

# SQUARE SUMMABLE POWER SERIES

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## PREFACE

An introduction to complex analysis is given in preparation for the proof of the Riemann hypothesis. Complex analysis is treated as a theory of functions analytic in the unit disk. The Hardy space for the unit disk is presented as a Hilbert space of functions represented by square summable power series. The factorization of functions which are analytic and bounded by one in the unit disk is applied to the construction of invariant subspaces for contractive transformations of a Hilbert space into itself. Hilbert spaces which are contained contractively in the Hardy space are essential to the relationship between factorization and invariant subspaces. Related Hilbert spaces of entire functions are applied in the proof of the Riemann hypothesis.

The Riemann mapping theorem is a persistent source of research in complex analysis. Best estimates of coefficients of Riemann mapping functions are stated in the Bieberbach conjecture, whose proof in 1984 prepares the final assault on the Riemann hypothesis. The proof of the Riemann hypothesis, which is now completed, gives insights into the nature of the Riemann mapping theorem. Classical themes of complex analysis retain their interest when they are formulated in the context indicated by the proof of the Riemann hypothesis.

A generalization of complex analysis applies to functions of a quaternion variable. The quaternion skew-plane is a skew-field which is rich in subfields isomorphic to the complex plane. An analytic function of a variable in a skew-plane is treated by analytic functions of variables in planes. Classical techniques of complex analysis are adapted to a new context in which generalizations of classical theorems are found. The Riemann mapping theorem admits a generalization in the skew-plane. The structure of Riemann mapping functions and the estimation of coefficients in the skew-plane are analogous to the results for the plane. The proof of the Riemann hypothesis stimulates new formulations of the Riemann mapping theorem.

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## CHAPTER 1. FACTORIZATION AND INVARIANT SUBSPACES

Complex numbers are constructed from the Gauss numbers

$$x + iy$$

whose coordinates  $x$  and  $y$  are rational numbers. The addition and multiplication of Gauss numbers is derived in the obvious way from the addition and multiplication of rational numbers and from the defining identity

$$i^2 = -1$$

of the imaginary unit  $i$ . Conjugation is the homomorphism of additive and multiplicative structure which takes

$$z = x + iy$$

into

$$z^- = x - iy.$$

The product

$$z^- z = x^2 + y^2$$

of a nonzero Gauss number and its conjugate is a positive rational number.

The Gauss plane admits topologies which are compatible with addition and multiplication. All of these topologies are relevant to complex analysis as it applies to the Riemann hypothesis. The Dedekind topology is derived from convex structure.

A convex combination

$$(1 - t)z + tw$$

of elements  $z$  and  $w$  of the Gauss plane is an element of the Gauss plane when  $t$  is a rational number in the interval  $[0, 1]$ . A subset of the Gauss plane is said to be preconvex if it contains all elements of the Gauss plane which are obtained as convex combinations of elements of the set. The preconvex span of a subset of the Gauss plane is defined as the smallest preconvex subset of the Gauss plane which contains the given set.

The closure in the Gauss plane of a preconvex subset  $B$  is the set  $B^-$  of elements  $w$  of the Gauss plane such that the set whose elements are  $w$  and the elements of  $B$  is preconvex. The closure of a preconvex set is a preconvex set which is its own closure.

A nonempty preconvex set is defined as open if it is disjoint from the closure of every disjoint nonempty preconvex set. The intersection of two nonempty preconvex sets is an open preconvex set when it is nonempty.

A subset of the Gauss plane is said to be open if it is a union of nonempty open preconvex sets. The empty set is open since it is an empty union of such sets. Unions of open subsets are open. Finite intersections of open sets are open.

An example of an open set is the complement in the Gauss plane of the closure of a nonempty preconvex set. A subset of the Gauss plane is said to be closed if it is the

complement in the Gauss plane of an open set. Intersections of closed sets are closed. Finite unions of closed sets are closed. The Gauss plane is a Hausdorff space in the topology whose open and closed sets are defined by convexity. These open and closed sets define the Dedekind topology of the Gauss plane.

If a nonempty open preconvex set  $A$  is disjoint from a nonempty preconvex set  $B$ , then a maximal preconvex set exists which contains  $B$  and is disjoint from  $A$ . The maximal convex set is closed and has a convex complement. The existence of the maximal preconvex set is an application of the Kuratowski–Zorn lemma. The construction of the Dedekind cut is a geometric formulation of the Hahn–Banach theorem.

Addition and multiplication are continuous as transformations of the Cartesian product of the Gauss plane with itself into the Gauss plane when the Gauss plane is given the Dedekind topology. The complex plane is the completion of the Gauss plane in the uniform Dedekind topology. Uniformity of topology refers to the determination of neighborhoods of a Gauss number by neighborhoods of the origin. If  $A$  is a neighborhood of the origin and if  $w$  is a Gauss number, then the set of sums of  $w$  with elements of  $A$  is a neighborhood of  $w$ . Every neighborhood of  $w$  is derived from a neighborhood of the origin.

A Cauchy class of closed subsets of the Gauss plane is a nonempty class of closed subsets such that the intersection of the members of any finite subclass is nonempty and such that for every neighborhood  $A$  of the origin some member  $B$  of the class exists such that all differences of elements of  $B$  belong to  $A$ .

A Cauchy class of closed subsets is contained in a maximal Cauchy class of closed subsets. A Cauchy sequence is a sequence of elements  $w_1, w_2, w_3, \dots$  of the Gauss plane such that a Cauchy class of closed subsets is defined whose members are indexed by the positive integers  $r$ . The  $r$ -th member is the Gauss closed convex span of the elements  $w_n$  with  $n$  not less than  $r$ . A Cauchy sequence determines a maximal Cauchy class. Every maximal Cauchy class is determined by a Cauchy sequence.

An element of the Gauss plane determines the maximal Cauchy class of closed sets which contain the element. An element of the complex plane is determined by a maximal Cauchy class. The Gauss plane is a subset of the complex plane.

If  $B$  is a closed subset of the Gauss plane, the closure  $B^-$  of  $B$  in the complex plane is the set of all elements of the complex plane whose maximal Cauchy class has  $B$  as a member. A subset of the complex plane is defined as open if it is disjoint from the closure in the complex plane of every disjoint closed subset of the Gauss plane. Unions of open subsets of the complex plane are open. Finite intersections of open subsets of the complex plane are open. A subset of the Gauss plane is open in the Dedekind topology of the Gauss plane if, and only if, it is the intersection with the Gauss plane of an open subset of the complex plane.

A subset of the complex plane is defined as closed if its complement in the complex plane is open. Intersections of closed subsets of the complex plane are closed. Finite unions of closed subsets of the complex plane are closed. The closure of a subset of the complex plane is defined as the smallest closed set containing the given subset. The Gauss closure

of a subset of the Gauss plane is the intersection with the Gauss plane of the closure of the subset in the complex plane.

The complex plane is a Hausdorff space in the topology whose open sets and closed sets are determined by convexity. These open sets and closed sets define the Dedekind topology of the complex plane.

The Gauss plane is dense in the complex plane. Addition and multiplication admit unique continuous extensions as transformations of the Cartesian product of the complex plane with itself into the complex plane. Conjugation admits a unique continuous extension as a transformation of the complex plane into itself.

Properties of addition in the Gauss plane are preserved in the complex plane. The associative law

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

holds for all complex numbers  $\alpha, \beta$ , and  $\gamma$ . The commutative law

$$\alpha + \beta = \beta + \alpha$$

holds for all complex numbers  $\alpha$  and  $\beta$ . The origin 0 of the Gauss plane satisfies the identities

$$0 + \gamma = \gamma = \gamma + 0$$

for every element  $\gamma$  of the complex plane. For every element  $\alpha$  of the complex plane a unique element

$$\beta = -\alpha$$

of the complex plane exists such that

$$\alpha + \beta = 0 = \beta + \alpha.$$

Conjugation is a homomorphism  $\gamma$  into  $\gamma^-$  of additive structure: The identity

$$(\alpha + \beta)^- = \alpha^- + \beta^-$$

holds for all complex numbers  $\alpha$  and  $\beta$ .

Multiplication by a complex number  $\gamma$  is a homomorphism of additive structure: The identity

$$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$$

holds for all complex numbers  $\alpha$  and  $\beta$ . The parametrization of homomorphisms is consistent with additive structure: The identity

$$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$$

holds for all complex numbers  $\alpha, \beta$ , and  $\gamma$ . Multiplication by  $\gamma$  is the homomorphism which annihilates every element of the complex plane when  $\gamma$  is the origin. Multiplication by  $\gamma$  is the identity homomorphism when  $\gamma$  is the unit 1 of the Gauss plane.

The composition of homomorphisms is consistent with multiplicative structure: The associative law

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

holds for all complex numbers  $\alpha, \beta$ , and  $\gamma$ . Multiplication is commutative: The identity

$$\alpha\beta = \beta\alpha$$

holds for all complex numbers  $\alpha$  and  $\beta$ . Conjugation is a homomorphism of multiplicative structure: The identity

$$(\alpha\beta)^- = \alpha^- \beta^-$$

holds for all complex numbers  $\alpha$  and  $\beta$ .

A real number is a complex number  $\gamma$  which is self-conjugate:

$$\gamma^- = \gamma.$$

Sums and products of real numbers are real. A real number is said to be nonnegative if it can be written

$$\gamma^- \gamma$$

for a complex number  $\gamma$ . The sum of two nonnegative numbers is nonnegative. A nonnegative number is said to be positive if it is nonzero. A positive number  $\alpha$  has a positive inverse

$$\beta = \alpha^{-1}$$

such that

$$\beta\alpha = 1 = \alpha\beta.$$

A nonzero complex number  $\alpha$  has a nonzero complex inverse

$$\beta = \alpha^{-1}$$

such that

$$\beta\alpha = 1 = \alpha\beta.$$

The complex plane is complete in the uniform Dedekind topology. Cauchy completion produces the same space. Closed and bounded subsets of the complex plane are compact. Boundedness of a set means that the elements  $\alpha$  of the set satisfy the inequality

$$\alpha^- \alpha \leq C$$

for some nonnegative number  $C$ . A nonempty class of closed sets has a nonempty intersection if the members of every finite subclass have a nonempty intersection and if some member of the class is bounded.

The space  $\mathcal{C}(z)$  of square summable power series is the set of power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots$$

with complex coefficients  $a_n$  such that the sum

$$\|f(z)\|^2 = |a_0|^2 + |a_1|^2 + |a_2|^2 + \dots$$

converges. If  $w$  is a complex number, the power series

$$wf(z) = wa_0 + wa_1z + wa_2z^2 + \dots$$

is square summable since the identity

$$\|wf(z)\|^2 = |w|^2\|f(z)\|^2$$

is satisfied. If power series

$$f(z) = a_0 + a_1z + a_2z^2 + \dots$$

and

$$g(z) = b_0 + b_1z + b_2z^2 + \dots$$

are square summable, then the convex combination

$$\begin{aligned} & (1-t)f(z) + tg(z) \\ &= [(1-t)a_0 + tb_0] + [(1-t)a_1 + tb_1]z + [(1-t)a_2 + tb_2]z^2 + \dots \end{aligned}$$

and the difference

$$g(z) - f(z) = (b_0 - a_0) + (b_1 - a_1)z + (b_2 - a_2)z^2 + \dots$$

are square summable power series since the convexity identity

$$\|(1-t)f(z) + tg(z)\|^2 + t(1-t)\|g(z) - f(z)\|^2 = (1-t)\|f(z)\|^2 + t\|g(z)\|^2$$

holds when  $t$  is in the interval  $[0, 1]$ .

The space of square summable power series is a vector space over the complex numbers which admits a scalar product. The scalar product of square summable power series

$$f(z) = a_0 + a_1z + a_2z^2 + \dots$$

and

$$g(z) = b_0 + b_1z + b_2z^2 + \dots$$

is the complex number

$$\langle f(z), g(z) \rangle = b_0^- a_0 + b_1^- a_1 + b_2^- a_2 + \dots$$

Convergence of the sum is a consequence of the identity

$$4\langle f(z), g(z) \rangle = \|f(z) + g(z)\|^2 - \|f(z) - g(z)\|^2 + i\|f(z) + ig(z)\|^2 - i\|f(z) - ig(z)\|^2.$$

Linearity of a scalar product states that the identity

$$\langle af(z) + bg(z), h(z) \rangle = a\langle f(z), h(z) \rangle + b\langle g(z), h(z) \rangle$$

holds for all complex numbers  $a$  and  $b$  when  $f(z)$ ,  $g(z)$ , and  $h(z)$  are square summable power series. Symmetry of a scalar product states that the identity

$$\langle g(z), f(z) \rangle = \langle f(z), g(z) \rangle^-$$

holds for all square summable power series  $f(z)$  and  $g(z)$ . Positivity of a scalar product states that the scalar self-product

$$\langle f(z), f(z) \rangle = \|f(z)\|^2$$

is positive for every nonzero square summable power series  $f(z)$ .

Linearity, symmetry, and positivity of a scalar product imply the Cauchy inequality

$$|\langle f(z), g(z) \rangle| \leq \|f(z)\| \|g(z)\|$$

for all square summable power series  $f(z)$  and  $g(z)$ . Equality holds in the Cauchy inequality if, and only if, the power series  $f(z)$  and  $g(z)$  are linearly dependent. The triangle inequality

$$\|h(z) - f(z)\| \leq \|g(z) - f(z)\| + \|h(z) - g(z)\|$$

follows for all square summable power series  $f(z)$ ,  $g(z)$ , and  $h(z)$ . The space of square summable power series is a metric space with

$$\|g(z) - f(z)\| = \|f(z) - g(z)\|$$

as the distance between square summable power series  $f(z)$  and  $g(z)$ . The space of square summable power series is a Hilbert space since the metric space is complete. If a Cauchy sequence  $f_0(z), f_1(z), f_2(z), \dots$  of square summable power series

$$f_n(z) = a_{n0} + a_{n1}z + a_{n2}z^2 + \dots$$

is given, the inequality

$$|a_{n0} - a_{m0}|^2 + \dots + |a_{nr} - a_{mr}|^2 \leq \|f_n(z) - f_m(z)\|^2$$

implies that a Cauchy sequence

$$a_{0k}, a_{1k}, a_{2k}, \dots$$

of complex numbers is obtained as coefficients of  $z^k$  for every nonnegative integer  $k$ . Since the complex plane is complete, a power series

$$f(z) = a_0 + a_1z + a_2z^2 + \dots$$

is defined whose coefficients

$$a_k = \lim a_{nk}$$

are limits. The triangle inequality implies that the sequence of distances

$$\|f_n(z) - f_0(z)\|, \|f_n(z) - f_1(z)\|, \|f_n(z) - f_2(z)\|, \dots$$

is Cauchy for every nonnegative integer  $n$ . A limit exists since the real line is complete. Since the inequality

$$|a_{n0} - a_0|^2 + \dots + |a_{nr} - a_r|^2 \leq \lim \|f_n(z) - f_m(z)\|^2$$

holds for every positive integer  $r$ , the power series

$$f_n(z) - f(z)$$

is square summable. Since the inequality

$$\|f_n(z) - f(z)\|^2 \leq \lim \|f_n(z) - f_m(z)\|^2$$

is satisfied, the square summable power series  $f(z)$  is the limit of the Cauchy sequence of square summable power series  $f_n(z)$ .

