THE MEASURE PROBLEM

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Abstract. A problem of Stefan Banach [1] is to determine the structure of a nonnegative measure, countably additive and with finite values, which is defined on all subsets of a set. Banach shows the existence of a countable subset whose complement has zero measure when the cardinality of the set is not greater than the least uncountable cardinal number. An Ulam measure is a countably additive measure which is defined on all subsets of a set and whose only nonzero value is one. A nonnegative measure, countably additive and with finite values, which is defined on all subsets of a set, is a countable sum of measures which are positive multiples of Ulam measures. An Ulam measure gives a nontrivial solution of the Banach problem if it annihilates every finite set. A set $\mathcal{S}$ of least cardinality which admits a nontrivial Ulam measure has remarkable properties. The set $\mathcal{S}$ is uncountable. No greatest cardinality exists which is less than the cardinality of $\mathcal{S}$. The set $\mathcal{S}$ is not a union of a class of smaller cardinality than $\mathcal{S}$ whose members are sets of smaller cardinality than $\mathcal{S}$. The nonexistence of nontrivial Ulam measures is proved by showing that no set has these properties.

A measure problem is treated as an issue in the relationship between topology and measure for a completely regular Hausdorff space $\mathcal{S}$. A function $f(s)$ of s in $\mathcal{S}$ is defined as Baire measurable if the limit

$$f(s) = \lim f_n(s)$$

holds for every element s of $\mathcal{S}$ with $f_n(s)$ a continuous function of s in $\mathcal{S}$ for every nonnegative integer n. A subset of $\mathcal{S}$ is defined as Baire measurable if its characteristic function is Baire measurable. A bounded nonnegative measure σ on the Baire measurable subsets of $\mathcal{S}$ is a function $\sigma(C)$ of Baire measurable subsets C of $\mathcal{S}$, whose values are nonnegative numbers, such that the measure of a countable union of disjoint sets is the sum of the measures of the sets. A Hilbert space is constructed from the equivalence classes of Baire measurable functions which are square integrable with respect to σ. Baire measurable functions $f(s)$ and $g(s)$ of s in $\mathcal{S}$ are considered equivalent if the set of elements s of $\mathcal{S}$ such that $f(s)$ and $g(s)$ are unequal has zero measure.

The space $\mathcal{C}_\sigma(\mathcal{S})$ of equivalence classes of square integrable continuous functions inherits a metric topology from the Hilbert space of equivalence classes of square

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integrable measurable functions. The integral

$$\int |f(s) - g(s)|^2 d\sigma$$

defines the square of the distance between the equivalence classes of continuous functions $f(s)$ and $g(s)$ of s in $\mathcal{S}$ when the functions are square integrable. The space $\mathcal{C}_\sigma(\mathcal{S})$ is given a locally convex topology which is derived by convexity from the metric topology.

A convex combination

$$f(1 - h) + gh$$

of elements f and g of $\mathcal{C}_\sigma(\mathcal{S})$ is defined by an element h of $\mathcal{C}_\sigma(\mathcal{S})$ whose values are taken in the interval $[0, 1]$. The definition of convexity depends on position in $\mathcal{S}$ as determined by the function $h(s)$ of s in $\mathcal{S}$. The convex combination is the function

$$f(s)[1 - h(s)] + g(s)h(s)$$

of s in $\mathcal{S}$.

A convex combination of continuous functions which are square integrable is a continuous function which is square integrable and whose equivalence class is determined by equivalence classes. The space $\mathcal{C}_\sigma(\mathcal{S})$ inherits a convex structure. The closure in $\mathcal{C}_\sigma(\mathcal{S})$ of a convex subset of $\mathcal{C}_\sigma(\mathcal{S})$ for the metric topology is a convex subset of $\mathcal{C}_\sigma(\mathcal{S})$ which contains the given convex set and whose closure in $\mathcal{C}_\sigma(\mathcal{S})$ is itself.

A topology is defined on $\mathcal{C}_\sigma(\mathcal{S})$ whose open sets are unions of open convex sets. A convex subset of $\mathcal{C}_\sigma(\mathcal{S})$ is defined as open if it is disjoint from the metric closure of every disjoint convex set. The complement of the metric closure of a nonempty convex subset B of $\mathcal{C}_\sigma(\mathcal{S})$ is a union of open convex sets. The proof is given by showing that a disjoint open convex set A which contains the origin exists whenever the origin does not belong to B.

