynomial

$$b(t)^- a(t)$$

is a linear combination of i, j, k, and 1 with polynomials whose coefficients are rational, polynomials $c(t)$ and $r(t)$ with hypercomplex coefficients exist such that

$$b(t)^- a(t) = b(t)^- b(t)c(t) + r(t)$$

and such that the degree of $r(t)$ is less than the degree of $b(t)^- b(t)$ if $r(t)$ is nonzero. Since the polynomial

$$d(t) = a(t)b(t) - c(t)$$

satisfies the identity

$$r(t) = b(t)^- d(t),$$

the degree of $d(t)$ is less than the degree of $b(t)$ if $d(t)$ is nonzero.

The Euclidean algorithm determines the structure of the right ideals in the algebra of polynomials with hypercomplex coefficients. A right ideal which contains a nonzero element contains a nonzero element $b(t)$ of least degree. If $a(t)$ belongs to the ideal, a polynomial $c(t)$ exists such that the degree of the polynomial

$$d(t) = a(t) - b(t)c(t)$$

is less than the degree of $b(t)$ if $d(t)$ is nonzero. Since $d(t)$ belongs to the ideal, it is zero. The elements of the ideal are the products

$$b(t)c(t)$$

of the generator $b(t)$ and polynomials $c(t)$ with hypercomplex coefficients.

A proper right ideal of the algebra of polynomials with hyperconvex coefficients which is closed under conjugation is also a left ideal. The quotient space of the polynomial algebra is an algebra which inherits a conjugation. If the ideal is a maximal proper right ideal, the quotient algebra is a skew–field. A proper ideal which is closed under conjugation and which is maximal among proper ideals is generated by a nonconstant polynomial $r(t)$ with rational coefficients. Polynomials $c_1(t), \ldots, c_n(t)$ belong to the ideal whenever the nonnegative polynomial

$$c_1^-(t)c_1(t) + \ldots + c_n^-(t)c_n(t)$$

belongs to the ideal. The quotient algebra is the quotient algebra of an algebra of rational functions. A rational function which belongs to the algebra is a ratio

$$\frac{f(t)}{g(t)}$$

of a polynomial $f(t)$ with hypercomplex coefficients and a polynomial $g(t)$ with rational coefficients which is not divisible by $r(t)$. The rational functions form an algebra which contains the polynomial algebra and which admits a conjugation extending the conjugation of the polynomial algebra. Elements

$$\frac{f_1(t)}{g_1(t)}$$

and

$$\frac{f_2(t)}{g_2(t)}$$

of the algebra are considered equivalent if the polynomial

$$f_1(t)g_2(t) - f_2(t)g_1(t)$$

is divisible by $r(t)$. The quotient space is an algebra of rational functions canonically isomorphic to the quotient algebra of the algebra of polynomials.

A polynomial with rational coefficients is said to be nonnegative if it is a sum of polynomials $f(t)^-f(t)$. A rational function

$$\frac{f(t)}{g(t)}$$

is said to be nonnegative if the polynomials $f(t)$ and $g(t)$ are nonnegative. Sums and products of nonnegative rational functions are nonnegative. A nonnegative rational function with numerator $f(t)$ and denominator $g(t)$ is said to be positive if a positive number ϵ exists such that the functions defined by the polynomials $f(t)$ and $g(t)$ if the polynomial

$$f(t) - \epsilon g(t)$$

is nonnegative.

A convex combination

$$f(t)[1 - h(t)] + g(t)h(t)$$

of rational functions $f(t)$ and $g(t)$ is defined by a nonnegative rational function $h(t)$ such that $1 - h(t)$ is nonnegative. A set of rational functions is said to be convex if it contains all convex combinations of $f(t)$ and $g(t)$ whenever $f(t)$ and $g(t)$ belong to the set. The convex span of a subset of the algebra of rational functions is the smallest convex subset of the algebra which contains the given set.

A topology on the quotient space of the algebra of rational functions is defined from a definition of closure for convex sets. A set of rational functions is said to be saturated with respect to the equivalence relation if every rational function which is equivalent to an element of the set is an element of the set. The saturated sets of rational functions are the inverse images of subsets of the quotient space under the canonical projection of the algebra onto the quotient space. The topology of the quotient space is defined from a definition of closure for saturated convex subsets of the algebra of rational functions.

A rational function $s(t)$ belongs to the closure of a saturated convex set C of rational functions if the set of rational functions which contains $s(t)$ and the elements of C is convex. The topology of the quotient space is defined from a definition of closure for saturated convex subsets of the algebra of rational functions.

A rational function $s(t)$ belongs to the closure of a saturated convex set C of rational functions if the set of rational functions which contains $s(t)$ and the elements of C is convex. The closure of a saturated convex set of rational functions is a saturated convex set of rational functions which is equal to its closure.

A nonempty saturated convex set of rational functions is said to be a disk if it is disjoint from the closure of every disjoint saturated convex set of rational functions. A nonempty saturated convex set A of rational functions is a disk if for every element $a(t)$ of A and every rational function $b(t)$, a positive rational function $h(t)$ exists such that $1 - h(t)$ is nonnegative and such that the convex combination

$$a(t)[1 - h(t)] + b(t)h(t)$$

belongs to A. If C is a saturated convex set of rational functions, the intersection of A with the closure of C is contained in the closure of the intersection of A with C.

The construction of disks has an application. If a rational function $a(t)$ does not belong to a saturated convex set B of rational functions, then $a(t)$ belongs to a disk A, which is disjoint from B, such that for every saturated convex set C of rational functions the intersection of A with the closure of C is contained in the closure of the intersection of A with C.

Every disk has the same property. If A is a disk and if C is a saturated convex set of rational functions, then the intersection of A with the closure of C is contained in the closure of the intersection of A with C. For an element of A which does not belong to the closure of the intersection of A with C belongs to a disk B, which is disjoint from the intersection of A with C, such that the intersection of B with the closure of C is contained in the closure of the intersection of B with C. Since A is disjoint from the intersection of B with C, A is disjoint from the closure of the intersection of B with C. Since A is disjoint from the intersection of B with the closure of C, the intersection of A and B is disjoint from the closure of C.