The distance from an element of a Hilbert space to a nonempty closed convex subset of a Hilbert space is attained by an element of the convex set. If  $f(z)$  is a square summable power series and if  $C$  is a nonempty closed convex set of square summable power series, then an element  $g(z)$  of  $C$  exists which minimizes the distance  $\delta$  from  $f(z)$  to elements of  $C$ . By definition

$$\delta = \inf \|f(z) - g(z)\|$$

is a greatest lower bound taken over the elements  $g(z)$  of  $C$ . It needs to be shown that

$$\delta = \|f(z) - g(z)\|$$

for some element  $g(z)$  of  $C$ . For every positive integer  $n$  an element  $g_n(z)$  of  $C$  exists which satisfies the inequality

$$\|f(z) - g_n(z)\|^2 \leq \delta^2 + n^{-2}.$$

Since

$$(1-t)g_m(z) + tg_n(z)$$

belongs to  $C$  when  $t$  belongs to the interval  $[0, 1]$ , the inequality

$$\delta \leq \|f(z) - (1-t)g_m(z) - tg_n(z)\|$$

is then satisfied. The convexity identity

$$\begin{aligned} & \|f(z) - (1-t)g_m(z) - tg_n(z)\|^2 \\ & + t(1-t)\|g_m(z) - g_n(z)\|^2 = (1-t)\|f(z) - g_m(z)\|^2 + t\|f(z) - g_n(z)\|^2 \end{aligned}$$

implies the inequality

$$t(1-t)\|g_m(z) - g_n(z)\|^2 \leq (1-t)m^{-2} + tn^{-2}.$$

Since the square summable power series  $g_n(z)$  form a Cauchy sequence, they converge to a square summable power series  $g(z)$ . Since  $C$  is closed,  $g(z)$  is an element of  $C$  which satisfies the identity

$$\delta = \|f(z) - g(z)\|.$$

A vector subspace of the space of square summable power series is an example of a nonempty convex subset. The orthogonal complement of a vector subspace  $\mathcal{M}$  of the space of square summable power series is the set of square summable power series  $f(z)$  which are orthogonal

$$\langle f(z), g(z) \rangle = 0$$

to every element  $g(z)$  of  $\mathcal{M}$ . The orthogonal complement of a vector subspace of the space of square summable power series is a closed vector subspace of the space of square summable power series. When  $\mathcal{M}$  is a closed vector subspace of the space of square summable power series, and when  $f(z)$  is a square summable power series, an element  $g(z)$  of  $\mathcal{M}$  which is nearest  $f(z)$  is unique and is characterized by the orthogonality of  $f(z) - g(z)$  to elements of  $\mathcal{M}$ . Then  $f(z) - g(z)$  belongs to the orthogonal complement of  $\mathcal{M}$ . Every square summable power series  $f(z)$  is the sum of an element  $g(z)$  of  $\mathcal{M}$ , called the orthogonal projection of  $f(z)$  in  $\mathcal{M}$ , and an element  $f(z) - g(z)$  of the orthogonal complement of  $\mathcal{M}$ . Orthogonal projection is linear. If  $f_0(z)$  and  $f_1(z)$  are elements of  $\mathcal{M}$  and if  $c_0$  and  $c_1$  are complex numbers, the orthogonal projection of

$$c_0f_0(z) + c_1f_1(z)$$

in  $\mathcal{M}$  is

$$c_0g_0(z) + c_1g_1(z)$$

with  $g_0(z)$  the orthogonal projection of  $f_0(z)$  in  $\mathcal{M}$  and  $g_1(z)$  the orthogonal projection of  $f_1(z)$  in  $\mathcal{M}$ . The closure of a vector subspace  $\mathcal{M}$  of the space of square summable power series is a vector subspace which is the orthogonal complement of the closed vector subspace which is the orthogonal complement of  $\mathcal{M}$ .

A linear functional on the space of square summable power series is a linear transformation of the space into the complex numbers. A linear functional is continuous if, and only if, its kernel is closed. If a continuous linear functional does not annihilate every square summable power series, then its kernel is a proper closed subspace of the space of square summable power series whose orthogonal complement has dimension one. An element  $g(z)$  of the orthogonal complement of the kernel exists such that the linear functional takes  $f(z)$  into the scalar product

$$\langle f(z), g(z) \rangle$$

for every square summable power series  $f(z)$ . If  $g(z)$  is a square summable power series, the linear functional which takes  $f(z)$  into

$$\langle f(z), g(z) \rangle$$

is continuous by the Cauchy inequality.

Multiplication by  $z$  in the space of square summable power series is the transformation which takes

$$f(z) = a_0 + a_1z + a_2z^2 + \dots$$

into

$$zf(z) = 0 + a_0z + a_1z^2 + \dots$$

whenever  $f(z)$ , and hence  $zf(z)$ , is square summable. Multiplication by  $z$  is an isometric transformation

$$\|zf(z)\| = \|f(z)\|$$

of the space of square summable power series into itself. The range of multiplication by  $z$  is the set of square summable power series with constant coefficient zero. The range is a closed vector subspace which is a Hilbert space in the inherited scalar product. The orthogonal complement of the range is the space of constants.

The space of square summable power series with complex coefficients is a fundamental example of a Hilbert space. The constructions made in this space apply in the Hilbert space of square summable power series with coefficients in a Hilbert space. Properties of topology in Hilbert spaces are applied in the generalization.

A Hilbert space is a vector space  $\mathcal{H}$  over the complex numbers which is given a scalar product for which self-products of nonzero elements are positive and which is complete in a resulting metric topology.

Addition is a transformation of the Cartesian product of the space with itself into the space whose properties are familiar from the complex plane. The associative law

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

holds for all elements  $\alpha, \beta$ , and  $\gamma$  of the space. The commutative law

$$\alpha + \beta = \beta + \alpha$$

holds for all elements  $\alpha$  and  $\beta$  of the space. The space contains an origin  $0$  which satisfies the identity

$$0 + \gamma = \gamma = \gamma + 0$$

for every element  $\gamma$  of the space. For every element  $\alpha$  of the space, a unique element

$$\beta = -\alpha$$

of the space exists such that

$$\alpha + \beta = 0 = \beta + \alpha.$$

Multiplication is a transformation of the Cartesian product of the complex plane with the space into the space whose properties are familiar from the complex plane. Multiplication by a complex number  $\gamma$  is a homomorphism of additive structure: The identity

$$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$$

holds for all elements  $\alpha$  and  $\beta$  of the space. The parametrization of homomorphisms is consistent with additive structure: The identity

$$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$$

holds for all complex numbers  $\alpha$  and  $\beta$  when  $\gamma$  is an element of the space. Multiplication by  $\gamma$  is the homomorphism which annihilates every element of the space when  $\gamma$  is the origin of the complex plane. Multiplication by  $\gamma$  is the identity homomorphism when  $\gamma$  is the unit of the complex plane. The composition of homomorphisms is consistent with multiplicative structure: The associative law

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

holds for all complex numbers  $\alpha$  and  $\beta$  when  $\gamma$  is an element of the space.

The scalar product is a transformation of the Cartesian product of the space with itself into the complex numbers whose properties are familiar from the complex plane. The scalar product defines homomorphisms of additive structure: The identities

$$\langle \alpha + \beta, \gamma \rangle = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$$

and

$$\langle \gamma, \alpha + \beta \rangle = \langle \gamma, \alpha \rangle + \langle \gamma, \beta \rangle$$

hold for all elements  $\alpha, \beta$ , and  $\gamma$  of the space. The scalar product defines a homomorphism and an anti-homomorphism of multiplicative structure for all elements  $\alpha$  and  $\beta$  of the space: The identities

$$\langle \gamma\alpha, \beta \rangle = \gamma\langle \alpha, \beta \rangle$$

and

$$\langle \alpha, \gamma\beta \rangle = \gamma^{-1}\langle \alpha, \beta \rangle$$

hold for every complex number  $\gamma$ . The scalar product is symmetric: The identity

$$\langle \beta, \alpha \rangle = \langle \alpha, \beta \rangle^{-1}$$

holds for all elements  $\alpha$  and  $\beta$  of the space. The scalar self-product

$$\langle c, c \rangle$$

is positive for every nonzero element  $c$  of the space.

A convex combination

$$(1 - t)\alpha + t\beta$$

of elements  $\alpha$  and  $\beta$  of the space is an element of the space defined by a real number  $t$  in the interval  $[0, 1]$ . A subset of the space is said to be convex if it contains all convex combinations of pairs of its elements. The convex span of a subset of the space is the smallest convex set which contains the given set.

The norm of an element  $\gamma$  of the space is the nonnegative solution  $\|\gamma\|$  of the equation

$$\|\gamma\|^2 = \langle \gamma, \gamma \rangle.$$

The Cauchy inequality

$$|\langle \alpha, \beta \rangle| \leq \|\alpha\| \|\beta\|$$

holds for all elements  $\alpha$  and  $\beta$  of the space. Equality holds when, and only when,  $\alpha$  and  $\beta$  are linearly dependent. The triangle inequality

$$\|\gamma - \alpha\| \leq \|\gamma - \beta\| + \|\beta - \alpha\|$$

holds for all elements  $\alpha, \beta$ , and  $\gamma$  of the space. The space is a metric space with

$$\|\beta - \alpha\| = \|\alpha - \beta\|$$

as the distance between elements  $\alpha$  and  $\beta$ .

The closure  $B^-$  of a nonempty convex subset  $B$  of the space is defined using the metric topology. The closure of a convex set is a convex set which is its own closure. The closed convex span of a set is the smallest closed convex set which contains the given set.

The metric is not used for the definition of an open set. A nonempty convex set is said to be open if it is disjoint from the closure of every nonempty disjoint convex set. If  $A$  is a nonempty open convex set and if  $B$  is a nonempty convex set, then the intersection of  $A$  with the closure of  $B$  is contained in the closure of the intersection of  $A$  with  $B$ . The intersection of two nonempty open convex sets is an open convex set if it is nonempty.

A subset of the space is said to be open if it is a union of nonempty open convex sets. The empty set is open since it is an empty union. Unions of open sets are open. Finite intersections of open sets are open.

An example of an open set is the complement of the closure of a convex set. A set is said to be closed if its complement is open. Intersections of closed sets are closed. Finite unions of closed sets are closed. The space is a Hausdorff space in the topology whose open and closed sets are defined by convexity. The open and closed sets define the Dedekind topology of the space.

The Dedekind topology is applied in the geometric formulation of the Hahn–Banach theorem: If a nonempty open convex set  $A$  is disjoint from a nonempty convex set  $B$ , then a maximal convex set exists which contains  $B$  and is disjoint from  $A$ . The maximal convex set is closed and has a convex complement.

If  $\beta$  is an element of the space, a continuous linear functional  $\beta^-$  is defined on the space by taking  $\alpha$  into

$$\beta^- \alpha = \langle \alpha, \beta \rangle$$

for every element  $\alpha$  of the space. If  $c$  is a real number, the set of elements  $\alpha$  which satisfy the inequality

$$\mathcal{R}\beta^- \alpha \geq c$$

is a closed convex set whose complement is convex.

Completeness is required of a Hilbert space so that every nonempty closed convex set whose complement is nonempty and convex is determined by an element of the space. A set which is open for the metric topology is open for the Dedekind topology since it is a union of convex sets which are open for the Dedekind topology. The convex set is defined by an element  $\gamma$  of the space and a positive number  $c$  as the set of elements  $\alpha$  which satisfy the inequality

$$(\alpha - \gamma)^-(\alpha - \gamma) < c.$$

When the space is complete in the metric topology, every set which is open for the Dedekind topology is open for the metric topology. The proof applies consequences of completeness in metric spaces observed by Baire.

It needs to be shown that every nonempty convex set which is open for the Dedekind topology is open for the metric topology. The convex set  $A$  can be assumed to contain the origin. Argue by contradiction assuming that the set is not open for the metric topology. Since the set is convex, it has an empty interior for the metric topology.

A convex set  $A_n$  which is open for the Dedekind topology is defined for every positive integer  $n$  as the set of products  $n\alpha$  with  $\alpha$  in  $A$ . The set is convex and has empty interior for the metric topology. A positive number  $c_n$  is chosen for every positive integer  $n$  so that the sum of the numbers converges. An element  $\alpha_n$  of  $A_n$  is chosen inductively so that the inequality

$$(\alpha_{n+1} - \alpha_n)^-(\alpha_{n+1} - \alpha_n) < c_n^2$$

holds for every positive integer  $n$ . The element  $\alpha_1$  of  $A_1$  is chosen arbitrarily. When  $\alpha_n$  is chosen in  $A_n$ , the element  $\alpha_{n+1}$  is chosen in  $A_{n+1}$  so that it does not belong to the closure of  $A_n$ . The element  $\alpha_n$  of the space converge to an element  $\alpha$  of the space since a Hilbert space is assumed metrically complete. Properties of open sets for the Dedekind topology are contradicted since the union of the set  $A_n$  does not contain every element of the space.

Since the Dedekind topology of a Hilbert space is identical with the metric topology, a Hilbert space is complete in the Dedekind topology.

If  $B$  is a nonempty closed convex subset of a Hilbert space and if  $\alpha$  is an element of the space which does not belong to  $B$ , then an element  $\gamma$  of the convex set exists which is closest to  $\alpha$ . The inequality

$$(\alpha - \gamma)^-(\alpha - \gamma) \leq (\alpha - \beta)^-(\alpha - \beta)$$

holds for every element  $\beta$  of  $B$ . The inequality

$$\mathcal{R}(\alpha - \gamma)^-(\beta - \gamma) \geq 0$$

holds for every element  $\beta$  of  $B$ . The set of elements  $\beta$  of the space which satisfy the inequality

$$\mathcal{R}(\alpha - \gamma)^-\beta \geq \mathcal{R}(\alpha - \gamma)^-\gamma$$

is a closed convex set which contains  $B$  and whose complement is convex and contains  $\alpha$ .

A subset  $B$  of a Hilbert space is said to be bounded if the linear functional  $\alpha^-$  maps  $B$  onto a bounded subset of the complex plane for every element  $\alpha$  of the space. If  $B$  is a nonempty closed and bounded convex subset of a Hilbert space, an element  $\gamma$  of  $B$  exists which satisfies the inequality

$$\beta^- \beta \leq \gamma^- \gamma$$

for every element  $\beta$  of  $B$ .

The closed convex span of a bounded subset of a Hilbert space is bounded. If  $B$  is a nonempty bounded subset of a Hilbert space, an element  $\gamma$  of its closed convex span exists which satisfies the inequality

$$\beta^- \beta \leq \gamma^- \gamma$$

for every element  $\beta$  of  $B$ .

A Hilbert space admits a topology which is compatible with addition and multiplication and for which every closed and bounded set is compact. The topology is defined as the weakest topology with respect to which the linear functional  $\alpha^-$  is continuous for every element  $\alpha$  of the space. An open set for the weak topology is a union of open convex sets. A convex set is closed for the weak topology if, and only if, it is closed for the Dedekind topology.

An orthonormal set is a set of elements such that the scalar self-product of every element is one but the scalar product of distinct elements is zero. A Hilbert space admits a maximal orthonormal set as an application of the Kuratowski-Zorn lemma. Two maximal orthonormal sets have the same cardinality. The dimension of the Hilbert space is defined as the cardinality of a maximal orthonormal set.

If  $\mathcal{S}$  is a maximal orthonormal set in a Hilbert space, the space determines a subspace of the Cartesian product of the complex numbers taken over the elements of  $\mathcal{S}$ . An element of the Cartesian product is written as a formal sum

$$\sum f(\iota)\iota$$

over the elements  $\iota$  of the orthonormal set with coefficients which are a complex valued function  $f(\iota)$  of  $\iota$  in the orthonormal set. An element of the Cartesian product determines an element of the Hilbert space if, and only if, the sum

$$\sum |f(\iota)|^2$$

over the elements of the orthonormal set converges. The weak topology of the Hilbert space agrees with the subspace topology of the Cartesian product topology on bounded sets. The compactness of closed and bounded subsets of the Cartesian product space follows from the compactness of closed and bounded subsets of the complex plane since a Cartesian product of compact Hausdorff spaces is a compact Hausdorff space. The weak compactness of closed and bounded convex subsets of a Hilbert space follows because a closed subset of a compact set is compact.