An upper semicontinuous function

$$k(s) = \inf |f(s)|$$

of s in $\mathcal{S}$ is defined as a greatest lower bound over the continuous functions $f(s)$ of s in $\mathcal{S}$ which represent elements of B. The greatest lower bound

$$\inf \int |f(s)|^2 d\sigma$$

is positive since the origin does not belong to B. A continuous function $f_n(s)$ of s in $\mathcal{S}$ exists for every positive integer n which represents an element of B and which satisfies the inequality

$$\int |f_n(s)|^2 d\sigma \leq n^{-1} + \inf \int |f(s)|^2 d\sigma.$$

Since the convex combination

$$f_m(1 - h) + f_n h$$

belongs to B, the convexity identity

$$|f_m(s)[1-h(s)]+f_n(s)h(s)|^2+|f_m(s)-f_n(s)|^2[1-h(s)]h(s)$$
$$=|f_m(s)|^2[1-h(s)]+|f_n(s)|^2h(s)$$

implies the inequality

$$\int |f_m(s)-f_n(s)|^2[1-h(s)]h(s)d\sigma \leq m^{-1}+n^{-1}$$

whenever $h(s)$ is a continuous function of s in $\mathcal{S}$ with values in the interval $[0,1]$.

A Baire measurable function $f_\infty(s)$ of s in $\mathcal{S}$ exists such that the integrals

$$\int |f_\infty(s)-f_n(s)|^2 d\sigma$$

converge to zero and such that the identity

$$\int |f_\infty(s)|^2 d\sigma = \inf \int |f(s)|^2 d\sigma$$

holds with the greatest lower bound taken over the element f of B. The identity

$$k(s)=|f_\infty(s)|$$

holds in a Baire measurable set of elements s of $\mathcal{S}$ whose complement has zero measure. The set A of continuous functions $f(s)$ of s in $\mathcal{S}$ which satisfy the inequality

$$|f(s)|<k(s)$$

in a Baire measurable set of elements s of $\mathcal{S}$ whose complement has zero measure represent elements of an open convex subset of $\mathcal{C}_\sigma(s)$ which contains the origin and which is disjoint from B.

If f is an element of an open convex set A and if g is an element of $\mathcal{C}_\sigma(\mathcal{S})$, the set of convex combinations

$$f(1-h)+gh$$

with $h(s)$ a continuous function of s in $\mathcal{S}$ with values in the interval $(0,1]$ is a convex set whose closure contains f. Since A is an open convex set, the convex combination belong to A for some h.

The intersection of an open convex set A with the metric closure of a convex set B is contained in the metric closure of the intersection of A with B. The intersection of two open convex sets is an open convex set. The space $\mathcal{C}_\sigma(\mathcal{S})$ is a Hausdorff space in a topology whose open sets are the unions of convex open sets and whose closed sets are the complements of open sets. The closure of a convex set coincides with its metric closure in $\mathcal{C}_\sigma(\mathcal{S})$.

If w is an element of $\mathcal{C}_\sigma(\mathcal{S})$ and if B is a nonempty convex subset of $\mathcal{C}_\sigma(\mathcal{S})$, then B is contained in the closure of a convex subset $B(w)$ of $\mathcal{C}_\sigma(\mathcal{S})$ constructed so that closure contains w. The set $B(w)$ is defined as the set of convex combinations

$$w(1-h)+ch$$

of w and elements c of B with h an element of $\mathcal{C}_\sigma(\mathcal{S})$ with values in the interval $(0,1)$. An open convex set which contains w or an element of B contains an element of $B(w)$.

The convexity of $B(w)$ remains to be verified. If

$$w(1-p)+ap$$

and

$$w(1-q)+bq$$

are elements of $B(w)$ constructed from elements a and b of B, then p and q are elements of $\mathcal{C}_\sigma(\mathcal{S})$ with values in the interval $(0,1)$. A convex combination

$$\begin{aligned}&[w(1-p)+ap](1-h)+[w(1-q)+bq]h\\&=w[(1-p)h+(1-q)h]+ap(1-h)+bqh\end{aligned}$$

of elements of $B(w)$ is defined by an element h of $\mathcal{C}_\sigma(\mathcal{S})$ with values in the interval $[0,1]$. An element

$$r=p(1-h)+qh$$

of $\mathcal{C}_\sigma(\mathcal{S})$ with values in the interval $(0,1)$ exists such that

$$1-r=(1-p)(1-h)+(1-q)h.$$

Since B is convex, an element c of B is obtained as solution of the equation

$$cr=ap(1-h)+bqh.$$

The convex combination of elements of $B(w)$ is the element

$$w(1-r)+cr$$

of $B(w)$.

The Hahn–Banach theorem separates an open convex subset of $\mathcal{C}_\sigma(\mathcal{S})$ from a disjoint convex subset of $\mathcal{C}_\sigma(\mathcal{S})$.