An application is made to the proof that the intersection of disks A and B is a disk if it is nonempty. The intersection of A and B is a saturated convex set. If a saturated convex set C is disjoint from the intersection of A and B, then the intersection of B with the closure of C is contained in the closure of the intersection of B with C. Since the intersection of B with C is a saturated convex set which is disjoint from A, A is disjoint from the closure of the intersection of B with C. Since A is disjoint from the intersection of B with the closure of C, the intersection of A and B is disjoint from the closure of C. The quotient space is a Hausdorff space whose open sets are determined by the unions of disks. A computation of topology is possible.

If C is a saturated convex set of rational functions and if $s(t)$ is a rational function, a saturated convex set $C(s)$ is constructed whose closure contains $s(t)$ and the elements of C. The set $C(s)$ is defined as the set of rational functions which are equivalent to convex combinations

$$s(t)[1 - h(t)] + c(t)h(t)$$

with $c(t)$ in C and $h(t)$ a positive rational function such that $1 - h(t)$ is positive. It is sufficient by a translation to prove convexity when $s(t)$ is the origin. A convex combination

$$a(t)p(t)[1 - h(t)] + b(t)q(t)h(t)$$

of elements

$$a(t)p(t)$$

and

$$b(t)q(t)$$

of $C(s)$ is defined by a nonnegative rational function $h(t)$ such that $1-h(t)$ is nonnegative. Since $p(t)$ and $q(t)$ are positive rational functions, a nonnegative rational function $k(t)$ such that $1 - k(t)$ is positive is defined by the equivalent equations

$$p(t)[1 - h(t)]k(t) + q(t)h(t)k(t) = q(t)h(t)$$

and

$$p(t)[1 - h(t)][1 - k(t)] + q(t)h(t)[1 - k(t)] = p(t)[1 - h(t)].$$

The convex combination

$$c(t) = a(t)[1 - k(t)] + b(t)k(t)$$

of elements $a(t)$ and $b(t)$ of C is an element of C. Since $1 - p(t)$ and $1 - q(t)$ are nonnegative, the sum of the positive rational functions

$$p(t)[1 - h(t)] + q(t)h(t)$$

and

$$[1 - p(t)][1 - h(t)] + [1 - q(t)]h(t)$$

is one. The convex combination

$$a(t)p(t)[1 - h(t)] + b(t)q(t)h(t) = c(t)p(t)[1 - h(t)] + c(t)q(t)h(t)$$

is an element of $C(s)$.

The Hahn–Banach theorem states that a disk which is disjoint from a saturated convex set is contained in a disk which has convex complement and which is disjoint from the convex set. The proof is an application of the Kuratowski–Zorn lemma.

A nonempty saturated convex set of rational functions which is disjoint from a disk A is contained in a maximal saturated convex set B of rational functions which is disjoint from A. Since the closure of B is a saturated convex set which contains B and is disjoint from A, the closure of B is equal to B. If a rational function $s(t)$ does not belong to B, then the closure of the saturated convex set $B(s)$ is a saturated convex set which properly contains B. Since the closure of $B(s)$ is not disjoint from A, $B(s)$ is not disjoint from A. The complement of B is shown to be a disk by showing that it is convex.

If rational functions $u(t)$ and $v(t)$ belong to the complement of B, the saturated convex sets $B(u)$ and $B(v)$ contain elements of A. A rational function $s(t)$ which is a convex combination of $u(t)$ and $v(t)$ is shown to lie in the complement of B by showing that the saturated convex set $B(s)$ contains an element of A. It can by a translation be assumed that $s(t)$ is the origin. An element of A which belongs to $B(s)$ is constructed as a convex combination of an element

$$[1 - p(t)]u(t) + p(t)a(t)$$

of A which belongs to $B(u)$ and an element

$$[1 - q(t)]v(t) + q(t)b(t)$$

of A which belongs to $B(v)$. The rational functions $a(t)$ and $b(t)$ are elements of B. The rational functions $p(t)$ and $q(t)$ are positive. The rational functions $1 - p(t)$ and $1 - q(t)$ are also positive. Since $s(t)$ is a convex combination of $u(t)$ and $v(t)$, a nonnegative rational function $h(t)$ such that $1 - h(t)$ is nonnegative exists such that

$$0 = u(t)[1 - p(t)][1 - h(t)] + v(t)[1 - q(t)]h(t).$$

A nonnegative rational function $k(t)$ such that $1 - k(t)$ is nonnegative is defined by the equivalent equations

$$p(t)[1 - h(t)]k(t) + q(t)h(t)k(t) = q(t)h(t)$$

and

$$p(t)[1 - h(t)][1 - k(t)] + q(t)h(t)[1 - k(t)] = p(t)[1 - h(t)].$$

The convex combination

$$c(t) = a(t)[1 - k(t)] + b(t)k(t)$$

of elements $a(t)$ and $b(t)$ of B is an element of B. The convex combination

$$u(t)[1 - p(t)][1 - h(t)] + a(t)p(t)[1 - h(t)] + \ v(t)[1 - q(t)]h(t) + b(t)q(t)h(t)$$

of elements

$$u(t)[1 - p(t)] + a(t)p(t)$$

and

$$v(t)[1 - q(t)] + b(t)q(t)$$

of elements of A is an element of A which is a convex combination of $s(t)$ and the element $c(t)$ of B.

The Hahn–Banach theorem determines the topology of the quotient space when the structure of disks is known. If a disk A has a nonempty convex complement, then the intersection of the closure of A with the complement of A is a convex set. Since addition of a rational function is a transformation which maps disks into disks, it is sufficient to determine the structure of a disk A which contains the origin. The disk and its complement are then invariant under multiplication by positive rational functions. Since multiplication by a nonzero hypercomplex number maps disks into disks, it can be assumed that the skew–conjugate hypercomplex numbers belong to the intersection of the closure of A with the complement of A. It follows that the skew–conjugate rational functions belong to the intersection of the closure of A with the complement of A. A rational function then belongs to A if, and only if, its self–conjugate component belongs to A.

The set of rational numbers α such that $t - \alpha$ belongs to A is a convex set which does not contain every rational number since the complement of A is nonempty. The set of rational numbers β such that $t - \beta$ belongs to the complement of A is a convex set which does not contain every rational number since A is nonempty. Since multiplication by minus one maps disk into disks, it can be assumed that α is less that β whenever α and β are rational numbers such that $t - \alpha$ belongs to A and $t - \beta$ belongs to the complement of B. A unique real number γ exists which belongs to the interval $(\alpha, \beta]$ whenever α and β are rational numbers such that $t - \alpha$ belongs to A and $t - \beta$ belongs to the complement of A. A polynomial $f(t)$ belongs to A when the real part of $f(\gamma)$ is positive and belongs to the complement of A when the real part of $f(\gamma)$ is nonpositive. Since A is saturated with respect to the equivalence relation defined by the generating polynomial $r(t)$, the real number γ is a zero of the polynomial.