The Hilbert space  $\mathcal{C}(z)$  of square summable power series with coefficients in a Hilbert space  $\mathcal{C}$  is advantageous for the relationship between factorization and invariant subspaces. The construction of the space imitates the construction made when  $\mathcal{C}$  is the complex numbers.

A vector is an element of the coefficient space  $\mathcal{C}$ . The scalar product

$$b^{-}a$$

of vectors  $a$  and  $b$  is taken in the complex plane. The norm  $|c|$  of a vector is the nonnegative solution of the equation

$$|c|^2 = c^{-}c.$$

The space  $\mathcal{C}(z)$  of square summable power series with coefficients in  $\mathcal{C}$  is the set of power series

$$f(z) = a_0 + a_1z + a_2z^2 + \dots$$

with vector coefficients such that the sum

$$\|f(z)\|^2 = |a_0|^2 + |a_1|^2 + |a_2|^2 + \dots$$

converges. If  $w$  is a complex number, the power series

$$wf(z) = wa_0 + wa_1z + wa_2z^2 + \dots$$

with vector coefficients is square summable since the identity

$$\|wf(z)\|^2 = |w|^2\|f(z)\|^2$$

is satisfied. If power series

$$f(z) = a_0 + a_1z + a_2z^2 + \dots$$

and

$$g(z) = b_0 + b_1z + b_2z^2 + \dots$$

with vector coefficients are square summable, then the convex combination

$$\begin{aligned} & (1-t)f(z) + tg(z) \\ &= [(1-t)a_0 + tb_0] + [(1-t)a_1 + tb_1]z + [(1-t)a_2 + tb_2]z^2 + \dots \end{aligned}$$

and the difference

$$g(z) - f(z) = (b_0 - a_0) + (b_1 - a_1)z + (b_2 - a_2)z^2 + \dots$$

are power series with vector coefficients which are square summable since the convexity identity

$$\|(1-t)f(z) + tg(z)\|^2 + t(1-t)\|g(z) - f(z)\|^2 = (1-t)\|f(z)\|^2 + t\|g(z)\|^2$$

holds when  $t$  is in the interval  $[0, 1]$ .

The space of square summable power series with vector coefficients is a vector space over the complex numbers which admits a scalar product. The scalar product

$$\langle f(z), g(z) \rangle = b_0^- a_0 + b_1^- a_1 + b_2^- a_2 + \dots$$

of square summable power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots$$

and

$$g(z) = b_0 + b_1 z + b_1 z^2 + \dots$$

with vector coefficients is a complex number. Convergence of the sum is a consequence of the identity

$$4\langle f(z), g(z) \rangle = \|f(z) + g(z)\|^2 - \|f(z) - g(z)\|^2 + i\|f(z) + ig(z)\|^2 - i\|f(z) - ig(z)\|^2.$$

Linearity

$$\langle af(z) + bg(z), h(z) \rangle = a\langle f(z), h(z) \rangle + b\langle g(z), h(z) \rangle$$

holds for all complex numbers  $a$  and  $b$  when  $f(z), g(z)$ , and  $h(z)$  are square summable power series with vector coefficients. Symmetry

$$\langle g(z), f(z) \rangle = \langle f(z), g(z) \rangle^-$$

holds for all square summable power series  $f(z)$  and  $g(z)$  with vector coefficients. The scalar self-product

$$\langle f(z), f(z) \rangle$$

of a nonzero square summable power series  $f(z)$  with vector coefficients is positive.

The norm of a square summable power series  $f(z)$  with vector coefficients is the non-negative solution  $\|f(z)\|$  of the equation

$$\|f(z)\|^2 = \langle f(z), f(z) \rangle.$$

The Cauchy inequality

$$|\langle f(z), g(z) \rangle| \leq \|f(z)\| \|g(z)\|$$

holds for all square summable power series  $f(z)$  and  $g(z)$  with vector coefficients. Equality holds in the Cauchy inequality if, and only if,  $f(z)$  and  $g(z)$  are linearly dependent. The triangle inequality

$$\|h(z) - f(z)\| \leq \|g(z) - f(z)\| + \|h(z) - g(z)\|$$

holds for all square summable power series  $f(z), g(z)$ , and  $h(z)$  with vector coefficients. The space of square summable power series with vector coefficients is a metric space with

$$\|g(z) - f(z)\| = \|f(z) - g(z)\|$$

as the distance between square summable power series  $f(z)$  and  $g(z)$  with vector coefficients. The space of square summable power series with vector coefficients is a Hilbert space since it is complete in the metric topology.

The multiplicative structure of the complex numbers is lacking in other coefficient spaces. Multiplicative structure is restored by introducing operators on the coefficient space. An operator is a continuous linear transformation of the coefficient space into itself. A complex number is treated as an operator since multiplication by a number is a continuous linear transformation. Since continuity is taken in the Dedekind topology, a continuous transformation is a constant multiple of a contractive transformation. The bound  $|\gamma|$  of an operator  $\gamma$  is the least nonnegative number such that the operator is the product of  $|\gamma|$  and a contractive transformation. When  $\gamma$  is a number, the bound is the absolute value of  $\gamma$ .

If  $W(z)$  is a power series with operator coefficients and if  $f(z)$  is a power series with vector coefficients, the product

$$W(z)f(z)$$

is a power series with vector coefficients. Multiplication by  $W(z)$  acts as a partially isometric transformation of the space  $\mathcal{C}(z)$  of square summable power series with vector coefficients onto a Hilbert space whose elements are power series with vector coefficients. The partially isometric character of the transformation refers to its isometric character on the orthogonal complement of its kernel. The space obtained is contained contractively in  $\mathcal{C}(z)$ , if, and only if, multiplication by  $W(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself. Multiplication by  $W(z)$  is a contractive transformation of the space into itself.

If a Hilbert space  $\mathcal{P}$  is contained contractively in a Hilbert space  $\mathcal{H}$ , a unique Hilbert space  $\mathcal{Q}$  exists, which is contained contractively in  $\mathcal{H}$ , such that the inequality

$$\|c\|_{\mathcal{H}}^2 \leq \|a\|_{\mathcal{P}}^2 + \|b\|_{\mathcal{Q}}^2$$

holds whenever

$$c = a + b$$

is the sum of an element  $a$  of  $\mathcal{P}$  and an element  $b$  of  $\mathcal{Q}$ , and such that every element  $c$  of  $\mathcal{H}$  is a sum for which equality holds. The space  $\mathcal{Q}$  is called the complementary space to  $\mathcal{P}$  in  $\mathcal{H}$ .

The construction of the space  $\mathcal{Q}$  and the verification of its properties are applications of convexity. The space  $\mathcal{Q}$  is defined as the set of elements  $b$  of  $\mathcal{H}$  for which a finite least upper bound

$$\|b\|_{\mathcal{Q}}^2 = \sup[\|a + b\|_{\mathcal{H}}^2 - \|a\|_{\mathcal{P}}^2]$$

is obtained over all elements  $a$  of  $\mathcal{P}$ . If  $b$  is an element of  $\mathcal{Q}$  and if  $w$  is a complex number, then  $wb$  is an element of  $\mathcal{Q}$  which satisfies the identity

$$\|wb\|_{\mathcal{Q}} = |w|\|b\|_{\mathcal{Q}}.$$

A convex combination

$$(1 - t)a + tb$$

of elements of  $\mathcal{Q}$  is an element of  $\mathcal{Q}$  which satisfies the convexity identity

$$\|(1-t)a + tb\|_{\mathcal{Q}}^2 + t(1-t)\|b - a\|_{\mathcal{Q}}^2 = (1-t)\|a\|_{\mathcal{Q}}^2 + t\|b\|_{\mathcal{Q}}^2$$

with  $t$  in the interval  $[0, 1]$ .

The space  $\mathcal{Q}$  is a vector space over the complex numbers. A scalar product is defined in the space by the identity

$$4\langle a, b \rangle_{\mathcal{Q}} = \|a + b\|_{\mathcal{Q}}^2 - \|a - b\|_{\mathcal{Q}}^2 + i\|a + ib\|_{\mathcal{Q}}^2 - i\|a - ib\|_{\mathcal{Q}}^2.$$

Immediate consequences of the definition are symmetry

$$\langle b, a \rangle_{\mathcal{Q}} = \langle a, b \rangle_{\mathcal{Q}}^{-}$$

and the identity

$$\langle \omega a, b \rangle = \omega \langle a, b \rangle$$

when  $\omega$  is a fourth root of unity if  $a$  and  $b$  are elements of  $\mathcal{Q}$ . An application of the convexity identity gives the identity

$$\|(1-t)a + tb\|_{\mathcal{Q}}^2 - \|(1-t)a - tb\|_{\mathcal{Q}}^2 = t(1-t)\|a + b\|_{\mathcal{Q}}^2 - t(1-t)\|a - b\|_{\mathcal{Q}}^2$$

which implies the identity

$$\langle (1-t)a, tb \rangle_{\mathcal{Q}} = t(1-t)\langle a, b \rangle_{\mathcal{Q}}$$

for all elements  $a$  and  $b$  of  $\mathcal{Q}$  when  $t$  is in the interval  $[0, 1]$ . Another application of the convexity identity gives the identity

$$\begin{aligned} & \|(1-t)(a+c) + t(b+c)\|_{\mathcal{Q}}^2 - \|(1-t)(a-c) + t(b-c)\|_{\mathcal{Q}}^2 \\ &= (1-t)\|a+c\|_{\mathcal{Q}}^2 - (1-t)\|a-c\|_{\mathcal{Q}}^2 + t\|b+c\|_{\mathcal{Q}}^2 - t\|b-c\|_{\mathcal{Q}}^2 \end{aligned}$$

which implies the identity

$$\langle (1-t)a + tb, c \rangle_{\mathcal{Q}} = (1-t)\langle a, c \rangle_{\mathcal{Q}} + t\langle b, c \rangle_{\mathcal{Q}}$$

for all elements  $a, b$ , and  $c$  of  $\mathcal{Q}$  when  $t$  belongs to the interval  $[0, 1]$ . Linearity of a scalar product follows.

Positivity of a scalar product is immediate from the definition. The inequality

$$\|b\|_{\mathcal{H}} \leq \|b\|_{\mathcal{Q}}$$

for elements  $b$  of  $\mathcal{Q}$  states that the space  $\mathcal{Q}$  is contained contractively in the space  $\mathcal{H}$ . The inequality is used to prove that the space  $\mathcal{Q}$  is complete in the metric topology defined by

the scalar product of  $\mathcal{Q}$ . A Cauchy sequence of elements  $b_n$  of  $\mathcal{Q}$  is a Cauchy sequence of elements of  $\mathcal{H}$  since the inequality

$$\|b_m - b_n\|_{\mathcal{H}} \leq \|b_m - b_n\|_{\mathcal{Q}}$$

holds for all nonnegative integers  $m$  and  $n$ . Since  $\mathcal{H}$  is a Hilbert space, an element  $b$  of  $\mathcal{H}$  exists which is the limit of the sequence in the metric topology of  $\mathcal{H}$ . Since the inequality

$$\|b_n\|_{\mathcal{Q}} \leq \|b_m\|_{\mathcal{Q}} + \|b_m - b_n\|_{\mathcal{Q}}$$

holds for all nonnegative integers  $m$  and  $n$ , the numbers  $\|b_n\|_{\mathcal{Q}}$  form a Cauchy sequence. A nonnegative number exists which is the limit of the sequence. If  $a$  is an element of  $\mathcal{P}$ , the element  $a + b$  of  $\mathcal{H}$  is the limit of the elements  $a + b_n$  in the metric topology of  $\mathcal{H}$ . Since the inequality

$$\|a + b_n\|_{\mathcal{H}}^2 \leq \|a\|_{\mathcal{P}}^2 + \|b_n\|_{\mathcal{Q}}^2$$

holds for every nonnegative integer  $n$ , the inequality

$$\|a + b\|_{\mathcal{H}}^2 \leq \|a\|_{\mathcal{P}}^2 + \lim \|b_n\|_{\mathcal{Q}}^2$$

is satisfied. Since the inequality

$$\|b\|_{\mathcal{Q}} \leq \lim \|b_n\|_{\mathcal{Q}}$$

follows,  $b$  belongs to  $\mathcal{Q}$ . Since the inequality

$$\|b - b_m\|_{\mathcal{Q}} \leq \lim \|b_n - b_m\|_{\mathcal{Q}}$$

holds for every nonnegative integer  $m$ ,  $b$  is the limit of the elements  $b_n$  in the metric topology of  $\mathcal{Q}$ .

The inequality

$$\|c\|_{\mathcal{H}}^2 \leq \|a\|_{\mathcal{P}}^2 + \|b\|_{\mathcal{Q}}^2$$

holds by the definition of  $\mathcal{Q}$  whenever

$$c = a + b$$

is the sum of an element  $a$  of  $\mathcal{P}$  and an element  $b$  of  $\mathcal{Q}$ . It will be shown that every element  $c$  of  $\mathcal{H}$  is the sum of an element  $a$  of  $\mathcal{P}$  and an element  $b$  of  $\mathcal{Q}$  for which equality holds. When  $c$  is given, a continuous linear functional on  $\mathcal{P}$  is defined by taking  $u$  into

$$\langle u, c \rangle_{\mathcal{H}}$$

since the inclusion of  $\mathcal{P}$  in  $\mathcal{H}$  is continuous.

The element  $a$  of  $\mathcal{P}$  which satisfies the identity

$$\langle u, c \rangle_{\mathcal{H}} = \langle u, a \rangle_{\mathcal{P}}$$

is obtained from  $c$  under the adjoint of the inclusion of  $\mathcal{P}$  in  $\mathcal{H}$ . It will be shown that the element

$$b = c - a$$

of  $\mathcal{H}$  belongs to  $\mathcal{Q}$ . Since the identity

$$\|u + b\|_{\mathcal{H}}^2 = \|c\|_{\mathcal{H}}^2 + \langle a, u - a \rangle_{\mathcal{P}} + \langle u - a, a \rangle_{\mathcal{P}} + \|u - a\|_{\mathcal{H}}^2$$

holds for every element  $u$  of  $\mathcal{P}$  and since the inequality

$$\|u - a\|_{\mathcal{H}} \leq \|u - a\|_{\mathcal{P}}$$

is satisfied, the inequality

$$\|u + b\|_{\mathcal{H}}^2 \leq \|c\|_{\mathcal{H}}^2 - \|a\|_{\mathcal{P}}^2 + \|u\|_{\mathcal{P}}^2$$

holds for every element  $u$  of  $\mathcal{P}$ . Since the inequality

$$\|b\|_{\mathcal{Q}}^2 \leq \|c\|_{\mathcal{H}}^2 - \|a\|_{\mathcal{P}}^2$$

holds by the definition of  $\mathcal{Q}$ ,  $b$  is an element of  $\mathcal{Q}$ . Equality holds since the reverse inequality holds by the definition of  $\mathcal{Q}$ .