Theorem 1. *If a nonempty open convex subset A of $\mathcal{C}_\sigma(\mathcal{S})$ is disjoint from a nonempty convex subset B of $\mathcal{C}_\sigma(\mathcal{S})$, then A is contained in an open convex set whose complement is convex and contains B.*

Proof of Theorem 1. Since a maximal convex subset of $\mathcal{C}_\sigma(\mathcal{S})$ which contains B and is disjoint from A exists by the Kuratowski–Zorn lemma, B can be assumed a maximal convex subset of $\mathcal{C}_\sigma(\mathcal{S})$ which is disjoint from A. Since A is an open convex set, $B(w)$ contains an element of A whenever an element w of $\mathcal{C}_\sigma(\mathcal{S})$ does not belong to B. The complement of B is shown to be an open convex set by showing convexity.

If u and v belong to the complement of B, then the convex sets $B(u)$ and $B(v)$ contain elements of A. Elements a and b of B exist such that the convex combinations

$$u(1-p)+ap$$

and

$$v(1-q)+bq$$

belong to A for elements p and q of $\mathcal{C}_\sigma(\mathcal{S})$ with values in the interval $(0,1)$.

A convex combination w of u and v is a solution of the equation

$$w[(1-p)(1-h)+(1-q)h]=u(1-p)(1-h)+v(1-q)h$$

for an element h of $\mathcal{C}_\sigma(\mathcal{S})$ with values in the interval $[0,1]$. Since B is convex, the equation

$$c[p(1-h)+qh]=ap(1-h)+bqh$$

admits a solution c in B. The convex combination

$$[u(1-p)+ap](1-h)+[v(1-q)+bq]h$$

of elements of A is an element of A equal to the element

$$w[(1-p)(1-h)+(1-q)h]+c[p(1-h)+qh]$$

of $B(w)$. The convex combination w of elements u and v of the complement of B is an element of the complement of B.

This completes the proof of the theorem.

A set which is treated as a Hausdorff space in the discrete topology is a completely regular Hausdorff space for which countable intersections of open sets are open and countable unions of closed sets are closed. Baire measurable sets for such a topology are sets which are both open and closed. Baire measurable functions for such a topology are continuous.

The measure problem is treated as a determination of the structure of nonnegative measures which are defined on the Baire measurable subsets of a completely regular Hausdorff space for which countable intersections of open sets are open and countable unions of closed sets are closed. This formulation motivates a solution of the measure problem without generalizing it since every such Hausdorff space has the discrete topology.

Theorem 2. *If a bounded nonnegative measure σ is defined on the Baire measurable subsets of a completely regular Hausdorff space $\mathcal{S}$ and if every square integrable function is equal almost everywhere with respect to σ to a continuous function, then σ is a countable sum of nonnegative measures on the Baire measurable subsets of $\mathcal{S}$ having only one nonzero value.*

Proof of Theorem 2. The space $\mathcal{C}_\sigma(\mathcal{S})$ is a Hilbert space since it contains a representative of every equivalence class of Baire measurable functions which are equivalent with respect to σ. If a nonempty convex subset A of $\mathcal{C}_\sigma(\mathcal{S})$ has a nonempty convex complement B, a continuous function $k(s)$ of s in $\mathcal{S}$, which represents an element of $\mathcal{C}_\sigma(\mathcal{S})$, exists such that the linear functional on $\mathcal{C}_\sigma(\mathcal{S})$ which takes f into

$$\int k(s)^- f(s)d\sigma$$

maps A and B into disjoint subsets of the complex plane. The kernel of the linear functional is convex since A and B are convex subsets of $\mathcal{C}_\sigma(\mathcal{S})$. The function $k(s)$ of s in $\mathcal{S}$ is equivalent to a constant on the set on which it is nonzero. The function can be chosen so that its values are zero and one. A measure σ_0 on the Baire measurable subsets of $\mathcal{S}$, which has only one nonzero value, is defined on a set as the integral

$$\int k(s) d\sigma$$

over the set.

If a continuous function $f(s)$ of s in $\mathcal{S}$ is square integrable with respect to σ, the function $k(s)f(s)$ of s in $\mathcal{S}$ is equivalent with respect to σ to a constant multiple of the function $k(s)$ of s in $\mathcal{S}$. The nonnegative measure on the Baire measurable subsets of $\mathcal{S}$ whose value on a set is the integral

$$\int k(s) d\sigma(s)$$

over the set has only one nonzero value. The measure σ_1 on the Baire measurable subsets of $\mathcal{S}$ whose value on a set is the integral

$$\int [1 - k(s) J d\sigma(s)$$

over the set has the property that every square integrable function with respect to σ_1 is equal almost everywhere with respect to σ_1 to a continuous function.