A computation of the disk topology of the quotient space results. The quotient space is mapped homeomorphically onto a subspace of a Cartesian product whose factors are spaces of hypercomplex numbers with real coordinates. The Cartesian product is indexed by the complex roots $\lambda_1, \dots, \lambda_\rho$ of the generating polynomial. A rational function $f(t)$ is mapped into the element

$$(f(\lambda_1), \dots, f(\lambda_\rho))$$

of the Cartesian product. Each factor space has the Euclidean topology of a vector space of finite dimension over the real numbers. Each equivalence class of rational functions contains a unique polynomial whose degree is less than the degree of the generating polynomial. Since the quotient space is a vector space over the rational numbers whose dimension is equal to the dimension of the Cartesian product space, every element of the Cartesian product space is the image of a unique element of the quotient space.

A hypercomplex number with rational coordinates is said to be integral if the coordinates are all integers or all halves of odd integers. Sums and products of integral hypercomplex numbers are integral. The conjugate of an integral hypercomplex number is integral. A polynomial whose coefficients are hypercomplex numbers with rational coordinates is said to be integral if the coordinates are integral hypercomplex numbers. Sums and products of integral polynomials are integral. The conjugate of an integral polynomial is integral.

An algebraic skew–field is the quotient space of the algebra of polynomials whose coefficients are hypercomplex numbers with rational coordinates. The quotient space is taken modulo an ideal generated by a nonconstant polynomial with real roots which is not a product of polynomials of smaller degree with rational coefficients. The definition of integral elements of the algebraic skew–field is made under additional hypotheses. A nonzero element ω of the algebraic skew–field is said to be a unit if ω^- is the inverse of ω. An element of the algebraic skew–field is said to be integral if it is a sum of units. An element of the algebraic skew–field is assumed to be ratiional if it commutes with every unit.

The Euclidean skew–plane associated with an algebraic skew–field is the completion of the algebraic skew–field in its disk topology. The Euclidean skew–plane is canonically isomorphic to a quotient algebra of the algebra of polynomials whose coefficients are hypercomplex numbers with real coordinates. An algebraic elements of the Euclidean skew–plane is an element which is represented by a polynomial whose coefficients are hypercomplex numbers with rational coordinates. The algebraic skew–field is identified with the set of algebraic elements of the Euclidean skew–plane.

An element ω of the Euclidean skew–plane is said to be a unit if its conjugate ω^- is equal to its inverse ω^{-1}. An invertible element of the Euclidean skew–plane is the product of a unit and a self–conjugate invertible element. If ω is a unit of the Euclidean skew–plane, an inner automorphism of the Euclidean skew–plane which commutes with conjugation is defined by taking ξ into

$$\omega^- \xi \omega.$$

Every automorphism of the Euclidean skew–plane which commutes with conjugation is of this form for a unit ω which is uniquely determined within a factor of a self–conjugate unit.

The adic skew–plane for an algebraic skew–field is the completion of the skew–field in a topology for which addition is continuous as a transformation of the Cartesian product of the skew–field with itself into the skew–field. A basic neighborhood of the origin for the adic topology is an ideal of the ring of integral elements which is generated by a positive integer. Since the ideal is invariant under automorphisms and the conjugation, the finite quotient algebra of the ring of integral elements inherits automorphisms and a conjugation. The completion of the ring of integral elements is a compact Hausdorff space since the adic topology of the ring of integral elements is the weakest topology with respect to which each projection onto a finite quotient algebra is continuous. An element of the adic skew–plane is said to be integral if it belongs to the closure of the ring of integral elements of the algebraic skew–field in the adic topology. The ring of integral elements of the adic skew–plane is a compact Hausdorff space which is a neighborhood of the origin for the adic topology of the adic skew–plane. Addition has a continuous extension as a transformation of the Cartesian product of the adic skew–plane with itself into the adic skew–plane. The automorphisms and the conjugation of the algebraic skew–field have continuous extensions as automorphisms and a conjugation of the adic skew–plane. Every element of the adic skew–plane is the sum of an element of the algebraic skew–field and an integral element of the adic skew–plane. An element of the adic skew–plane is said to be algebraic if it is an element of the algebraic skew–field. Multiplication by an algebraic element of the adic skew–plane has a continuous extension as a transformation of the adic skew–plane into itself. An invertible integral element of the adic skew–plane is said to be a unit if its inverse is integral. An element of the adic skew–plane is invertible if, and only if, it is the product of a unit of the adic skew–plane and a nonzero algebraic element of the adic skew–plane.

The p–adic skew–plane is a quotient algebra of the adic skew–plane which is defined for every prime p. The p–adic skew–plane is constructed as the completion of a quotient space of the algebraic skew–field in a topology for which addition is continuous as a transformation of the Cartesian product of the skew–field with itself into the skew–field. A basic neighborhood of the origin for the p–adic topology is an ideal of the ring of integral elements which is generated by a power of the prime p. Since the ideal is invariant under automorphisms and the conjugation, the finite quotient algebra of the ring of integral elements inherits automorphisms and a conjugation. The completion of the ring of integral elements is a compact Hausdorff space since the p–adic topology of the ring of integral elements is the weakest topology with respect to which each projection onto a finite quotient algebra is continuous. An element of the p–adic skew–plane is said to be integral if it belongs to the closure of the ring of integral elements of the algebraic skew–field in the p–adic topology. The ring of integral elements of the p–adic skew–plane is a compact Hausdorff space which is a neighborhood of the origin for the p–adic topology of the p–adic skew–plane. Addition has a continuous extension as a transformation of the Cartesian product of the p–adic skew–plane with itself into the p–adic skew–plane. The automorphisms and the conjugation of the algebraic skew–field have continuous extensions as automorphisms and a conjugation of the p–adic skew–plane. Every element of the p–adic skew–plane is the sum of an element of the algebraic skew–field, whose product by some power of p is integral, and an integral element of the p–adic skew–plane. An element of the p–adic skew–plane is said to be algebraic if it is an element of the algebraic skew–field. Multiplication by an algebraic element of the p–adic skew–plane has a continuous extension as a transformation of the p–adic skew–plane into itself. An invertible integral element of the p–adic skew–plane is said to be a unit if its inverse is integral. An element of the p–adic skew–plane is invertible if, and only if, it is the product of a unit of the p–adic skew–plane and a nonzero algebraic element of the p–adic skew–plane.