The intersection of  $\mathcal{P}$  and  $\mathcal{Q}$  is a Hilbert space  $\mathcal{P} \wedge \mathcal{Q}$  with scalar product

$$\langle a, b \rangle_{\mathcal{P} \wedge \mathcal{Q}} = \langle a, b \rangle_{\mathcal{P}} + \langle a, b \rangle_{\mathcal{Q}}.$$

The inclusions of  $\mathcal{P} \wedge \mathcal{Q}$  and in  $\mathcal{P}$  and in  $\mathcal{Q}$  are continuous. If an element

$$c = a + b$$

of  $\mathcal{H}$  is the sum of an element  $a$  of  $\mathcal{P}$  and an element  $b$  of  $\mathcal{Q}$  such that equality holds in the inequality

$$\|c\|_{\mathcal{H}}^2 \leq \|a\|_{\mathcal{P}}^2 + \|b\|_{\mathcal{Q}}^2,$$

then the inequality

$$\|a\|_{\mathcal{P}}^2 + \|b\|_{\mathcal{Q}}^2 \leq \|a + u\|_{\mathcal{P}}^2 + \|b - u\|_{\mathcal{Q}}^2$$

holds for every element  $u$  of the intersection of  $\mathcal{P}$  and  $\mathcal{Q}$ . Since  $u$  can be replaced by  $wu$  for every complex number  $w$ , the identity

$$\langle a, u \rangle_{\mathcal{P}} = \langle b, u \rangle_{\mathcal{Q}}$$

holds for every element  $u$  of  $\mathcal{P} \wedge \mathcal{Q}$ . The identity implies uniqueness of the elements  $a$  and  $b$  in the minimal decomposition of an element  $c$  of  $\mathcal{H}$ . The element  $a$  of  $\mathcal{P}$  is obtained from  $c$  under the adjoint of the inclusion of  $\mathcal{P}$  in  $\mathcal{H}$ . The element  $b$  of  $\mathcal{Q}$  is obtained from  $c$  under the adjoint of the inclusion of  $\mathcal{Q}$  in  $\mathcal{H}$ . The identity

$$\langle c, c' \rangle_{\mathcal{H}} = \langle a, a' \rangle_{\mathcal{P}} + \langle b, b' \rangle_{\mathcal{Q}}$$

holds whenever an element

$$c' = a' + b'$$

of  $\mathcal{H}$  is the sum of an element  $a'$  of  $\mathcal{P}$  and an element  $b'$  of  $\mathcal{Q}$ .

Uniqueness of the complementary space  $\mathcal{Q}$  to  $\mathcal{P}$  in  $\mathcal{H}$  is a consequence of properties of minimal decompositions. The adjoint of the inclusion of  $\mathcal{Q}$  in  $\mathcal{H}$  is the same for every complementary space. The elements of a complementary space  $\mathcal{Q}$  obtained from the adjoint of the inclusion of  $\mathcal{Q}$  in  $\mathcal{H}$  are dense in  $\mathcal{Q}$  and have scalar self-products in  $\mathcal{Q}$  which are independent of the choice of complementary space  $\mathcal{Q}$ . These properties imply uniqueness of the complementary space to  $\mathcal{P}$  in  $\mathcal{H}$ . The space  $\mathcal{P}$  is the complementary space to  $\mathcal{Q}$  in  $\mathcal{H}$ . The Hilbert space

$$\mathcal{H} = \mathcal{P} \vee \mathcal{Q}$$

is said to be the complementary sum of  $\mathcal{P}$  and  $\mathcal{Q}$ . The complementary sum is an orthogonal sum if, and only if, the intersection space  $\mathcal{P} \wedge \mathcal{Q}$  of  $\mathcal{P}$  and  $\mathcal{Q}$  contains no nonzero element.

Complementation is preserved under surjective partially isometric transformations. If Hilbert spaces  $\mathcal{P}$  and  $\mathcal{Q}$  are contained contractively as complementary subspaces of a Hilbert space  $\mathcal{P} \vee \mathcal{Q}$  and if  $T$  is a partially isometric transformation of  $\mathcal{P} \vee \mathcal{Q}$  onto a Hilbert space  $\mathcal{H}$ , then  $T$  acts as a partially isometric transformation of  $\mathcal{P}$  onto a Hilbert space  $\mathcal{P}'$  which is contained contractively in  $\mathcal{H}$ ,  $T$  acts as a partially isometric transformation of  $\mathcal{Q}$  onto a Hilbert space  $\mathcal{Q}'$  which is contained contractively in  $\mathcal{H}$ , and the space  $\mathcal{P}'$  and  $\mathcal{Q}'$  are complementary subspaces of

$$\mathcal{H} = \mathcal{P}' \vee \mathcal{Q}'.$$

The extension space  $\text{ext } \mathcal{C}(z)$  of  $\mathcal{C}(z)$  is the Hilbert space of square summable Laurent series with vector coefficients. The space  $\mathcal{C}(z)$  is contained isometrically in  $\text{ext } \mathcal{C}(z)$ . An isometric transformation of  $\mathcal{C}(z)$  onto its orthonormal complement in  $\text{ext } \mathcal{C}(z)$  is defined by taking  $f(z)$  into  $z^{-1}f(z^{-1})$ .

Multiplication by  $z$  and division by  $z$  are isometric transformations of  $\text{ext } \mathcal{C}(z)$  into itself. If a Hilbert space  $\mathcal{M}$  is contained contractively in  $\mathcal{C}(z)$ , multiplication by  $z$  is a contractive transformation of  $\mathcal{M}$  into itself if, and only if, division by  $z$  is a contractive transformation of the complementary space to  $\mathcal{M}$  in  $\text{ext } \mathcal{C}(z)$  into itself.

A Hilbert space  $\mathcal{H}$  whose elements are power series with vector coefficients is said to satisfy the inequality for difference quotients if  $[f(z) - f(0)]/z$  belongs to the space whenever  $f(z)$  belongs to the space and if the inequality

$$\|[f(z) - f(0)]/z\|_{\mathcal{H}}^2 \leq \|f(z)\|_{\mathcal{H}}^2 - |f(0)|^2$$

is satisfied. A Hilbert space whose elements are power series with vector coefficients is said to satisfy the identity for difference quotients if it satisfies the inequality for difference quotients and if equality always holds.

A Hilbert space of power series with vector coefficients which satisfies the inequality for difference quotients is contained contractively in  $\mathcal{C}(z)$ . The extension space of a Hilbert space  $\mathcal{H}$  which satisfies the inequality for difference quotients is the Hilbert space  $\text{ext } \mathcal{H}$

which is contained contractively in  $\text{ext } \mathcal{C}(z)$  such that the complementary space to  $\text{ext } \mathcal{H}$  in  $\text{ext } \mathcal{C}(z)$  is isometrically equal to the complementary space to  $\mathcal{H}$  in  $\mathcal{C}(z)$ . Division by  $z$  is a contractive transformation of  $\text{ext } \mathcal{H}$  into itself.

A Hilbert space  $\mathcal{H}$  which is contained contractively in  $\mathcal{C}(z)$  satisfies the inequality for difference quotients if, and only if, multiplication by  $z$  is a contractive transformation of the complementary space to  $\mathcal{H}$  in  $\mathcal{C}(z)$  into itself.

A theorem which is due to Beurling [1] when the coefficient space is the complex numbers applies to spaces which satisfy the inequality for difference quotients and are contained isometrically in  $\mathcal{C}(z)$ . An isometric inclusion in  $\mathcal{C}(z)$  is unnecessary for the conclusion of the theorem as is the restriction on the coefficient space. A weaker hypothesis is sufficient.

**Theorem 1.** *A Hilbert space  $\mathcal{H}$  of power series with vector coefficients which satisfies the inequality for difference quotients is isometrically equal to the space  $\mathcal{H}(W)$  for a power series  $W(z)$  with operator coefficients such that multiplication by  $W(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself if multiplication by  $z$  is an isometric transformation of the complementary space to  $\mathcal{H}$  in  $\mathcal{C}(z)$  into itself.*

*Proof of Theorem 1.* The complementary space in  $\mathcal{C}(z)$  to a Hilbert space  $\mathcal{H}$  which satisfies the inequality for difference quotients is a Hilbert space  $\mathcal{M}$  such that multiplication by  $z$  is a contractive transformation of  $\mathcal{M}$  into itself. If multiplication by  $z$  is an isometric transformation of  $\mathcal{M}$  into itself, the orthogonal complement of the range of multiplication by  $z$  is a Hilbert space  $\mathcal{B}$  which is contained isometrically in  $\mathcal{M}$  and which is mapped isometrically onto orthogonal subspaces of  $\mathcal{M}$  under multiplication by  $z^m$  and multiplication by  $z^n$  for unequal nonnegative integers  $m$  and  $n$ . The closed span of these subspaces of  $\mathcal{M}$  is the full space.

A power series  $W(z)$  with operator coefficients acts as a partially isometric multiplication of  $\mathcal{C}$  onto  $\mathcal{B}$  if, and only if, it acts as a partially isometric multiplication of  $\mathcal{C}(z)$  onto  $\mathcal{M}$ . If an element of  $\mathcal{C}(z)$  belongs to the kernel of multiplication by  $W(z)$ , then all coefficients of the element belong to the kernel of multiplication by  $W(z)$ .

Such a power series exists if, and only if, the dimension of  $\mathcal{B}$  is less than or equal to the dimension of  $\mathcal{C}$ . The dimension inequality is satisfied when  $\mathcal{C}$  has infinite dimension since the dimension of  $\mathcal{C}(z)$  is equal to the dimension of  $\mathcal{C}$  and since the dimension of  $\mathcal{B}$  is less than or equal to the dimension of  $\mathcal{C}(z)$ . It remains to verify the dimension inequality when  $\mathcal{C}$  has finite dimension  $r$ .

Argue by contradiction assuming that  $\mathcal{B}$  contains an orthonormal set of  $r + 1$  elements

$$f_0(z), \dots, f_r(z).$$

If  $c_0, \dots, c_r$  are corresponding vectors, then the square matrix which has entry

$$c_i^- f_j(z)$$

in the  $i$ -th row and  $j$ -column has vanishing determinant. Expanding the determinant along the  $i$ -th row produces the vanishing power series

$$c_i^- f_0(z)g_0(z) + \dots + c_i^- f_r(z)g_r(z)$$

with

$$(-1)^k g_k(z)$$

the determinant of the matrix obtained by deleting the  $i$ -th row and  $k$ -th column of the starting matrix. Since  $c_i$  is arbitrary and since each power series  $g_k(z)$  with complex coefficients is square summable, the element

$$f_0(z)g_0(z) + \dots + f_r(z)g_r(z)$$

of  $\mathcal{M}$  vanishes identically. The power series

$$g_0(z), \dots, g_r(z)$$

vanish since the elements

$$z^n f_k(z)$$

of  $\mathcal{M}$  form an orthonormal set. An inductive argument shows that the determinants of all square submatrices of the starting matrix vanish. A contradiction is obtained since the starting matrix does not vanish identically for all vectors  $c_0, \dots, c_r$ .

This completes the proof of the theorem.

The conclusion of Theorem 1 applies under another hypothesis.

**Theorem 2.** *A Hilbert space  $\mathcal{H}$  of power series with vector coefficients which satisfies the identity for difference quotients is isometrically equal to a space  $\mathcal{H}(W)$  for a power series  $W(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself.*

*Proof of Theorem 2.* As in the proof of Theorem 1 it needs to be shown that the complementary space  $\mathcal{B}$  to  $\mathcal{H}$  in the augmented space  $\mathcal{H}'$  is less than or equal to the dimension of the coefficient space  $\mathcal{C}$ . The conclusion is immediate if  $\mathcal{C}$  has infinite dimension. It remains to treat the case in which the dimension of  $\mathcal{C}$  is finite. The proof applies the Hilbert space  $\mathcal{B}(z)$  of power series with coefficients in the Hilbert space  $\mathcal{B}$ .

Division by  $z$  is an isometric transformation of ext  $\mathcal{H}'$  into itself since  $\mathcal{H}$  satisfies the identity for difference quotients. An isometric transformation of  $\mathcal{B}(z)$  into ext  $\mathcal{H}$  is defined by taking a power series

$$\sum b_n z^n$$

whose coefficients are elements  $b_n(z)$  of  $\mathcal{B}$  into the element

$$\sum b_n(z) z^{-n}$$

of ext  $\mathcal{H}$ . The set of elements of  $\mathcal{B}(z)$  which are mapped into  $\mathcal{H}$  is a Hilbert space which is contained isometrically in  $\mathcal{B}(z)$  and which satisfies the identity for difference quotients. A power series  $V(z)$  whose coefficients are contractive transformations of  $\mathcal{B}$  into itself exists by Theorem 1 such that multiplication by  $V(z)$  is a partially isometric transformation of  $\mathcal{B}(z)$

onto the set of elements which map into the orthogonal complement of  $\mathcal{H}$  in  $\text{ext } \mathcal{H}$ . The choice of  $V(z)$  can be made so that multiplication by  $V(z)$  is an isometric transformation of  $\mathcal{C}(z)$  into itself since no nonzero element  $f(z)$  of  $\mathcal{B}(z)$  exists such that  $z^n f(z)$  is orthogonal to the range of multiplication by  $V(z)$  for every nonnegative integer  $n$ .

The transformation of  $\mathcal{B}(z)$  into  $\text{ext } \mathcal{H}$  takes the range of multiplication by  $V(z)$  onto a Hilbert space  $\mathcal{M}$  which is contained isometrically in the orthogonal complement of  $\mathcal{H}$  in  $\text{ext } \mathcal{H}$ . Division by  $z$  is an isometric transformation of  $\mathcal{M}$  into itself. The image of  $\mathcal{B}$  is the orthogonal complement in  $\mathcal{M}$  of the range of division by  $z$ . The elements of the image of  $\mathcal{B}$  are elements

$$z^{-1} f(z^{-1})$$

of  $\text{ext } \mathcal{C}(z)$  such that  $f(z)$  belongs to  $\mathcal{C}(z)$ . An injective transformation of the image of  $\mathcal{B}$  into  $\mathcal{C}$  is defined by taking  $f(z)$  into  $f(0)$ .

The dimension of  $\mathcal{B}$  is less than or equal to the dimension of  $\mathcal{C}$  since an injective transformation exists of  $\mathcal{B}$  into  $\mathcal{C}$ .

This completes the proof of the theorem.

Linear systems are a mechanism for the construction of invariant subspaces of transformations by factorization of analytic functions called transfer functions of linear systems. For transformations which take a Hilbert space contractively into itself, the state space and external spaces of the linear system are Hilbert spaces.

The linear system is a square matrix whose four entries are continuous linear transformations. The matrix acts on a Hilbert space which is the Cartesian product of the state space and the external space. The elements of the Cartesian product are realized as column vectors with upper entry in the state space and lower entry in the external space.

The upper left entry of the matrix is the main transformation, which takes the state space into itself. The upper right entry is the input transformation, which takes the external space into the state space. The lower left entry is the output transformation, which takes the state space into the external space. The lower right entry is the external operator, which takes the external space into itself.

The matrix of the linear system is assumed to have an isometric adjoint. A canonical model of the linear system is constructed in a Hilbert space of power series with vector coefficients for a coefficient space which is isometrically equal to the external space.

When the linear system has matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with isometric adjoint, a power series

$$\sum a_n z^n$$

with vector coefficients is defined by

$$a_n = CA^n f$$

for every element  $f$  of the state space. The set of elements of the state space for which the power series vanishes is a closed invariant subspace for the main transformation in which its restriction has an isometric adjoint. Invariant subspaces of transformations with isometric adjoint are constructed without the use of the factorization. The subspace is assumed to contain no nonzero element for the construction of the canonical model.

A canonical linear system whose matrix has isometric adjoint is assumed to have as state space a Hilbert space whose elements are power series with vector coefficients. The power series

$$f(z) = \sum a_n z^n$$

associated with an element of the space is assumed to be constructed by iteration of the main transformation and the action of the output transformation. The external operator then takes a power series  $f(z)$  into its constant coefficient  $f(0)$ . The main transformation takes  $f(z)$  into

$$[f(z) - f(0)]/z.$$

The state space  $\mathcal{H}$  of the linear system is a Hilbert space of power series with vector coefficients whose structure is determined by the isometric property of the adjoint matrix. The augmented space  $\mathcal{H}'$  is the Hilbert space of power series  $f(z)$  with vector coefficients such that  $[f(z) - f(0)]/z$  belongs to  $\mathcal{H}$  with scalar product determined by the identity

$$\|[f(z) - f(0)]/z\|_{\mathcal{H}}^2 = \|f(z)\|_{\mathcal{H}'}^2 - |f(0)|^2.$$

The coefficient space  $\mathcal{C}$  is contained isometrically in the augmented space  $\mathcal{H}'$ . Multiplication by  $z$  is an isometric transformation of the space  $\mathcal{H}$  onto the orthogonal complement of  $\mathcal{C}$  in  $\mathcal{H}'$ .