The proof of the theorem is completed by induction. Nonnegative measures σ_n are defined on the Baire measurable subsets of $\mathcal{S}$ either for n less than or equal to some nonnegative integer r or for all nonnegative integers n. Every square integrable function with respect to σ_n is equivalent with respect to σ_n to a continuous function. The measure σ_n is equal to σ when n is zero. If the measure σ_n has more than one nonzero value, a continuous function k_n with values zero and one is defined so that the function $k_n(s)f(s)$ of s in $\mathcal{S}$ is equivalent with respect to σ_n to a constant multiple of the function $k_n(s)$ of s in $\mathcal{S}$ for every continuous function $f(s)$ of s in $\mathcal{S}$ which is square integrable with respect to σ_n. The measure σ_{n+1} is defined on a Baire measurable set as the integral

$$\int k_n(s) d\sigma_n(s)$$

over the set.

Since the construction is made so that the integral

$$\int k_n(s) d\sigma_n(s)$$

over $\mathcal{S}$ is positive, the inequality

$$\int d\sigma_{n+1}(s) < \int d\sigma_n(s)$$

holds with integration over $\mathcal{S}$. The identity

$$\int d\sigma(s) = \sum \int k_n(s) d\sigma_n(s)$$

holds with integration over every Baire measurable set. Summation is with n less than or equal to r when the measures σ_n are defined only when n is less than or equal to r. Otherwise summation is over all nonnegative integers n. The measure whose value on a Baire measurable set is the integral

$$\int k_n(s) d\sigma_n(s)$$

over the set has only one nonzero value.

This completes the proof of the theorem.

The measure problem is reduced to a determination of structure for Ulam measures. A theorem of Ulam [4] reduces the structure problem for Ulam measures to an issue in the cardinality of uncountable sets.

A construction of Cantor [3] implies that no transformation maps a set onto the class of all its subsets. If a transformation T maps elements of a set $\mathcal{S}$ into subsets of $\mathcal{S}$, then a subset $\mathcal{S}_\infty$ is constructed which is not in the range of T. The set $\mathcal{S}_\infty$ is the set of elements s of $\mathcal{S}$ for which no elements s_n of $\mathcal{S}$ can be chosen for every nonnegative integer n so that s_0 is equal to s and so that s_n belongs to Js_{n-1} when n is positive. An element s of $\mathcal{S}$ belongs to S_∞ if, and only if, Ts is contained in S_∞. No element s of $\mathcal{S}$ exists such that Ts is equal to S_∞.

A continuum of order γ is defined for a cardinal number γ as a set of least cardinality which has the same cardinality as the class of its subsets which are of cardinality less than γ. The empty set and a set with one element are continua with order equal to cardinality. No finite set containing more than one element is a continuum. A set of least infinite cardinality is a continuum with order equal to cardinality. The order of a continuum is less than or equal to the cardinality of a continuum.

The class of all subsets of a set of infinite cardinality γ is a set $\mathcal{S}$ which as the same cardinality as the class of its subsets of cardinality at most γ. The set $\mathcal{S}$ is a continuum whose order is the least cardinal number greater than γ. Every infinite cardinal number is the order of a continuum. The cardinality of a continuum of order γ is less than or equal to the cardinality of the class of all subsets of a set of cardinality γ.

A parametrization of a continuum $\mathcal{S}$ of order γ is an injective transformation J of the continuum onto the class of its subsets of cardinality less than γ such that no elements s_n of $\mathcal{S}$ can be chosen for every nonnegative integer n so that s_n belongs to Ts_{n-1} when n is positive. A continuum $\mathcal{S}$ of order γ admits an essentially unique parametrization J. If an injective transformation T maps $\mathcal{S}$ onto the class of its subsets which are continua of order less than γ, then an injective transformation W of $\mathcal{S}$ onto $\mathcal{S}_\infty$ exists such that TWa is always the set of elements Wb such that b belongs to Ja.

If $\mathcal{S}$ is a continuum of order γ with parametrization J and if a cardinal number α less than or equal to γ is the order of a continuum, then a unique subset of the continuum exists which is a continuum of order α with parametrization a restriction of J. A canonical subcontinuum of order α is contained in a canonical subcontinuum of order β if, and only if, α is less than or equal to β.