The adic completion of the algebraic skew–field is canonically isomorphic to a subalgebra of the Cartesian product of the p–adic completions of the algebraic skew–field taken over all primes p. An element of the Cartesian product determines an element of the adic completion if, and only if, it is integral for all but a finite number of primes p. For every prime p the p–adic completion of the algebraic skew–field is a Cartesian product of skew–fields isomorphic to the hypercomplex numbers with coefficients in a field, which contains the p–adic numbers and is a vector space of dimension σ over the p–adic numbers with σ the greatest power of p which is a divisor of ρ. The number of factors in the Cartesian product is ρ/σ.

An automorphism of the algebraic skew–field which permutes roots of the generating polynomial admits a continuous extension as an automorphism of the p–adic skew–plane. The automorphism of the p–adic skew–plane determines isomorphisms of component p–adic skew–fields. The automorphisms of the p–adic skew–plane which determine automorphisms of component p–adic skew–fields represent the exterior automorphisms of the p–adic skew–field. The character for the algebraic skew–field is the product of induced characters for component p–adic skew–fields.

The compactification of the Euclidean skew–plane is a quotient space of the Cartesian product of the Euclidean skew–plane and the adic skew–plane. An element ξ of the Cartesian product has an Euclidean component ξ_+, which is an element of the Euclidean skew–plane, and an adic component ξ_-, which is an element of the adic skew–plane. An algebraic element of the Euclidean skew–plane is an element of the adic skew–plane. Elements ξ and η of the Cartesian product are considered equivalent if

$$\eta_+ = \xi_+ + \gamma$$

and

$$\eta_- = \xi_- - \gamma$$

for an algebraic element γ of the Euclidean skew–plane. The sum of elements α and β of the quotient space is the element

$$\gamma = \alpha + \beta$$

of the quotient space whose Euclidean component

$$\gamma_+ = \alpha_+ + \beta_+$$

is the sum of the Euclidean components of α and β and whose adic component

$$\gamma_- = \alpha_- + \beta_-$$

is the sum of the adic components of α and β. The definition does not depend on the choice of representatives in equivalence classes.

The product

$$\eta = \gamma\xi$$

of an algebraic element γ of the Euclidean skew–plane and an element ξ of the quotient space is the element η of the quotient space whose Euclidean component

$$\eta_+ = \gamma\xi_+$$

is the product of γ with the Euclidean component of ξ and whose adic component

$$\eta_- = \gamma\xi_-$$

is the product of γ with the adic component of ξ. The definition does not depend on the choice of representatives in equivalence classes.

The product

$$\eta = \xi\gamma$$

of an algebraic element γ of the Euclidean skew–plane and an element ξ of the quotient space is the element η of the quotient space whose Euclidean component

$$\eta_+ = \xi_+\gamma$$

is the product of γ with the Euclidean component of ξ and whose adic component

$$\eta_- = \xi_-\gamma$$

is the product of γ with the adic component of ξ. The definition does not depend on the choice of representatives in equivalence classes.

The quotient space inherits a conjugation. The quotient space is a compact Hausdorff space in the quotient topology of the Cartesian product space. A computation of topology results from the construction of a fundamental domain for the equivalence relation on the Cartesian product space. Hypercomplex numbers $\xi_1, \dots, \xi_\rho$ with real coordinates are defined for each element ξ of the Euclidean skew–plane by the roots $\lambda_1, \dots, \lambda_\rho$ of the generating polynomial $r(t)$. If a polynomial $f(t)$, whose coefficients are hypercomplex numbers with real coordinates, represents ξ, then

$$\xi_1 = f(\lambda_1), \dots, \xi_\rho = f(\lambda_\rho).$$

If ξ is an element of the Euclidean skew–plane, a nearest integral element η of the Euclidean skew–plane is found by minimizing the nonnegative number

$$(\xi_1 - \eta_1)^-(\xi_1, -\eta_1) + \dots + (\xi_\rho - \eta_\rho)^-(\xi_\rho - \eta_\rho).$$

The fundamental domain for the Cartesian product space is the set of elements with integral adic component for which the integral element of the Euclidean skew–plane nearest the

Euclidean component is unique and is the origin. The fundamental domain is an open set whose closure is compact. Addition by an element of the Cartesian product space equivalent to the origin maps the fundamental region onto an open set containing the element. Open set containing distinct elements equivalent to the origin are disjoint. The Cartesian product space is the union of the closures of the open sets.

The Euclidean skew–plane, the adic skew–plane, the Cartesian product, and the quotient space are examples of locally compact additive groups. In Fourier analysis a character is defined as a continuous homomorphism of the group into the multiplicative group of complex numbers of absolute value one. A character for the Euclidean skew–plane is a function

$$\exp(\pi i\langle\xi,\eta\rangle+\pi i\langle\eta,\xi\rangle)$$

of ξ in the Euclidean skew–plane determined for an element η of the Euclidean skew–plane by the scalar product

$$\langle\xi,\eta\rangle=\eta_1^-\xi_1+\ldots+\eta_\rho^-\xi_\rho.$$

The scalar product is a hypercomplex number with rational coordinates when ξ and η are algebraic elements of the Euclidean skew–plane. The scalar product is integral when ξ and η are integral. The sum

$$\langle\xi,\eta\rangle+\langle\eta,\xi\rangle$$

is an integer when ξ and η are integral elements of the Euclidean skew–plane. An element ξ of the Euclidean skew–plane is integral or half–integral if the sum is integral for every element η of the Euclidean skew–plane.