The matrix of the linear system is realized as a transformation of the augmented space  $\mathcal{H}'$  into itself. The matrix takes an element  $f(z)$  of  $\mathcal{H}'$  with constant coefficient zero into

$$f(z)/z.$$

The matrix takes an element  $c$  of the coefficient space into

$$W(z)c$$

for a power series  $W(z)$  with operator coefficients which defines the transfer function of the linear system. The matrix takes an element  $f(z)$  of  $\mathcal{H}'$  into

$$[f(z) - f(0)]/z + W(z)f(0).$$

Since the matrix has an isometric adjoint, it is a partially isometric transformation of the augmented Hilbert space onto itself. If  $f(z)$  is an element of  $\mathcal{H}$  and if  $c$  is a vector, then

$$g(z) = f(z) + W(z)c$$

is an element of  $\mathcal{H}'$ . The inequality

$$\|g(z)\|_{\mathcal{H}'}^2 \leq \|f(z)\|_{\mathcal{H}}^2 + |c|^2$$

is satisfied. Every element  $g(z)$  of  $\mathcal{H}'$  admits a representation for which equality hold. The space  $\mathcal{H}$  satisfies the inequality for difference quotients.

The space  $\mathcal{H}$  is contained contractively in the space  $\mathcal{H}'$ . Multiplication by  $W(z)$  is a partially isometric transformation of  $\mathcal{C}$  onto the complementary space to  $\mathcal{H}$  in  $\mathcal{H}'$ . Multiplication by  $W(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself. Multiplication by  $W(z)$  acts as a partially isometric transformation of  $\mathcal{C}(z)$  onto a Hilbert space which is contained contractively in  $\mathcal{C}(z)$ . The space  $\mathcal{H}$  is isometrically equal to the complementary space  $\mathcal{H}(W)$  in  $\mathcal{C}(z)$  of the range of multiplication by  $W(z)$  as it acts on  $\mathcal{C}(z)$ .

If  $W(z)$  is a power series with operator coefficients such that multiplication by  $W(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself, then multiplication by  $W(z)$  has a unique continuous extension as a contractive transformation of ext  $\mathcal{C}(z)$  into itself which commutes with multiplication by  $z$ . The conjugate power series

$$W^*(z) = W_0^- + W_1^- z + W_2^- z^2 + \dots$$

is defined with operator coefficients which are adjoints of the operator coefficients of the power series

$$W(z) = W_0 + W_1 z + W_2 z^2 + \dots$$

Multiplication by  $W^*(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself which has a contractive extension as a transformation of ext  $\mathcal{C}(z)$  into itself. Multiplication by  $W^*(z^{-1})$  is the contractive transformation of ext  $\mathcal{C}(z)$  into itself which takes  $f(z)$  into  $g(z)$  whenever multiplication by  $W^*(z)$  takes  $f(z^{-1})$  into  $g(z^{-1})$ . The adjoint of multiplication by  $W(z)$  as a transformation of ext  $\mathcal{C}(z)$  into itself is multiplication by  $W^*(z^{-1})$ . The adjoint of multiplication by  $W(z)$  as a transformation of  $\mathcal{C}(z)$  into itself takes an element  $f(z)$  of  $\mathcal{C}(z)$  into the power series which has the same coefficient of  $z^n$  as

$$W^*(z^{-1})f(z)$$

for every nonnegative integer  $n$ .

A construction of invariant subspaces is due to David Hilbert for contractive transformations of a Hilbert space into itself whose adjoint is isometric. A canonical model of such transformations appears in an expository treatment of the construction by Gustav Herglotz [9].

A Herglotz space is a Hilbert space of power series with vector coefficients such that a contractive transformation of the space into itself with isometric adjoint is defined by taking  $f(z)$  into  $[f(z) - f(0)]/z$  and such that a continuous transformation of the space into the coefficient space is defined by taking a power series  $f(z)$  into its constant coefficient  $f(0)$ .

A transformation  $T$  with domain and range in a Hilbert space  $\mathcal{H}$  is said to be dissipative if

$$\langle Tf, f \rangle_{\mathcal{H}} + \langle f, Tf \rangle_{\mathcal{H}}$$

is nonnegative for every element  $f$  of the domain of  $T$ . The transformation is said to be maximal dissipative if every element of the Hilbert space is a sum

$$f + Tf$$

with  $f$  in the domain of  $T$ .

If  $\phi(z)$  is a power series with operator coefficients such that a maximal dissipative transformation in  $\mathcal{C}(z)$  is defined by taking  $f(z)$  into

$$\phi(z)f(z)$$

whenever  $f(z)$  and  $\phi(z)f(z)$  belong to  $\mathcal{C}(z)$ , then a power series

$$W(z) = [1 - \phi(z)]/[1 + \phi(z)]$$

with operator coefficients is defined such that multiplication by  $W(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself. The Herglotz space  $\mathcal{L}(\phi)$  is a Hilbert space of power series with vector coefficients such that an isometric transformation of the space onto the space  $\mathcal{H}(W)$  is defined by taking  $f(z)$  into

$$[1 + W(z)]f(z).$$

A contractive transformation of the space  $\mathcal{L}(\phi)$  into itself which has an isometric adjoint is defined by taking  $f(z)$  into  $[f(z) - f(0)]/z$ . A continuous transformation of the space  $\mathcal{L}(\phi)$  into the coefficient space is defined by taking a power series  $f(z)$  into its constant coefficient  $f(0)$ . The space  $\mathcal{L}(\phi)$  is a Herglotz space. A Herglotz space is isometrically equal to a space  $\mathcal{L}(\phi)$  for some Herglotz function  $\phi(z)$ . The Herglotz spaces associated with two Herglotz functions are isometrically equal if, and only if, the Herglotz functions differ by a constant which is a skew-conjugate operator.

The extension space  $\mathcal{E}(\phi)$  of a Herglotz space  $\mathcal{L}(\phi)$  is a Hilbert space whose elements are Laurent series with vector coefficient such that multiplication by  $z$  is an isometric transformation of the space onto itself and such that a partially isometric transformation of  $\mathcal{E}(\phi)$  onto  $\mathcal{L}(\phi)$  is defined by taking a Laurent series into the power series which has the same coefficient of  $z^n$  for every nonnegative integer  $n$ .

The Herglotz space  $\mathcal{L}(\phi)$  is isometrically equal to  $\mathcal{C}(z)$  when  $\phi(z)$  is the constant one. The extension space of the space  $\mathcal{L}(\phi)$  is then isometrically equal to  $\text{ext } \mathcal{C}(z)$ . A Hilbert space whose elements are Laurent series is isometrically equal to the extension space of a Herglotz space if multiplication by  $z$  is an isometric transformation of the space onto itself and if a continuous transformation of the space into the coefficient space is defined by taking a Laurent series into its constant coefficient.

A convex structure applies to the Hilbert spaces which are contained contractively in a given Hilbert space. If Hilbert spaces  $\mathcal{P}$  and  $\mathcal{Q}$  are contained contractively in the given Hilbert space, and if  $t$  is in the interval  $[0, 1]$ , a unique Hilbert space

$$\mathcal{H} = (1 - t)\mathcal{P} + t\mathcal{Q}$$

exists, which is contained contractively in the given Hilbert space, such that the convex combination

$$c = (1 - t)a + tb$$

belongs to  $\mathcal{H}$  whenever  $a$  belongs to  $\mathcal{P}$  and  $b$  belongs to  $\mathcal{Q}$ , such that the inequality

$$\|c\|_{\mathcal{H}}^2 \leq (1 - t)\|a\|_{\mathcal{P}}^2 + t\|b\|_{\mathcal{Q}}^2$$

is satisfied, and such that every element  $c$  of  $\mathcal{H}$  is a convex combination for which equality holds.

If  $\mathcal{E}(\phi)$  and  $\mathcal{E}(\psi)$  are extension spaces of Herglotz spaces  $\mathcal{L}(\phi)$  and  $\mathcal{L}(\psi)$ , then a Herglotz space

$$\mathcal{L}(\phi + \psi) = \mathcal{L}(\phi) \vee \mathcal{L}(\psi)$$

exists in which the Herglotz spaces  $\mathcal{L}(\phi)$  and  $\mathcal{L}(\psi)$  are contained contractively as complementary spaces. The extension spaces  $\mathcal{E}(\phi)$  and  $\mathcal{E}(\psi)$  are contained contractively as complementary space in the extension space

$$\mathcal{E}(\phi + \psi) = \mathcal{E}(\phi) \vee \mathcal{E}(\psi).$$

A Herglotz space  $\mathcal{L}(\theta)$  exists such that the extension space

$$\mathcal{E}(\theta) = \mathcal{E}(\phi) \wedge \mathcal{E}(\psi)$$

is the intersection spaces of the extension spaces  $\mathcal{E}(\phi)$  and  $\mathcal{E}(\psi)$ .

A Herglotz space  $\mathcal{L}(\phi - \theta)$  exists such that the extension space  $\mathcal{E}(\phi - \theta)$  is the complementary space in the extension space  $\mathcal{E}(\phi)$  of the extension space  $\mathcal{E}(\theta)$ . A Herglotz space  $\mathcal{L}(\psi - \theta)$  exists such that the extension space  $\mathcal{E}(\psi - \theta)$  is the complementary space in the extension space  $\mathcal{E}(\psi)$  of the extension space  $\mathcal{E}(\theta)$ . A Herglotz space  $\mathcal{L}(\phi + \theta)$  exists such that the extension space  $\mathcal{E}(\phi + \theta)$  is the complementary space in the extension space  $\mathcal{E}(\phi + \psi)$  of the extension space  $\mathcal{E}(\psi - \theta)$ . A Herglotz space  $\mathcal{L}(\psi + \theta)$  exists such that the extension space  $\mathcal{E}(\psi + \theta)$  is the complementary space in the extension space  $\mathcal{E}(\phi + \psi)$  of the extension space  $\mathcal{E}(\phi - \theta)$ . The extension space

$$\mathcal{E}(\phi) = (1 - t)\mathcal{E}(\phi + \theta) + t\mathcal{E}(\psi - \theta)$$

is a convex combination of the extension spaces  $\mathcal{E}(\phi + \theta)$  and  $\mathcal{E}(\psi - \theta)$  with  $t$  and  $1 - t$  equal. The extension space

$$\mathcal{E}(\psi) = (1 - t)\mathcal{E}(\psi + \theta) + t\mathcal{E}(\phi - \theta)$$

is a convex combination of the extension spaces  $\mathcal{E}(\psi + \theta)$  and  $\mathcal{E}(\psi - \theta)$  with  $t$  and  $1 - t$  equal.

The Herglotz space  $\mathcal{L}(\theta)$  is contained contractively in the Herglotz spaces  $\mathcal{L}(\phi)$  and  $\mathcal{L}(\psi)$ . The Herglotz space  $\mathcal{L}(\phi - \theta)$  is the complementary space in the Herglotz space  $\mathcal{L}(\phi)$  of the Herglotz space  $\mathcal{L}(\theta)$ . The Herglotz space  $\mathcal{L}(\psi - \theta)$  is the complementary space in the Herglotz space  $\mathcal{L}(\psi)$  of the Herglotz space  $\mathcal{L}(\theta)$ . The Herglotz space  $\mathcal{L}(\phi + \theta)$  is the complementary space in the Herglotz space  $\mathcal{L}(\phi + \psi)$  of the Herglotz space  $\mathcal{L}(\psi - \theta)$ . The Herglotz space  $\mathcal{L}(\psi + \theta)$  is the complementary space in the Herglotz space  $\mathcal{L}(\phi + \psi)$  of the Herglotz space  $\mathcal{L}(\phi - \theta)$ . The Herglotz space

$$\mathcal{L}(\phi) = (1 - t)\mathcal{L}(\phi + \theta) + t\mathcal{L}(\phi - \theta)$$

is a convex combination of the Herglotz spaces  $\mathcal{L}(\phi + \theta)$  and  $\mathcal{L}(\phi - \theta)$  with  $t$  and  $1 - t$  equal. The Herglotz space

$$\mathcal{L}(\psi) = (1 - t)\mathcal{L}(\psi + \theta) + t\mathcal{L}(\psi - \theta)$$

is a convex combination of the Herglotz spaces  $\mathcal{L}(\psi + \theta)$  and  $\mathcal{L}(\psi - \theta)$  with  $t$  and  $1 - t$  equal.

An extreme point of a convex set is an element of the set which is not a convex combination

$$(1 - t)a + tb$$

of distinct elements of the set with  $t$  and  $1 - t$  positive. An element of the convex set of Herglotz spaces which are contained contractively in a given Herglotz space  $\mathcal{L}(\psi)$  is an extreme point if, and only if, it is a Herglotz space whose extension space is contained isometrically in the extension space  $\mathcal{E}(\psi)$ .

A convex combination of Hilbert spaces of power series with vector coefficients which satisfy the inequality for difference quotients is a Hilbert space of power series with vector coefficients which satisfies the inequality for difference quotients. A Hilbert space of power series with vector coefficients is said to satisfy the identity for difference quotients if it satisfies the inequality for difference quotients and if equality always holds.

A Hilbert space  $\mathcal{H}$  of power series with vector coefficients which satisfies the identity for difference quotients is an extreme point of the convex set of Hilbert spaces of power series with vector coefficients which satisfy the inequality for difference quotients. If

$$\mathcal{H} = (1 - t)\mathcal{P} + t\mathcal{Q}$$

is a convex combination with  $t$  and  $1 - t$  positive of Hilbert space  $\mathcal{P}$  and  $\mathcal{Q}$  of power series with vector coefficients which satisfy the inequality for difference quotients, then every element

$$h(z) = (1 - t)f(z) + tg(z)$$

of  $\mathcal{H}$  is a convex combination of elements  $f(z)$  of  $\mathcal{P}$  and  $g(z)$  of  $\mathcal{Q}$  such that equality holds in the inequality

$$\|h(z)\|_{\mathcal{H}}^2 \leq (1 - t)\|f(x)\|_{\mathcal{P}}^2 + t\|g(z)\|_{\mathcal{Q}}^2.$$

Since the inequality

$$\|[h(z) - h(0)]/z\|_{\mathcal{H}}^2 \leq (1-t)\|[f(z) - f(0)]/z\|_{\mathcal{P}}^2 + t\|[g(z) - g(0)]/z\|_{\mathcal{Q}}^2$$

holds with

$$\|[f(z) - f(0)]/z\|_{\mathcal{P}}^2 \leq \|f(z)\|_{\mathcal{P}}^2 - |f(0)|^2$$

and

$$\|[g(z) - g(0)]/z\|_{\mathcal{Q}}^2 \leq \|g(z)\|_{\mathcal{Q}}^2 - |g(0)|^2,$$

the inequality

$$\|[h(z) - h(0)]/z\|_{\mathcal{H}}^2 \leq \|h(z)\|_{\mathcal{H}}^2 - (1-t)|f(0)|^2 - t|g(0)|^2$$

is satisfied. The inequality reads

$$\|[h(z) - h(0)]/z\|_{\mathcal{H}}^2 \leq \|h(z)\|_{\mathcal{H}}^2 - |h(0)|^2 - t(1-t)|g(0) - f(0)|^2$$

since the convexity identity

$$|(1-t)f(0) + tg(0)|^2 + t(1-t)|g(0) - f(0)|^2 = (1-t)|f(0)|^2 + t|g(0)|^2$$

is satisfied. Since the space  $\mathcal{H}$  satisfies the identity for difference quotients, the constant coefficients in  $f(z)$ ,  $g(z)$ , and  $h(z)$  are equal. An inductive argument shows that the  $n$ -th coefficients of  $f(z)$ ,  $g(z)$ , and  $h(z)$  are equal for every nonnegative integer  $n$ . Since  $f(z)$ ,  $g(z)$ , and  $h(z)$  are always equal, the spaces  $\mathcal{P}$ ,  $\mathcal{Q}$ , and  $\mathcal{H}$  are isometrically equal.