A well–ordering of a continuum $\mathcal{S}$ is said to be compatible with a parametrization J if the inequality $a < b$ holds whenever a belongs to Jb. The open interval defined by elements a and b of the well–ordered set such that $a < b$ is the set (a, b) of elements c such that $a < c$ and $c < b$. The well–ordered set is a Hausdorff space in a topology whose open sets are the unions of open intervals. The well–ordered set is locally compact in the order topology. The set is compact if, and only if, it contains a greatest element. If the well–ordered set is not compact, it is compactified by supplying an additional infinite element which is greater than the elements of the given set. The elements of the given set are recovered as the finite elements of the compactification.

A parametrized continuum admits a well–ordering which is compatible with its parametrization. The continuum is compact in its order topology only when it is finite. The compactification of the continuum in a related topology permits the construction of a continuum of greater order.

If a continuum $\mathcal{S}$ with parametrization J is treated in a compatible well–ordering, a regressive function is defined as a function $\pi(s)$ of s in $\mathcal{S}$ whose values are elements of the continuum less than s when s is not the least element and whose value is s otherwise. The regression topology of the continuum is the weakest topology for which every regressive function is continuous when values are given the order topology. The continuum is a completely regular Hausdorff space in the regression topology. The continuum admits a unique compactification $\mathcal{S}^\wedge$ to which every regressive function admits a unique continuous extension when values are given the order topology and an infinite value is admitted. A regressive function $\pi(s)$ of s in $\mathcal{S}$ admits a unique continuous extension as a function $\pi(s)$ of s in $\mathcal{S}^\wedge$ whose values are taken in $\mathcal{S}^\wedge$ and which is continuous in the regression topology of $\mathcal{S}^\wedge$.

Baire measurable subsets of $\mathcal{S}$ for the regression topology are applied in an integral representation of elements of $\mathcal{S}^\wedge$. An element c of $\mathcal{S}^\wedge$ determines a nonnegative measure γ on the Baire measurable subsets of $\mathcal{S}$ which satisfies the identity

$$f(c) = \int f d\gamma$$

for every continuous function of s in $\mathcal{S}$ which admits a continuous extension to $\mathcal{S}^\wedge$ for the regression topology. The only nonzero value of a representing measure is one.

If a regressive function π takes an element b of $\mathcal{S}^\wedge$ with representing measure β into an element a of $\mathcal{S}^\wedge$ with representing measure α, then the value of α of a Baire measurable set C is equal to the value of β on the set of elements s of $\mathcal{S}$ such that $\pi(s)$ belongs to C.

If an element c_n of $\mathcal{S}^\wedge$ with representing measure γ_n is defined for every nonnegative integer n so that the identity

$$c_n = \pi_n c_{n-1}$$

holds when n is positive for a regressive function π_n, then for large n the measure γ_n has value one on sets containing the origin and has value zero on other sets. The element c_n of $\mathcal{S}^\wedge$ is the least element of $\mathcal{S}$ when n is sufficiently large.

A partial ordering of $\mathcal{S}^\wedge$ is defined by $a < b$ if b is not the least element of $\mathcal{S}$ and if the identity

$$a = \pi b$$

holds for a regressive function π. The partial ordering of $\mathcal{S}^\wedge$ extends the well–ordering of $\mathcal{S}$. No elements c_n of $\mathcal{S}^\wedge$ can be defined for every nonnegative integer n so that the inequality

$$c_n < c_{n-1}$$

holds when n is positive.

If b is an element of $\mathcal{S}^\wedge$, define Jb as the set of elements a of $\mathcal{S}^\wedge$ which are obtained as images

$$a = \pi(b)$$

of b under some regressive function π such that $\pi(c)$ belongs to Jc for every element c of $\mathcal{S}$ such that Jc is nonempty. The inequality $a < b$ holds for elements a and b of $\mathcal{S}^\wedge$ if, and only if, elements $c_0, \dots, c_r$ of $\mathcal{S}^\wedge$ exist for some positive integer r such that $c_0 = a,\ c_r = b$, and C_{n-1} belongs to Jc_n when n is positive.

If the order of the continuum is a cardinal number τ such that a set of cardinality τ is not a union of a class of ordinality less than τ of sets of cardinality less than τ, then for every element c of the continuum the cardinality of the set of elements less than c is less than τ.

A regressive function π is defined so that

$$a = \pi b$$

is the least element of Jb such that the cardinality of Jb is less than or equal to the cardinality of a continuum whose order is less than or equal to the cardinality of the set of elements of Jb less than or equal to a.

A least element c of $\mathcal{S}^\wedge$ which does not belong to $\mathcal{S}$ exists since $\mathcal{S}$ does not contain every element of $\mathcal{S}^\wedge$.

These properties of the continuum $\mathcal{S}$ are used to show that the continuum is not the union of canonical subcontinua of smaller order.

References

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