If η is an algebraic element of the Euclidean skew–plane, the function

$$\exp(\pi i\langle\xi,\eta\rangle+\pi i\langle\eta,\xi\rangle)$$

of algebraic elements ξ of the Euclidean skew–plane is continuous with respect to the adic topology. The function has a unique continuous extension as a function

$$\exp(\pi i\langle\xi,\eta\rangle+\pi i\langle\eta,\xi\rangle)$$

of elements ξ of the adic skew–plane. If η is an element of the adic skew–plane, the function

$$\exp(\pi i\langle\xi,\eta\rangle+\pi i\langle\eta,\xi\rangle)$$

of algebraic elements ξ of the Euclidean skew–plane is continuous with respect to the adic topology. The function has a unique continuous extension as a function

$$\exp(\pi i\langle\xi,\eta\rangle+\pi i\langle\eta,\xi\rangle)$$

of ξ in the adic skew–plane.

A Fourier character for the Cartesian product space is determined by an element η of the Cartesian product space as a function

$$\exp(\pi i\langle\xi_+,\eta_+\rangle+\pi i\langle\eta_+,\xi_+\rangle)\exp(\pi i\langle\xi_-,\eta_-\rangle+\pi i\langle\eta_-,\xi_-\rangle)$$

of ξ in the Cartesian product space. A Fourier character for the quotient space is then determined when the Euclidean and adic components of η are equal elements of the algebraic skew–field.

A dense set of elements of the fundamental domain in the Cartesian product space have integral algebraic elements as adic component. Since these elements of the fundamental domain are equivalent to elements of the Cartesian product space with vanishing adic component, they are determined by elements of the Euclidean skew–plane. The quotient space is isomorphic as a locally compact additive group to the completion of the Euclidean skew–plane in a topology with respect to which addition is continuous as a transformation of the Cartesian product of the Euclidean skew–plane with itself into the Euclidean skew–plane. The topology is the weakest topology with respect to which

$$\exp(\pi i\langle\xi,\eta\rangle + \pi i\langle\eta,\xi\rangle)$$

is a continuous function of ξ in the Euclidean skew–plane for every algebraic element η of the Euclidean skew–plane.

Haar measure for a locally compact additive group is an essentially unique nonnegative measure on the Baire subsets of the group which is invariant under translation, has finite values on compact sets, and has positive values on nonempty open sets. The measure is unique within a constant factor. Since the Euclidean skew–plane is a vector space of finite dimension over the real numbers, Haar measure is a normalization of Lebesgue measure. The measure is normalized so that measure one is assigned to the set of elements of the Euclidean skew–plane which have the origin as nearest with integral components element. Haar measure for the adic skew–plane is normalized so that measure one is assigned to the set of elements with integral components. Haar measure for the Cartesian product space is the Cartesian product of Haar measure for the Euclidean skew–plane and Haar measure for the adic skew–plane. The fundamental domain of the Cartesian product space has measure one. Haar measure for the quotient space is determined by Haar measure on the fundamental domain.

The Fourier transformation for the Euclidean skew–plane is an isometric transformation of the space of square integrable functions with respect to Haar measure onto itself. The transformation takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into a function $g(\eta)$ of η in the Euclidean skew–plane when the integral

$$g(\eta) = \int \exp(\pi i\langle\xi,\eta\rangle + \pi i\langle\eta,\xi\rangle)f(\xi)d\xi$$

is absolutely convergent. The inverse transformation takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into a function $g(\eta)$ of η in the Euclidean skew–plane when the Fourier transformation takes the conjugate function $f(\xi)^-$ of ξ in the Euclidean skew–plane into the conjugate function $g(\eta)^-$ of η in the Euclidean skew–plane.

The Fourier transformation for the adic skew–plane is an isometric transformation of the space of square integrable functions with respect to Haar measure onto itself. The transformation takes a function $f(\xi)$ of ξ in the adic skew–plane into a function $g(\eta)$ of η in the adic skew–plane when the integral

$$g(\eta) = \int \exp(\pi i\langle\xi,\eta\rangle + \pi i\langle\eta,\xi\rangle)f(\xi)d\xi$$

is absolutely convergent. The inverse transformation takes a function $f(\xi)$ of ξ in the adic skew–plane into a function $g(\eta)$ of η in the adic skew–plane when the Fourier transformation takes the conjugate function $f(\xi)^-$ of ξ in the adic skew–plane into the conjugate function $g(\eta)^-$ of η in the adic skew–plane.

The Fourier transformation for the Cartesian product space is an isometric transformation of the space of square integrable functions with respect to Haar measure onto itself. The transformation takes a function $f(\xi)$ of ξ in the Cartesian product space into a function $g(\eta)$ of η in the Cartesian product space when the integral

$$g(\eta) = \int \exp(\pi i\langle\xi_+, \eta_+\rangle + \pi i\langle\eta_+, \xi_+\rangle) \exp(\pi i\langle\xi_-, \eta_-\rangle + \pi i\langle\eta_-, \xi_-\rangle) f(\xi) d\xi$$

is absolutely convergent. The inverse transformation takes a function $f(\xi)$ of ξ in the Cartesian product space into a function $g(\eta)$ of η in the Cartesian product space when the Fourier transformation takes the conjugate function $f(\xi)^-$ of ξ in the Cartesian product space into the conjugate function $g(\eta)^-$ of η in the Cartesian product space.

The Fourier transformation for the quotient space is an isometric transformation of the space of square integrable functions with respect to Haar measure for the quotient space onto the space of square summable functions of algebraic elements of the Euclidean skew–plane. The transformation takes a function $f(\xi)$ of ξ in the quotient space into a function $g(\eta)$ of algebraic elements of the Euclidean skew–plane defined by the integral

$$g(\eta) = \int \exp(\pi i\langle\xi_+, \eta_+\rangle + \pi i\langle\eta_+, \xi_+\rangle) \exp(\pi i\langle\xi_-, \eta_-\rangle + \pi i\langle\eta_-, \xi_-\rangle) f(\xi) d\xi$$

with respect to Haar measure for the quotient space. The identity

$$\int |f(\xi)|^2 d\xi = \sum |g(\eta)|^2$$

holds with integration with respect to Haar measure for the quotient space and with summation over the algebraic elements of the Euclidean skew–plane. The inverse transformation takes a function $f(\xi)$ of algebraic elements of the Euclidean skew–plane into a function $g(\eta)$ of η in the quotient space defined by the orthogonal expansion

$$g(\eta)^- = \sum \exp(\pi i\langle\xi_+, \eta_+\rangle + \pi i\langle\eta_+, \xi_+\rangle) \exp(\pi i\langle\xi_-, \eta_-\rangle + \pi i\langle\eta_-, \xi_-\rangle) f(\xi)^-$$

of conjugate functions.