If a Hilbert space  $\mathcal{H}$  of power series with vector coefficients satisfies the inequality for difference quotients, the augmented space is the Hilbert space  $\mathcal{H}'$  of power series  $f(z)$  with vector coefficients such that  $[f(z) - f(0)]/z$  belongs to  $\mathcal{H}$  with scalar product determined by the identity

$$\|[f(z) - f(0)]/z\|_{\mathcal{H}}^2 = \|f(z)\|_{\mathcal{H}'}^2 - |f(0)|^2.$$

The space  $\mathcal{H}$  is contained contractively in the space  $\mathcal{H}'$ . The complementary space  $\mathcal{B}$  to  $\mathcal{H}$  in  $\mathcal{H}'$  is a Hilbert space which is contained contractively in the complementary space  $\mathcal{M}$  to  $\mathcal{H}$  in  $\mathcal{C}(z)$ . Multiplication by  $z$  is an isometric transformation of  $\mathcal{M}$  onto the complementary space to  $\mathcal{B}$  in  $\mathcal{M}$ .

If  $W(z)$  is a power series with operator coefficients such that multiplication by  $W(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself, then multiplication by  $W(z)$  admits a unique continuous extension as a contractive transformation of ext  $\mathcal{C}(z)$  into itself which commutes with multiplication by  $z$ .

The conjugate power series

$$W^+(z) = W_0^- + W_1^- z + W_2^- z^2 + \dots$$

is defined with operator coefficients which are adjoints of the operator coefficients of

$$W(z) = W_0 + W_1 z + W_2 z^2 + \dots$$

Multiplication by  $W^*(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself. The adjoint of multiplication by  $W(z)$  as a transformation of  $\text{ext } \mathcal{C}(z)$  into itself takes  $f(z)$  into  $g(z)$  when multiplication by  $W^*(z)$  takes  $z^{-1}f(z^{-1})$  into  $z^{-1}g(z^{-1})$ . The adjoint of multiplication by  $W(z)$  as a transformation of  $\mathcal{C}(z)$  into itself takes an element  $f(z)$  of  $\mathcal{C}(z)$  into the element  $g(z)$  of  $\mathcal{C}(z)$  which has the same coefficient of  $z^n$  for every nonnegative integer  $n$  as the element of  $\text{ext } \mathcal{C}(z)$  obtained from  $f(z)$  under the adjoint of multiplication by  $W(z)$  as a transformation of  $\text{ext } \mathcal{C}(z)$  into itself.

If  $U(z)$  and  $V(z)$  are power series with operator coefficients such that multiplication by  $U(z)$  and multiplication by  $V(z)$  are contractive transformations of  $\mathcal{C}(z)$  into itself, then multiplication by

$$W(z) = U(z)V(z)$$

is a continuous transformation of  $\mathcal{C}(z)$  into itself. The space  $\mathcal{H}(U)$  is contained contractively in the space  $\mathcal{H}(W)$ . Multiplication by  $U(z)$  is a partially isometric transformation of the space  $\mathcal{H}(V)$  onto the complementary space to the space  $\mathcal{H}(U)$  in the space  $\mathcal{H}(W)$ .

A factorization is derived from a contractive inclusion.

**Theorem 3.** *If  $U(z)$  and  $W(z)$  are power series with operator coefficients such that multiplication by  $U(z)$  and multiplication by  $W(z)$  are contractive transformations of  $\mathcal{C}(z)$  into itself and if the space  $\mathcal{H}(U)$  is contained contractively in the space  $\mathcal{H}(W)$ , then*

$$W(z) = U(z)V(z)$$

*for a power series  $V(z)$  with operator coefficients such that multiplication by  $V(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself and such that the range of multiplication by  $V(z)$  as a transformation of  $\text{ext } \mathcal{C}(z)$  into itself is orthogonal to the kernel of multiplication by  $U(z)$  as a transformation of  $\text{ext } \mathcal{C}(z)$  into itself.*

*Proof of Theorem 3.* A Hilbert space  $\mathcal{C}_r(z)$ , which is contained isometrically in  $\text{ext } \mathcal{C}(z)$ , is defined for every nonnegative integer  $r$  as the set of elements  $f(z)$  such that  $z^r f(z)$  belongs to  $\mathcal{C}(z)$ . Multiplication by  $U(z)$  and multiplication by  $W(z)$  are contractive transformations of  $\mathcal{C}_r(z)$  into itself. The adjoint transformations of  $\mathcal{C}_r(z)$  into itself are compositions of the adjoints of multiplication by  $U(z)$  and of multiplication by  $W(z)$  as transformations of  $\text{ext } \mathcal{C}(z)$  into itself with the orthogonal projection of  $\text{ext } \mathcal{C}(z)$  onto  $\mathcal{C}_r(z)$ .

A space  $\mathcal{H}(U_r)$  is defined with

$$U_r(z) = z^r U(z).$$

The set of elements  $f(z)$  of  $\mathcal{C}_r(z)$  such that  $z^r f(z)$  belongs to the space  $\mathcal{H}(U_r)$  is a Hilbert space  $\mathcal{H}_r(U)$  which is mapped isometrically onto the space  $\mathcal{H}(U_r)$  on multiplication by  $z^r$ . The space  $\mathcal{H}_r(U)$  is contained contractively in  $\mathcal{C}_r(z)$ . Multiplication  $U(z)$  is a partially isometric transformation of  $\mathcal{C}_r(z)$  onto the complementary space to the space  $\mathcal{H}_r(U)$  in  $\mathcal{C}_r(z)$ . If  $h(z)$  is an element of  $\mathcal{C}_r(z)$ , a unique element  $f(z)$  of  $\mathcal{C}(z)$  exists such that

$$h(z) - U(z)f(z)$$

belongs to the space  $\mathcal{H}_r(U)$  and such that equality holds in the inequality

$$\|h(z)\|^2 \leq \|h(z) - U(z)f(z)\|_{\mathcal{H}_r(U)}^2 + \|f(z)\|^2.$$

The element  $f(z)$  of  $\mathcal{C}_r(z)$  is obtained from  $h(z)$  under the adjoint of multiplication by  $U(z)$  as a transformation of  $\mathcal{C}(z)$  into itself.

A space  $\mathcal{H}(W_r)$  is defined with

$$W_r(z) = z^r W(z).$$

The set of elements  $f(z)$  of  $\mathcal{C}_r(z)$  such that  $z^r f(z)$  belongs to the space  $\mathcal{H}(W_r)$  is a Hilbert space  $\mathcal{H}_r(W)$  which is mapped isometrically onto the space  $\mathcal{H}(W_r)$  on multiplication by  $z^r$ . The space  $\mathcal{H}_r(W)$  is contained contractively in  $\mathcal{C}_r(z)$ . Multiplication by  $W(z)$  is a partially isometric transformation of  $\mathcal{C}_r(z)$  onto the complementary space to the space  $\mathcal{H}_r(W)$  in  $\mathcal{C}_r(z)$ . If  $h(z)$  is an element of  $\mathcal{C}_r(z)$  a unique element  $g(z)$  of  $\mathcal{C}_r(z)$  exists such that

$$h(z) - W(z)g(z)$$

belongs to the space  $\mathcal{H}_r(W)$  and such that equality holds in the inequality

$$\|h(z)\|^2 \leq \|h(z) - W(z)g(z)\|_{\mathcal{H}_r(W)}^2 + \|g(z)\|^2.$$

The element  $g(z)$  of  $\mathcal{C}_r(z)$  is obtained from  $h(z)$  under the adjoint of multiplication by  $W(z)$  as a transformation of  $\mathcal{C}_r(z)$  into itself.

The space  $\mathcal{H}_r(U)$  is contained contractively in the space  $\mathcal{H}_r(W)$  since the space  $\mathcal{H}(U_r)$  is contained contractively in the space  $\mathcal{H}(W_r)$ . Since the element

$$h(z) - U(z)f(z)$$

of the space  $\mathcal{H}_r(U)$  is obtained from  $h(z)$  under the adjoint of the inclusion of the space in  $\mathcal{C}_r(z)$  and since the element

$$h(z) - W(z)g(z)$$

of the space  $\mathcal{H}_r(W)$  is obtained from  $h(z)$  under the adjoint of the inclusion of the space in  $\mathcal{C}_r(z)$ , the element

$$h(z) - U(z)f(z)$$

of the space  $\mathcal{H}_r(U)$  is obtained from the element

$$h(z) - W(z)g(z)$$

of the space  $\mathcal{H}_r(W)$  under the adjoint of the inclusion of the space  $\mathcal{H}_r(U)$  in the space  $\mathcal{H}_r(W)$ . The inequality

$$\|g(z)\| \leq \|f(z)\|$$

follows from the inequality

$$\|h(z) - U(z)f(z)\|_{\mathcal{H}_r(U)} \leq \|h(z) - W(z)g(z)\|_{\mathcal{H}_r(W)}.$$

If the adjoint of multiplication by  $U(z)$  as a transformation of  $\text{ext } \mathcal{C}(z)$  into itself takes  $h(z)$  into  $f(z)$  and if the coefficient of  $z^n$  in  $h(z)$  vanishes when  $n$  is less than  $r$ , then the coefficient of  $z^n$  in  $f(z)$  vanishes when  $n$  is less than  $r$ . If the adjoint of multiplication by  $W(z)$  as a transformation of  $\text{ext } \mathcal{C}(z)$  into itself takes  $h(z)$  into  $g(z)$  and if the coefficient of  $z^n$  in  $h(z)$  vanishes when  $n$  is less than  $r$ , then the coefficient of  $z^n$  in  $g(z)$  vanishes when  $n$  is less than  $r$ .

If  $h(z)$  is an element of  $\text{ext } \mathcal{C}(z)$ , a unique element  $h_r(z)$  of  $\mathcal{C}_r(z)$  exists such that the coefficient of  $z^n$  in

$$h(z) - h_r(z)$$

vanishes when  $n$  is not less than  $-r$ . If the adjoint of multiplication by  $U(z)$  as a transformation of  $\text{ext } \mathcal{C}(z)$  into itself takes  $h(z)$  into  $f(z)$ , an element  $f_r(z)$  of  $\mathcal{C}_r(z)$  exists such that the coefficient of  $z^n$  in

$$f(z) - f_r(z)$$

vanishes when  $n$  is not less than  $-r$ . If the adjoint of multiplication by  $W(z)$  as a transformation of  $\text{ext } \mathcal{C}(z)$  into itself takes  $h(z)$  into  $g(z)$ , an element  $g_r(z)$  of  $\mathcal{C}_r(z)$  exists such that the coefficient of  $z^n$  in

$$g(z) - g_r(z)$$

vanishes when  $n$  is not less than  $-r$ . The element

$$h_r(z) - U(z)f_r(z)$$

of  $\mathcal{C}_r(z)$  belongs to the space  $\mathcal{H}_r(U)$  and equality holds in the inequality

$$\|h_r(z)\|^2 \leq \|h_r(z) - U(z)f_r(z)\|_{\mathcal{H}_r(U)}^2 + \|f_r(z)\|^2.$$

The element

$$h_r(z) - W(z)g_r(z)$$

of  $\mathcal{C}_r(z)$  belongs to the space  $\mathcal{H}_r(W)$  and equality holds in the inequality

$$\|h_r(z)\|^2 \leq \|h_r(z) - W(z)g_r(z)\|_{\mathcal{H}_r(W)}^2 + \|g_r(z)\|^2.$$

The inequality

$$\|g(z)\| \leq \|f(z)\|$$

holds since the inequality

$$\|g_r(z)\| \leq \|f_r(z)\|$$

holds for every nonnegative integer  $r$ .

The transformation which takes  $f(z)$  into  $g(z)$  admits a unique continuous extension as a contractive transformation of  $\text{ext } \mathcal{C}(z)$  into itself which annihilates the kernel of multiplication by  $U(z)$ . The transformation commutes with multiplication by  $z$  and takes elements of  $\text{ext } \mathcal{C}(z)$  whose coefficient of  $z^n$  vanishes when  $n$  is not less than  $-r$  into elements of  $\text{ext } \mathcal{C}(z)$  whose coefficient of  $z^n$  vanishes when  $n$  is not less than  $-r$ .

A unique power series  $V(z)$  with operator coefficients exists such that multiplication by  $V(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself and such that the adjoint of multiplication by  $V(z)$  takes  $f(z)$  into  $g(z)$  for every element  $f(z)$  of ext  $\mathcal{C}(z)$ . The identity

$$W(z) = U(z)V(z)$$

is satisfied since

$$W^*(z) = V^*(z)U^*(z).$$

This completes the proof of the theorem.

If  $W(z)$  is a power series with operator coefficienty such that multiplication by  $W(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself, then multiplication by  $W(z)$  is an isometric transformation of  $\mathcal{C}(z)$  into itself when, and only when,

$$W^*(z)W(z^{-1}) = 1.$$

Multiplication by  $W^*(z)$  is an isometric transformation of  $\mathcal{C}(z)$  into itself when, and only when,

$$W(z^{-1})W^*(z) = 1.$$

If  $W(z)$  is a power series with operator coefficients such that multiplication by  $W(z)$  and multiplication by  $W^*(z)$  are isometric transformations of  $\mathcal{C}(z)$  into itself, then an isometric transformation of ext  $\mathcal{C}(z)$  onto itself is defined by taking  $f(z)$  into

$$W(z)z^{-1}f(z^{-1}).$$

The inverse transformation takes  $f(z)$  into

$$W^*(z)z^{-1}f(z^{-1}).$$

An isometric transformation of the space  $\mathcal{H}(W)$  onto the space  $\mathcal{H}(W^*)$  is defined by taking  $f(z)$  into

$$W^*(z)z^{-1}f(z^{-1}).$$

The inverse transformation takes  $f(z)$  into

$$W(z)z^{-1}f(z^{-1}).$$

The adjoint of the transformation of the space  $\mathcal{H}(W)$  into itself which takes  $f(z)$  into  $[f(z) - f(0)]/z$  is unitarily equivalent to the transformation of the space  $\mathcal{H}(W^*)$  into itself which takes  $f(z)$  into  $[f(z) - f(0)]/z$ . The adjoint of the transformation of the space  $\mathcal{H}(W^*)$  into itself which takes  $f(z)$  into  $[f(z) - f(0)]/z$  is unitarily equivalent to the transformation of the space  $\mathcal{H}(W)$  into itself which takes  $f(z)$  into  $[f(z) - f(0)]/z$ .

When  $W(z)$  is a power series with operator coefficients such that multiplication by  $W(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself. The isometric transformation of the space  $\mathcal{H}(W)$  onto the space  $\mathcal{H}(W^*)$  is replaced by a relation whose graph is a Hilbert space  $\mathcal{D}(W)$  of pairs  $(f(z), g(z))$  of elements  $f(z)$  of the space  $\mathcal{H}(W)$  and  $g(z)$  of the space  $\mathcal{H}(W^*)$ .

**Theorem 4.** *If  $W(z)$  is a power series with operator coefficients such that multiplication by  $W(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself, then a unique Hilbert space  $\mathcal{D}(W)$  exists whose elements are pairs  $(f(z), g(z))$  of power series  $f(z)$  and  $g(z)$  with vector coefficients such that*

$$([f(z) - f(0)]/z, zg(z) - W^*(z)f(0))$$

and

$$(zf(z) - W(z)g(0), [g(z) - g(0)]/z)$$

belong to the space whenever  $(f(z), g(z))$  belong to the space and such that the identities

$$\begin{aligned} & \|([f(z) - f(0)]/z, zg(z) - W^*(z)f(0))\|_{\mathcal{D}(W)}^2 \\ &= \|(f(z), g(z))\|_{\mathcal{D}(W)}^2 - |f(0)|^2 \end{aligned}$$

and

$$\begin{aligned} & \|(zf(z) - W(z)g(0), [g(z) - g(0)]/z)\|_{\mathcal{D}(W)}^2 \\ &= \|(f(z), g(z))\|_{\mathcal{D}(W)}^2 - |g(0)|^2 \end{aligned}$$

are satisfied. A partially isometric transformation of the space  $\mathcal{D}(W)$  onto the space  $\mathcal{H}(W)$  is defined by taking  $(f(z), g(z))$  into  $f(z)$ . A partially isometric transformation of the space  $\mathcal{D}(W)$  onto the space  $\mathcal{H}(W^*)$  is defined by taking  $(f(z), g(z))$  into  $g(z)$ . An isometric transformation of the space  $\mathcal{D}(W)$  onto the space  $\mathcal{D}(W^*)$  is defined by taking  $(f(z), g(z))$  into  $(g(z), f(z))$ .

*Proof of Theorem 4.* The construction of the space  $\mathcal{D}(W)$  applies properties of the space  $\mathcal{H}(W_r)$  defined by

$$W_r(z) = z^r W(z)$$

for every nonnegative integer  $r$ . Multiplication by  $z^r$  is an isometric transformation of the space  $\mathcal{H}(W)$  into the space  $\mathcal{H}(W_r)$ . The orthogonal complement of the image of the space  $\mathcal{H}(W)$  in the space  $\mathcal{H}(W_r)$  is contained isometrically in  $\mathcal{C}(z)$  and consists of the elements of  $\mathcal{C}(z)$  which are polynomials of degree less than  $r$ . The space  $\mathcal{H}(W)$  is contained contractively in the space  $\mathcal{H}(W_r)$ . Multiplication by  $W(z)$  is a partially isometric transformation of the Hilbert space of polynomial elements of  $\mathcal{C}(z)$  of degree less than  $r$  onto the complementary space to the space  $\mathcal{H}(W)$  in the space  $\mathcal{H}(W_r)$ .

A Hilbert space  $\mathcal{D}(W)$  is defined as the set of pairs  $(f(z), g(z))$  of elements  $f(z)$  of the space  $\mathcal{H}(W)$  and polynomial elements

$$g(z) = a_0 + a_1z + \dots + a_{r-1}z^{r-1}$$

of  $\mathcal{C}(z)$  of degree less than  $r$  such that the element

$$z^r f(z) - W(z)(a_0z^{r-1} + \dots + a_{r-1})$$

of the space  $\mathcal{H}(W_r)$  belongs to the space  $\mathcal{H}(W)$ . The scalar product in the space  $\mathcal{D}(W)$  is determined by the identity

$$\begin{aligned} & \|z^r f(z) - W(z)(a_0 z^{r-1} + \dots + a_{r-1})\|_{\mathcal{H}(W)}^2 \\ &= \|(f(z), g(z))\|_{\mathcal{D}_r(W)}^2 - |a_0|^2 - \dots - |a_{r-1}|^2. \end{aligned}$$

If  $f(z)$  is an element of the space  $\mathcal{H}(W)$ , a unique polynomial element  $g(z)$  of  $\mathcal{C}(z)$  of degree less than  $r$  exists such that

$$\|f(z)\|_{\mathcal{H}(W)} = \|(f(z), g(z))\|_{\mathcal{D}_r(W)}.$$

A partially isometric transformation of the space  $\mathcal{D}_{r+1}(W)$  onto the space  $\mathcal{D}_r(W)$  is defined by taking  $(f(z), g(z))$  into  $(f(z), h(z))$  with  $h(z)$  the polynomial of degree less than  $r$  whose coefficient of  $z^n$  is equal to the coefficient of  $z^n$  in  $g(z)$  when  $n$  is less than  $r$ .

The pair

$$(zf(z) - W(z)g(0), [g(z) - g(0)]/z)$$

belongs to the space  $\mathcal{D}_r(W)$  whenever  $(f(z), g(z))$  belongs to the space  $\mathcal{D}_{r+1}(W)$  and the identity

$$\begin{aligned} & \|(zf(z) - W(z)g(0), [g(z) - g(0)]/z)\|_{\mathcal{D}_r(W)}^2 \\ &= \|(f(z), g(z))\|_{\mathcal{D}_{r+1}(W)}^2 - |g(0)|^2 \end{aligned}$$

is satisfied.

The adjoint of the transformation of the space  $\mathcal{H}(W)$  into  $\mathcal{C}$  which takes

$$f(z) = \sum a_n z^n$$

into

$$a_r$$

for a nonnegative integer  $r$  is the transformation which takes a vector  $c$  into the element

$$z^r c - W(z)(W_0^- z^r + \dots + W_r^-)c$$

of the space  $\mathcal{H}(W)$ .

The adjoint of the transformation of the space  $\mathcal{D}_{r+1}(W)$  into  $\mathcal{C}$  which takes  $(f(z), g(z))$  into  $f(0)$  is the transformation which takes a vector  $c$  into the element

$$([1 - W(z)W(0)^-]c, (W_1^- + W_2^- z + \dots + W_r^- z^r)c)$$

of the space  $\mathcal{D}_{r+1}(W)$ .

The space  $\mathcal{D}(W)$  is defined as the set of pairs  $(f(z), g(z))$  of elements  $f(z)$  of the space  $\mathcal{H}(W)$  and power series  $g(z)$  with vector coefficients such that an element  $(f(z), g_r(z))$  of the space  $\mathcal{D}_r(W)$  is defined for every nonnegative integer  $r$  with  $g_r(z)$  the polynomial of

degree less than  $r$  whose coefficient of  $z^n$  is equal to the coefficient of  $z^n$  in  $g(z)$  when  $n$  is less than  $r$ , and such that the sequence of numbers

$$\|(f(z), g_r(z))\|_{\mathcal{D}_r(W)}^2$$

is bounded. A limit exists since the sequence is nondecreasing. The scalar product of the space  $\mathcal{D}(W)$  is determined by the requirement that the limit is equal to

$$\|(f(z), g(z))\|_{\mathcal{D}(W)}^2.$$

A contractive transformation of the space  $\mathcal{D}(W)$  into itself is defined by taking  $(f(z), g(z))$  into

$$(zf(z) - W(z)g(0), [g(z) - g(0)]/z).$$

The identity

$$\begin{aligned} & \|(zf(z) - W(z)g(0), [g(z) - g(0)]/z)\|_{\mathcal{D}(W)}^2 \\ &= \|(f(z), g(z))\|_{\mathcal{D}(W)}^2 - |g(0)|^2. \end{aligned}$$

The adjoint of the transformation of the space  $\mathcal{D}(W)$  into  $\mathcal{C}$  which takes  $(f(z), g(z))$  into  $f(0)$  is the transformation which takes a vector  $c$  into the element

$$([1 - W(z)W(0)^-]c, [W^*(z) - W(0)^-]/z)$$

of the space  $\mathcal{D}(W)$ .

The adjoint of the transformation of the space  $\mathcal{D}(W)$  into itself which takes  $(f(z), g(z))$  into

$$(zf(z) - W(z)g(0), [g(z) - g(0)]/z)$$

is the transformation which takes  $(f(z), g(z))$  into

$$([f(z) - f(0)]/z, zg(z) - W^*(z)f(0)).$$

The identity

$$\begin{aligned} & \|([f(z) - f(0)]/z, zg(z) - W^*(z)f(0))\|_{\mathcal{D}(W)}^2 \\ &= \|(f(z), g(z))\|_{\mathcal{D}(W)}^2 - |f(0)|^2 \end{aligned}$$

is satisfied.

Assume that a Hilbert space  $\mathcal{D}$  whose elements are pairs  $(f(z), g(z))$  of power series  $f(z)$  and  $g(z)$  with vector coefficients has the required properties: The pairs

$$([f(z) - f(0)]/z, zg(z) - W^*(z)f(0))$$

and

$$([zf(z) - W(z)g(0), [g(z) - g(0)]/z)$$

belong to the space whenever  $(f(z), g(z))$  belongs to the space and the identities

$$\begin{aligned} & \|([f(z) - f(0)]/z, zg(z) - W^*(z)f(0))\|_{\mathcal{D}}^2 \\ &= \|(f(z), g(z))\|_{\mathcal{D}}^2 - |f(0)|^2 \end{aligned}$$

and

$$\begin{aligned} & \|(zf(z) - W(z)g(0), [g(z) - g(0)]/z)\|_{\mathcal{D}}^2 \\ &= \|(f(z), g(z))\|_{\mathcal{D}}^2 - |g(0)|^2 \end{aligned}$$

are satisfied.

A partially isometric transformation of  $\mathcal{D}$  onto a Hilbert space  $\mathcal{H}$  of power series with vector coefficients which satisfies the inequality for difference quotients is defined by taking  $(f(z), g(z))$  into  $f(z)$ . The space  $\mathcal{H}$  is isometrically equal to the space  $\mathcal{H}(W)$  since it is the state space of a canonical linear system which is conjugate isometric and has transfer function  $W(z)$ . An inductive argument constructs a partially isometric transformation of the space  $\mathcal{D}$  onto the space  $\mathcal{D}_r(W)$  for every nonnegative integer  $r$ . The transformation takes  $(f(z), g(z))$  into  $(f(z), g_r(z))$  with  $g_r(z)$  the polynomial of degree less than  $r$  whose coefficient of  $z^n$  is equal to the coefficient of  $z^n$  in  $g(z)$  when  $n$  is less than  $r$ . The space  $\mathcal{D}$  is isometrically equal to the space  $\mathcal{D}(W)$  by the construction of the space  $\mathcal{D}(W)$ .

The isometric transformation of the space  $\mathcal{D}(W)$  onto the space  $\mathcal{D}(W^*)$  which takes  $(f(z), g(z))$  into  $(g(z), f(z))$  is a consequence of the characterization of the spaces.

This completes the proof of the theorem.

The relationship between spaces  $\mathcal{H}(W)$  and  $\mathcal{H}(W^*)$  is well behaved in factorization. Assume that  $U(z)$  and  $V(z)$  are power series with operator coefficients such that multiplication by  $u9z$ ) and multiplication by  $V(z)$  are contractive transformations of  $\mathcal{C}(z)$  into itself. Then

$$W(z) = U(z)V(z)$$

is a power series with operator coefficients such that multiplication by  $W(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself. A partially isometric transformation of the space  $\mathcal{D}(U)$  onto a Hilbert space which is contained contractively in the space  $\mathcal{D}(W)$  is defined by taking  $(f(z), g(z))$  into

$$(f(z), V^*(z)g(z)).$$

A partially isometric transformation of the space  $\mathcal{D}(V)$  onto a Hilbert space which is contained contractively in the space  $\mathcal{D}(W)$  is defined by taking  $(f(z), g(z))$  into

$$(U(z)f(z), g(z)).$$

The image of the space  $\mathcal{D}(U)$  and the image of the space  $\mathcal{D}(V)$  are complementary subspaces of the space  $\mathcal{D}(W)$ .

A Herglotz space is associated with a power series  $W(z)$  with operator coefficients such that multiplication by  $W(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself. Since the

adjoint of multiplication by  $W(z)$  as a transformation of  $\mathcal{C}(z)$  into itself takes  $[f(z) - f(0)]/z$  into  $[g(z) - g(0)]/z$  whenever it takes  $f(z)$  into  $g(z)$ , the adjoint acts as a partially isometric transformation of  $\mathcal{C}(z)$  onto a Herglotz space  $\mathcal{L}(\phi)$  which is contained contractively in  $\mathcal{C}(z)$ . The adjoint of multiplication by  $W(z)$  as a transformation of  $\text{ext } \mathcal{C}(z)$  into itself acts as a partially isometric transformation of  $\text{ext } \mathcal{C}(z)$  onto  $\text{ext } \mathcal{L}(\phi)$ . The complementary space to the space  $\mathcal{L}(\phi)$  in  $\mathcal{C}(z)$  is the Herglotz space  $\mathcal{L}(1 - \phi)$ . The complementary space to  $\text{ext } \mathcal{L}(\phi)$  in  $\text{ext } \mathcal{C}(z)$  is  $\text{ext } \mathcal{L}(1 - \phi)$ .

An element  $f(z)$  of  $\mathcal{C}(z)$  belongs to the Herglotz space  $\mathcal{L}(1 - \phi)$  if, and only if,  $W(z)f(z)$  belongs to the space  $\mathcal{H}(W)$ . The identity

$$\|f(z)\|_{\mathcal{L}(1-\phi)}^2 = \|f(z)\|^2 + \|W(z)f(z)\|_{\mathcal{H}(W)}^2$$

is satisfied. An element

$$f(z) + z^{-1}g(z^{-1})$$

of  $\text{ext } \mathcal{C}(z)$  belongs to  $\text{ext } \mathcal{L}(1 - \phi)$  if, and only if,  $f(z)$  and  $g(z)$  are elements of  $\mathcal{C}(z)$  such that

$$(W(z)f(z), -g(z))$$

belongs to the space  $\mathcal{D}(W)$ . The identity

$$\|f(z) + z^{-1}g(z^{-1})\|_{\text{ext } \mathcal{L}(1-\phi)}^2 = \|f(z)\|^2 + \|(W(z)f(z), -g(z))\|_{\mathcal{D}(W)}^2$$

is satisfied.

An element  $f(z)$  of  $\mathcal{C}(z)$  belongs to the space  $\mathcal{H}(W)$  if, and only if, the adjoint of multiplication by  $W(z)$  as a transformation of  $\mathcal{C}(z)$  into itself takes  $f(z)$  into an element  $h(z)$  of the Herglotz space  $\mathcal{L}(1 - \phi)$ . The identity

$$\|f(z)\|_{\mathcal{H}(W)}^2 = \|f(z)\|^2 + \|h(z)\|_{\mathcal{L}(1-\phi)}^2$$

is satisfied. A pair  $(f(z), g(z))$  of elements  $f(z)$  and  $g(z)$  of  $\mathcal{C}(z)$  belongs to  $\mathcal{D}(W)$  if, and only if, the adjoint of multiplication by  $W(z)$  as a transformation of  $\mathcal{C}(z)$  into itself takes  $f(z)$  into  $h(z)$  for an element

$$h(z) + z^{-1}g(z^{-1})$$

of  $\text{ext } \mathcal{L}(1 - \phi)$ . The identity

$$\|(f(z), g(z))\|_{\mathcal{D}(W)}^2 = \|f(z)\|^2 + \|h(z) + z^{-1}g(z^{-1})\|_{\text{ext } \mathcal{L}(1-\phi)}^2$$

is satisfied.

If  $U(z), V(z)$ , and

$$W(z) = U(z)V(z)$$

are power series with operator coefficients such that multiplication by  $U(z), V(z)$ , and  $W(z)$  are contractive transformations of  $\mathcal{C}(z)$  into itself and if the adjoint of multiplication by  $U(z)$  acts as a partially isometric transformation of  $\mathcal{C}(z)$  onto a Herglotz space  $\mathcal{L}(\phi)$ ,

the adjoint of multiplication by  $V(z)$  acts as a partially isometric transformation of  $\mathcal{C}(z)$  onto a Herglotz space  $\mathcal{L}(\theta)$ , and the adjoint of multiplication by  $W(z)$  acts as a partially isometric transformation of  $\mathcal{C}(z)$  onto a Herglotz space  $\mathcal{L}(\psi)$ , then the space  $\mathcal{L}(\theta)$  is contained contractively in the space  $\mathcal{L}(\psi)$  and the adjoint of multiplication by  $V(z)$  acts as a partially isometric transformation of the space  $\mathcal{L}(\phi)$  onto the complementary space to the space  $\mathcal{L}(\theta)$  in the space  $\mathcal{L}(\psi)$ . The space  $\mathcal{L}(1 - \theta)$  is contained contractively in the space  $\mathcal{L}(1 - \psi)$ . The adjoint of multiplication by  $V(z)$  acts as a partially isometric transformation of the space  $\mathcal{L}(1 - \phi)$  onto the complementary space to the space  $\mathcal{L}(1 - \theta)$  in the space  $\mathcal{L}(1 - \psi)$ .