Poisson summation constructs an integrable function with respect to Haar measure on the quotient space from an integrable function with respect to Haar measure on the Cartesian product space. The Poisson sum

$$f'(\xi) = \sum f(\xi + \eta)$$

of a function $f(\xi)$ of ξ in the Cartesian product space is a function $f'(\xi)$ of ξ in the quotient space defined by summation over the elements η of the Cartesian product space which are equivalent to the origin. The inequality

$$\int |f'(\xi)| d\xi \leq \int |f(\xi)| d\xi$$

holds with integration on the left with respect to Haar measure for the quotient space and with integration on the right with respect to Haar measure for the Cartesian product space. If γ is an element of the Cartesian product space whose Euclidean component γ_+ and whose adic component γ_- are equal elements of the algebraic skew–field, then the integral

$$\int \exp(\pi i\langle\gamma_+, \xi_+\rangle + \pi i\langle\xi_+, \gamma_+\rangle) \exp(\pi i\langle\gamma_-, \xi_-\rangle + \pi i\langle\xi_-, \gamma_-\rangle) f'(\xi) d\xi$$

with respect to Haar measure for the quotient space is equal to the integral

$$\int \exp(\pi i\langle\gamma_+, \xi_+\rangle + \pi i\langle\xi_+, \gamma_+\rangle) \exp(\pi i\langle\gamma_-, \xi_-\rangle + \pi i\langle\xi_-, \gamma_-\rangle) f(\xi) d\xi$$

with respect to Haar measure for the Cartesian product space.

The Poisson formula

$$\sum f(\eta) = \sum g(\eta)$$

states that the Poisson sum of a bounded integrable function $f(\xi)$ of ξ in the Cartesian product space has the same value at the origin as the Poisson sum of its Fourier transform when the Fourier transform is a bounded integrable function $g(\xi)$ of ξ in the Cartesian product space. Boundedness and integrability imply the square integrability condition under which the Fourier transformation has been defined. The function values applied in the Poisson formula are meaningful since the Fourier transform of an integrable function is continuous. The Poisson formula is obtained by Fourier analysis on the quotient space. The Fourier coefficients of a function of ξ on the quotient space are determined by the values $g(\eta)$ when η is equivalent to zero since this condition means that the Euclidean component η_+ and the adic component η_- are algebraic elements with sum zero. The sum

$$\sum g(\eta)$$

taken over the elements η of the Cartesian product space which are equivalent to zero is the value at the origin of a function

$$\sum \exp(\pi i\langle\xi_+, \eta_+\rangle + \pi i\langle\eta_+, \xi_+\rangle) \exp(-\pi i\langle\xi_-, \eta_-\rangle - \pi i\langle\eta_-, \xi_-\rangle) g(\eta)$$

of ξ in the Cartesian product space which has the same value at the origin as $f'(\xi)$.

The elements of an Euclidean skew–plane are ρ–tuples $(\xi_1, \ldots, \xi_\rho)$ of elements of the Euclidean skew–field whose elements are hypercomplex numbers with real coordinates. Function on the Euclidean skew–plane are applied which are sums

$$f_1(\xi_1) + \ldots + f_\rho(\xi_\rho)$$

of functions defined on component Euclidean skew–fields.

A function of

$$\xi = t + ix + jy + kz$$

is an Euclidean skew–field is said to be a homogeneous polynomial of degree ν if it is a linear combination of monomials

$$x^a y^b z^c t^d$$

whose exponents a, b, c, and d are nonnegative integers with sum ν. The space of homogeneous polynomials of degree ν is a Hilbert space which admits the monomials as an orthogonal basis. The scalar self–product of the monomial with exponents a, b, c, and c is

$$\frac{a!\ b!\ c!}{\nu!}.$$

Isometric transformations of the space of homogeneous polynomials of degree ν into itself are defined by taking a function $f(\xi)$ of ξ in the Euclidean skew–field into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the Euclidean skew–field for every unit ω of the Euclidean skew–field.

The Laplacian

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial t^2}$$

acts as a linear transformation of the space of homogeneous polynomials of degree ν into the space of homogeneous polynomials of degree $\nu - 2$ when ν is greater than one and annihilates homogeneous polynomials of smaller degree. The Laplacian commutes with the transformations which take a function $f(\xi)$ of ξ in the Euclidean skew–field into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the Euclidean skew–field for every unit ω of the Euclidean skew–field.

A homogeneous polynomial of degree ν is said to be harmonic if it is annihilated by the Laplacian. The homogeneous harmonic polynomials of degree ν form an invariant subspace of dimension $1 + 2\nu$ for the transformation which takes $f(\xi)$ into $f(\omega^{-}\xi\omega)$ for every unit ω of the Euclidean skew–field.

A harmonic polynomial of degree ν is a homogeneous polynomial of degree ν which is annihilated by the Laplacian. The harmonic polynomials of degree ν form an invariant subspace for the transformations which take a function $f(\xi)$ of ξ in the Euclidean skew–field into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the Euclidean skew–field for every unit ω of the Euclidean kew–field. The dimension of the space of homogeneous polynomials of degree ν is

$$(\nu + 1)(\nu + 2)(\nu + 3)/6.$$

The dimension of the space of harmonic polynomials of degree ν is

$$(\nu + 1)^2.$$

Commuting self–adjoint transformations $\Delta(n)$ are defined in the space of harmonic polynomials of degree ν for every positive integer n. The transformation $\Delta(n)$ takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into the function $g(\xi)$ of ξ in the Euclidean skew–plane defined by the summations

$$n^{\frac{1}{2}\nu} g(\xi) = \sum f(\omega^- \xi)$$

over the integral elements ω of the algebraic skew–field such that

$$n = \omega^- \omega.$$

The identity

$$\Delta(m)\Delta(n) = \sum \Delta(mn/k^2)\Delta(1)$$

holds for all positive integers m and n with summation over the common divisors k of m and n. Each transformation $\Delta(n)$ commutes with the transformation which takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into the function

$$f^*(\xi) = f(\xi^-)^-$$

of ξ in the Euclidean skew–plane.

The transformations $\Delta(n)$ take harmonic polynomials of degree ν into harmonic polynomials of degree ν. The transformations are self–adjoint in the space of harmonic polynomials of degree v. The space is the orthogonal sum of simultaneous invariant subspaces for the transformations $\Delta(n)$. Each invariant subspace is invariant under the transformation $f(\xi)$ into $f^*(\xi)$. The nonzero elements of an invariant subspace are characterized as eigenfunctions of $\Delta(n)$ for a real eigenvalue $\tau(n)$ for every positive integer n. The identity

$$\tau(m)\tau(n) = \sum \tau(mn/k^2)$$

holds for all positive integers m and n with summation over the common divisors k of m and n.

A nonzero harmonic polynomial of degree ν, which is an eigenfunction of the transformation $\Delta(n)$ for some eigenvalue $\tau(n)$ for every positive integer n, is said to be primitive modulo ν if no nonzero harmonic polynomial of smaller degree exists which is an eigenfunction of the transformation $\Delta(n)$ for the same eigenvalue $\tau(n)$ for every positive integer n.

The Poisson summation formula is applied to functions of ξ in the Cartesian product of the Euclidean skew–plane and the adic skew–plane which vanish when the adic component of ξ is not integral and whose value is independent of the adic component of ξ when the adic component of ξ is not integral. Such a function on the Cartesian product space is determined by the function $f(\xi_+)$ of ξ_+ in the Euclidean skew–plane which is obtained when the adic component of ξ is integral. The function of ξ in the Cartesian product space is square integrable with respect to Haar measure for the Cartesian product space if, and only if, the function $f(\xi_+)$ of ξ_+ in the Euclidean skew–plane is square integrable with respect to Haar measure for the Euclidean skew–plane. The Fourier transform for the Cartesian product space of the square integrable function of ξ in the Cartesian product space is a function of ξ in the Cartesian product space which vanishes when the adic component of ξ is not integral and whose value is independent of ξ when the adic component of ξ is integral. The function obtained on the Cartesian product is determined by the function $g(\xi_+)$ of ξ_+ in the Euclidean skew–plane obtained when the adic component of ξ is integral. The function $g(\xi)$ of ξ in the Euclidean skew–plane is the Fourier transform for the Euclidean skew–plane of the function $f(\xi)$ of ξ in the Euclidean skew–plane. The functions $f(\xi)$ and $g(\xi)$ of ξ in the Euclidean skew–plane are integrable with respect to Haar measure for the Euclidean skew–plane if, and only if, the functions determined on the Cartesian product space are integrable with respect to Haar measure on the Cartesian product space. The Poisson summation formula

$$\sum f(\eta) = \sum g(\eta)$$

then applies with summation over the integral elements η of the algebraic skew–field.

A computation of Fourier transforms is made when the Euclidean skew–plane is the Euclidean skew–field of hypercomplex numbers with real coordinates. If z is in the upper half–plane, the function

$$\varphi(\xi)\exp(\pi iz\xi^-\xi)$$

of ξ in the Euclidean skew–plane is square integrable with respect to Haar measure for the Euclidean skew–plane whenever $\varphi(\xi)$ is a harmonic polymonial of degree ν. The Fourier transform for the Euclidean skew–plane is a function

$$(i/z)^{2+4\nu}\psi(\xi)\exp(-\pi iz^{-1}\xi^-\xi)$$

of ξ in the Euclidean skew–plane for a harmonic polynomial $\psi(\xi)$ of degree ν. If the harmonic polynomial $\varphi(\xi)$ is an eigenfunctio of $\Delta(n)$ for the eigenvalue $\tau(n)$ for every positive integer n, then the harmonic polynomail $\psi(\xi)$ is an eigenfunction of $\Delta(n)$ for the eigenvalue $\tau(n)$ for every positive integer n. If the harmonic polynomial $\varphi(\xi)$ is primitive for the given eigenvalues, then the polynomials $\varphi(\xi)$ and $\psi(\xi)$ are linearly dependent. The identity

$$\psi(\xi) = \sigma\varphi(\xi)$$

holds for a real number σ of absolute value one.

The Poisson summation formula

$$(i/z)^{2+\nu}\sum\phi(\xi)\exp(-\pi iz^{-1}\xi^-\xi) = \sigma\sum\phi(\xi)\exp(\pi iz\xi^-\xi)$$

holds with summation over the integral elements ξ of the algebraic skew–field when z is in the upper half–plane. Since the identity

$$\tau(n)n^{\frac{1}{2}\nu}\phi(1) = \sum\varphi(\xi^-\xi)$$

holds for every positive integer n with summation over the integral elements ξ of the algebraic skew–field such that

$$n = \xi^-\xi,$$

the Poisson summation formula reads

$$(i/z)^{2+\nu} \sum \theta 6au(n) n^{\frac{1}{2}\nu} \exp(-\pi i n z^{-1}) = \sigma \sum \tau(n) n^{\frac{1}{2}\nu} \exp(\pi i n z)$$

with summation over the positive integers n when z is in the upper half–plane.

The modular group is the set of matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with integer entries and determinant one. The signature for the modular group is a homomorphism of the group into the fourth roots of unity whose action on the matrix is written

$$\operatorname{sgn} \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

has signature minus one. The matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has signature i. The identity

$$\theta(z) = \operatorname{sgn}^{\nu} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \frac{1}{(Cz+D)^{2+\nu}} \, \theta \left(\frac{Az+B}{Cz+D} \right)$$

holds for every element

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of the modular group when

$$\theta(z) = \sum \tau(n) n^{\frac{1}{2}\nu} \exp(\pi i n z).$$

The zeta function

$$\zeta(s) = \sum n^{-\frac{1}{2}} \tau(n) n^{-s}$$

is defined as a sum over the positive integers n in the half–plane $\mathcal{R}s > 1$. The zeta function has an analytic extension as a function of s in the complex plane which satisfies the functional identity

$$\begin{aligned} &\pi^{-\frac{1}{2}\nu - \frac{1}{2} + s - 1} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} + 1 - s) \zeta(1-s) \\ &= \sigma \pi^{-\frac{1}{2}\nu - \frac{1}{2} - s} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} + s) \zeta(s). \end{aligned}$$

The Euler product

$$\zeta(s)^{-1} = \prod (1 - p^{-\frac{1}{2}} \tau(p) p^{-s}),$$

taken over the primes p, converges in the half–plane $\mathcal{R}s > 1$.

The domain of the Laplace transformation of order ν and harmonic ϕ for the Euclidean skew–plane is the space of functions $f(\xi)$ of ξ in the Euclidean skew–plane which satisfy the identity

$$\phi(\xi) f(\xi\omega) = \phi(\xi\omega) f(\xi)$$

for every unit ω of the Euclidean skew–plane and which are square integrable with respect to Haar measure for the Euclidean skew–plane. The Laplace transform of the function $f(\xi)$ of ξ in the Euclidean skew–plane is the analytic function

$$g(z) = \int \phi(\xi)^{-} f(\xi) \exp(\pi i z \xi^{-} \xi) d\xi$$

of z in the upper half–plane defined by integration with respect to Haar measure for the Euclidean skew–plane. When w is in the upper half–plane, the Laplace transform of the function

$$\phi(\xi) \exp(-\pi i w^{-} \xi^{-} \xi)$$

of ξ in the upper half–plane is the function

$$\pi \frac{\|\phi\|^2 \Gamma(2+\nu)}{(\pi i w^{-} - \pi i z)^{2+\nu}}$$

of z in the upper half–plane with the norm of the harmonic polynomial ϕ taken in the Hilbert space of homogeneous polynomials of degree ν. An analytic function $g(z)$ of z in the upper half–plane is a Laplace transform of order ν and harmonic ϕ if, and only if, the least upper bound

$$\sup \int_0^{\infty} \int_{-\infty}^{+\infty} |g(x+iy)|^2 y^{\nu} dx dy,$$

taken over all positive numbers y, is finite. The least upper bound is then equal to the integral

$$(2\pi)^{-\nu} \Gamma(1+\nu) \|\phi\|^2 \int |f(\xi)|^2 d\xi$$

with respect to Haar measure for the Euclidean skew–plane.

An isometric transformation of the range of the Laplace transformation onto itself is defined by taking an analytic function $g(z)$ of z in the upper half–plane into the analytic function

$$(i/z)^{2+\nu} g(-1/2)$$

of z in the upper half–plane. The Hankel transformation of order ν and harmonic ϕ for the Euclidean skew–plane is an isometric transformation of the domain of the Laplace transformation onto itself which takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into a function $g(\xi)$ of ξ in the Euclidean skew–plane when the identity

$$\int \phi(\xi)^- g(\xi) \exp(\pi i z \xi^- \xi) d\xi$$
$$= (i/z)^{2+\nu} \int \phi(\xi)^- f(\xi) \exp(-\pi i z^{-1} \xi^- \xi) d\xi$$

holds for z in the upper half–plane with integration with respect to Haar measure for the Euclidean plane. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to Haar measure for the Euclidean skew–plane. The function $f(\xi)$ of ξ in the Euclidean skew–plane is the Hankel transform of order ν and harmonic ϕ for the Euclidean skew–plane of the function $g(\xi)$ of ξ in the Euclidean skew–plane.

The Mellin transformation of order ν and harmonic ϕ for the Euclidean skew–plane is a spectral theory for the Laplace transformation of order ν and harmonic ϕ for the Euclidean skew–plane. The domain of the Mellin transformation of order ν and harmonic ϕ for the Euclidean skew–plane is the space of functions which belong to the domain of the Laplace transformation of order ν and harmonic ϕ for the Euclidean skew–plane and which vanish in a neighborhood of the origin. The Laplace transform of a function $f(\xi)$ of ξ in the Euclidean skew–plane is an analytic function

$$g(z) = \int \phi(\xi)^- f(\xi) \exp(\pi i z \xi^- \xi) d\xi$$

of z in the upper half–plane defined by integration with respect to Haar measure for the Euclidean skew–plane. The Mellin transform of order ν and harmonic ϕ for the Euclidean skew–plane of the function $f(\xi)$ of ξ in the Euclidean skew–plane is the analytic function

$$F(z) = \int_0^\infty g(it) t^{\frac{1}{2}\nu - iz} dt$$

of z in the upper half–plane. Since the function

$$W(z) = \pi^{-\frac{1}{2}\nu - \frac{1}{2} + iz} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz)$$

admits the integral representation

$$W(z) = (\xi^- \xi)^{\frac{1}{2}\nu + 1 - iz} \int_0^\infty \exp(-\pi t \xi^- \xi) t^{\frac{1}{2}\nu - iz} dt$$

when z is in the upper half–plane, the identity

$$F(z)/W(z) = \int \phi(\xi)^- f(\xi) (\xi^- \xi)^{iz - 1 - \frac{1}{2}\nu} d\xi$$

holds when z is in the upper half–plane with integration with respect to Haar measure for the Euclidean skew–plane. If $f(\xi)$ vanishes when

$$\xi^-\xi < a$$

for some positive number a, then the least upper bound

$$\sup \int_{-\infty}^{+\infty} a^y |F(x+iy)/W(x+iy)|^2 dN$$

taken over all positive numbers y is equal to the integral

$$2\pi^2 \|\phi\|^2 \int |f(\xi)|^2 d\xi$$

with respect to Haar measure for the Euclidean skew–plane.

The Radon transformation of order ν and harmonic ϕ for the Euclidean skew–plane is a maximal dissipative transformation in the domain of the Laplace transformation of order ν and harmonic ϕ for the Euclidean skew–plane. The transformation takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into a function $g(\xi)$ of ξ in the Euclidean skew–plane when the identity

$$g(\xi) = \int f(\xi + \eta)|\eta|^{-\frac{1}{2}} d\eta$$

holds formally with integration with respect to a normalization of Haar measure for the space of elements η of the Euclidean skew–plane such that

$$\eta^-\xi + \xi^-\eta = 0.$$

Haar measure is normalized so that the set of elements η such that $\eta^-\eta < 1$ has measure $4\pi/3$. The integral is accepted as the definition when

$$f(\xi) = \phi(\xi) \exp(\pi i z \xi^- \xi)$$

with z in the upper half–plane, in which case

$$g(\xi = (2i/z) f(\xi).$$

The adjoint of the Radon transformation of order ν and harmonic ϕ for the Euclidean skew–plane takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into a function $g(\xi)$ of ξ in the Euclidean skew–plane when the identity

$$\int \phi(\xi)^- g(\xi) \exp(\pi i z \xi^- \xi) d\xi$$
$$= (2i/z) \int \phi(\xi)^- f(\xi) \exp(\pi i z \xi^- \xi) d\xi$$

holds when z is in the upper half–plane with integration with respect to Haar measure for the Euclidean skew–plane.