A factorization of a power series  $W(z)$  with operator coefficients applies when multiplication by  $W(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself but the space  $\mathcal{H}(W)$  does not satisfy the identity for difference quotients.

**Theorem 5.** *If multiplication by a power series  $W(z)$  with operator coefficients is a contractive transformation of  $\mathcal{C}(z)$  into itself, then*

$$W(z) = U(z)V(z)$$

*for power series  $U(z)$  and  $V(z)$  with operator coefficients such that multiplication by  $U(z)$  and multiplication by  $V(z)$  are contractive transformations of  $\mathcal{C}(z)$  into itself, such that the space  $\mathcal{H}(U)$  is contained isometrically in the space  $\mathcal{H}(W)$  and satisfies the identity for difference quotients, such that multiplication by  $U(z)$  is a partially isometric transformation of the space  $\mathcal{H}(V)$  onto the orthogonal complement of the space  $\mathcal{H}(U)$  in the space  $\mathcal{H}(W)$  whose kernel contains  $[f(z) - f(0)]/z$  whenever it contains  $f(z)$ , and such that the orthogonal complement of the kernel in the space  $\mathcal{H}(V)$  is the closure of the set of products  $V(z)f(z)$  with  $f(z)$  a polynomial element of  $\mathcal{C}(z)$ .*

*Proof of Theorem 5.* The adjoint of multiplication by  $W(z)$  as a transformation of  $\mathcal{C}(z)$  into itself acts as a partially isometric transformation of  $\mathcal{C}(z)$  onto a Herglotz space  $\mathcal{L}(\psi)$  which is contained contractively in  $\mathcal{C}(z)$ . The complementary space to the space  $\mathcal{L}(\psi)$  in  $\mathcal{C}(z)$  is a Herglotz space  $\mathcal{L}(1 - \psi)$  whose elements are the elements  $f(z)$  of  $\mathcal{C}(z)$  such that  $W(z)f(z)$  belongs to the space  $\mathcal{H}(W)$ . The identity

$$\|f(z)\|_{(1-\psi)}^2 = \|f(z)\|^2 + \|W(z)f(z)\|_{\mathcal{H}(W)}^2$$

is satisfied.

A Herglotz space  $\mathcal{L}(\eta)$  is defined which is contained contractively in  $\mathcal{C}(z)$  and whose complementary space in  $\mathcal{C}(z)$  is a Herglotz space  $\mathcal{L}(1 - \eta)$  which is contained isometrically in the space  $\mathcal{L}(1 - \psi)$  and is the closure of the polynomial elements of the space. The space  $\mathcal{L}(\psi)$  is the closure of its polynomial elements and is contained contractively in the space  $\mathcal{L}(\eta)$ . A Herglotz space  $\mathcal{L}(\theta)$  is defined which is contained isometrically in the space  $\mathcal{L}(\eta)$  and is the closure of the polynomial elements of the space. The space  $\mathcal{L}(\psi)$  is contained contractively in the space  $\mathcal{L}(\theta)$ .

The orthogonal complement of the space  $\mathcal{L}(\theta)$  in the space  $\mathcal{L}(\eta)$  is a Herglotz space  $\mathcal{L}(\eta - \theta)$  which contains no nonzero polynomial. The space  $\mathcal{L}(1 - \eta)$  is contained contractively in

the space  $\mathcal{L}(1 - \theta)$ . The inclusion is isometric since the space  $\mathcal{L}(1 - \eta)$  is the closure of its polynomial elements and since the complementary space to the space  $\mathcal{L}(1 - \eta)$  in the space  $\mathcal{L}(1 - \theta)$  is the space  $\mathcal{L}(\eta - \theta)$  which contains no nonzero polynomial. The space  $\mathcal{L}(\eta - \theta)$  is contained isometrically in the space  $\mathcal{L}(1 - \theta)$  and is the orthogonal complement of the space  $\mathcal{L}(1 - \eta)$ .

Since the space  $\mathcal{L}(\eta - \theta)$  is contained isometrically in the space  $\mathcal{L}(\eta)$  and is contained contractively in the space  $\mathcal{L}(\eta - \psi)$  which is contained contractively in the space  $\mathcal{L}(\eta)$ , the space  $\mathcal{L}(\eta - \theta)$  is contained isometrically in the space  $\mathcal{L}(\eta - \psi)$ . The orthogonal complement of the space  $\mathcal{L}(\eta - \theta)$  in the space  $\mathcal{L}(\eta - \psi)$  is the Herglotz space  $\mathcal{L}(\theta - \psi)$ .

Since the space  $\mathcal{L}(1 - \eta)$  is contained isometrically in the space  $\mathcal{L}(1 - \psi)$ , the space  $\mathcal{L}(\eta - \psi)$  is contained isometrically in the space  $\mathcal{L}(1 - \psi)$ . The space  $\mathcal{L}(\eta - \theta)$  is contained isometrically in the space  $\mathcal{L}(1 - \psi)$  since it is contained isometrically in the space  $\mathcal{L}(\eta - \psi)$ . Since the orthogonal complement of the space  $\mathcal{L}(\eta - \theta)$  in the space  $\mathcal{L}(1 - \theta)$  is the space  $\mathcal{L}(1 - \eta)$  which is contained isometrically in the space  $\mathcal{L}(1 - \theta)$ , the space  $\mathcal{L}(1 - \theta)$  is contained isometrically in the space  $\mathcal{L}(1 - \psi)$ .

Since the space  $\mathcal{L}(\theta)$  is contained contractively in  $\mathcal{C}(z)$  and is the closure of its polynomial elements, a power series  $V(z)$  with operator coefficients exists such that multiplication by  $V(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself, such that the adjoint of multiplication by  $V(z)$  as a transformation of  $\mathcal{C}(z)$  into itself acts as a partially isometric transformation of  $\mathcal{C}(z)$  onto the space  $\mathcal{L}(\theta)$ , and such that the kernel of the transformation contains  $zf(z)$  whenever it contains  $f(z)$ . The complementary space to the space  $\mathcal{L}(\theta)$  in  $\mathcal{C}(z)$  is a Herglotz space  $\mathcal{L}(1 - \theta)$  whose elements are the elements  $f(z)$  of  $\mathcal{C}(z)$  such that  $V(z)f(z)$  belongs to the space  $\mathcal{H}(V)$ . The identity

$$\|f(z)\|_{\mathcal{L}(1-\theta)}^2 = \|f(z)\|^2 + \|V(z)f(z)\|_{\mathcal{H}(V)}^2.$$

The kernel of the adjoint of multiplication by  $V(z)$  as a transformation of  $\mathcal{C}(z)$  into itself is contained isometrically in the space  $\mathcal{H}(V)$  and is the orthogonal complement of the image of the space  $\mathcal{L}(1 - \theta)$  in the space  $\mathcal{H}(V)$ . A partially isometric transformation of the space  $\mathcal{H}(V)$  into the space  $\mathcal{H}(W)$  exists which takes  $V(z)f(z)$  into  $W(z)f(z)$  for every element  $f(z)$  of the space  $\mathcal{L}(1 - \theta)$  and whose kernel is the kernel of the adjoint of multiplication by  $V(z)$  as a transformation of  $\mathcal{C}(z)$  into itself.

An element

$$f(z) + z^{-1}g(z^{-1})$$

of  $\text{ext } \mathcal{C}(z)$  belongs to  $\text{ext } \mathcal{L}(1 - \psi)$  if, and only if,  $f(z)$  and  $g(z)$  are elements of  $\mathcal{C}(z)$  such that  $(W(z)f(z), -g(z))$  belongs to the space  $\mathcal{D}(W)$ . The identity

$$\|f(z) + z^{-1}g(z^{-1})\|_{\text{ext } \mathcal{L}(1-\psi)}^2 = \|f(z)\|^2 + \|(W(z)f(z), -g(z))\|_{\mathcal{D}(W)}^2$$

is satisfied.

An element

$$f(z) + z^{-1}g(z^{-1})$$

of  $\text{ext } \mathcal{C}(z)$  belongs to  $\text{ext } \mathcal{L}(1 - \theta)$  if, and only if,  $f(z)$  and  $g(z)$  are elements of  $\mathcal{C}(z)$  such that  $(V(z)f(z), -g(z))$  belongs to the space  $\mathcal{D}(V)$ . The identity

$$\|f(z) + z^{-1}g(z^{-1})\|_{\text{ext } \mathcal{L}(1-\theta)}^2 = \|f(z)\|^2 + \|(V(z)f(z), -g(z))\|_{\mathcal{D}(V)}^2$$

is satisfied. An isometric transformation of the kernel of the adjoint of multiplication by  $V(z)$  as a transformation of  $\mathcal{C}(z)$  into itself into the space  $\mathcal{D}(V)$  is defined by taking  $f(z)$  into  $(f(z), \theta)$ . The image of the kernel is the orthogonal complement in the space  $\mathcal{D}(V)$  of the image of  $\text{ext } \mathcal{L}(1 - \theta)$ .

Since the space  $\text{ext } \mathcal{L}(1 - \theta)$  is contained isometrically in the space  $\text{ext } \mathcal{L}(1 - \psi)$ , a partially isometric transformation of the space  $\mathcal{D}(V)$  into the space  $\mathcal{D}(W)$  exists which takes  $(V(z)f(z), -g(z))$  into  $(W(z)f(z), -g(z))$  whenever  $f(z)$  and  $g(z)$  are elements of  $\mathcal{C}(z)$  such that

$$f(z) + z^{-1}g(z^{-1})$$

belongs to the space  $\text{ext } \mathcal{L}(1 - \theta)$  and whose kernel is the set of elements  $(f(z), 0)$  with  $f(z)$  in the kernel of the adjoint of multiplication by  $V(z)$  as a transformation of  $\mathcal{C}(z)$  into itself.

The space  $\mathcal{H}(V^*)$  is continued isometrically in the space  $\mathcal{H}(W^*)$ . A power series  $U(z)$  with operator coefficients exists by Theorem 3 such that multiplication by  $U(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself, such that

$$W(z) = U(z)V(z),$$

and such that the range of multiplication by  $V(z)$  as a transformation of  $\text{ext } \mathcal{C}(z)$  into itself is orthogonal to the kernel of multiplication by  $U(z)$  as a transformation of  $\text{ext } \mathcal{C}(z)$  into itself.

Since the kernel of  $V^*(z)$  as a transformation of  $\text{ext } \mathcal{C}(z)$  into itself is orthogonal to the range of multiplication by  $U^*(z)$  as a transformation of  $\text{ext } \mathcal{C}(z)$  into itself. The kernel of the adjoint of multiplication by  $V(z)$  as a transformation of  $\mathcal{C}(z)$  into itself is orthogonal to the range of multiplication by  $U^*(z)$  as a transformation of  $\mathcal{C}(z)$  into itself and is contained isometrically in the space  $\mathcal{H}(U^*)$ . Multiplication by  $U(z)$  annihilates the kernel of the adjoint of multiplication by  $V(z)$  as a transformation of  $\mathcal{C}(z)$  into itself. The space  $\mathcal{H}(U)$  is contained isometrically in the space  $\mathcal{H}(W)$  since multiplication by  $U(z)$  is a partially isometric transformation of the space  $\mathcal{H}(V)$  into the space  $\mathcal{H}(W)$ .

The adjoint of multiplication by  $U(z)$  as a transformation of  $\mathcal{C}(z)$  into itself acts as a partially isometric transformation of  $\mathcal{C}(z)$  onto a Herglotz space  $\mathcal{L}(\phi)$  which is contained contractively in  $\mathcal{C}(z)$ . The complementary space to the space  $\mathcal{L}(\phi)$  in  $\mathcal{C}(z)$  is a Herglotz space  $\mathcal{L}(1 - \phi)$  whose elements are the elements  $f(z)$  of  $\mathcal{C}(z)$  such that  $U(z)f(z)$  belongs to the space  $\mathcal{H}(U)$ . The identity

$$\|f(z)\|_{\mathcal{L}(1-\phi)}^2 = \|f(z)\|^2 + \|U(z)f(z)\|_{\mathcal{H}(U)}^2$$

is satisfied.

The adjoint of multiplication by  $V(z)$  as a transformation of  $\mathcal{C}(z)$  into itself acts as a partially isometric transformation of the space  $\mathcal{L}(1 - \phi)$  onto the orthogonal complement of the space  $\mathcal{L}(1 - \theta)$  in the space  $\mathcal{L}(1 - \psi)$ . The kernel of the adjoint of multiplication by  $V(z)$  as a transformation of  $\mathcal{C}(z)$  into itself is contained isometrically in the space  $\mathcal{L}(1 - \phi)$ . Since the orthogonal complement of the kernel is mapped isometrically onto the orthogonal complement of the space  $\mathcal{L}(1 - \theta)$  in the space  $\mathcal{L}(1 - \psi)$ , since the transformation takes  $[f(z) - f(0)]/z$  into  $[g(z) - g(0)]/z$  whenever it takes  $f(z)$  into  $g(z)$ , and since the orthogonal complement of the space  $\mathcal{L}(1 - \theta)$  in the space  $\mathcal{L}(1 - \psi)$  contains no nonzero polynomial, the polynomial elements of the space  $\mathcal{L}(1 - \phi)$  belong to the kernel of the adjoint of multiplication by  $V(z)$  as a transformation of  $\mathcal{C}(z)$  into itself. Since the kernel is contained isometrically in the space  $\mathcal{L}(1 - \phi)$ , the space  $\mathcal{H}(U)$  satisfies the identity for difference quotients.

This completes the proof of the theorem.

An application of commutant lifting [6] is made to the structure of a contractive transformation of a space  $\mathcal{H}(A)$  into a space  $\mathcal{H}(B)$  which takes  $[f(z) - f(0)]/z$  into  $[g(z) - g(0)]/z$  whenever it takes  $f(z)$  into  $g(z)$ .

**Theorem 6.** *If a contractive transformation  $T$  of a space  $\mathcal{H}(A)$  which satisfies the identity for difference quotients into a space  $\mathcal{H}(B)$  takes  $[f(z) - f(0)]/z$  into  $[g(z) - g(0)]/z$  whenever it takes  $f(z)$  into  $g(z)$  for power series  $A(z)$  and  $B(z)$  with operator coefficients such that multiplication by  $A(z)$  and multiplication by  $B(z)$  are contractive transformations of  $\mathcal{C}(z)$  into itself, then a contractive transformation  $T'$  of the augmented space  $\mathcal{H}(A')$ ,*

$$A'(z) = zA(z)$$

*into the augmented space  $\mathcal{H}(B')$ ,*

$$B'(z) = zB(z),$$

*exists which extends  $T$  and takes  $[f(z) - f(0)]/z$  into  $[g(z) - g(0)]/z$  whenever it takes  $f(z)$  into  $g(z)$ .*

*Proof of Theorem 6.* The construction of  $T'$  is an application of complementation. The coefficient space  $\mathcal{C}$  is contained isometrically in the spaces  $\mathcal{H}(A')$  and  $\mathcal{H}(B')$ . Multiplication by  $z$  is an isometric transformation of the space  $\mathcal{H}(A)$  onto the orthogonal complement of  $\mathcal{C}$  in the space  $\mathcal{H}(A')$  and of the space  $\mathcal{H}(B)$  onto the orthogonal complement of  $\mathcal{C}$  in  $\mathcal{H}(B')$ . The space  $\mathcal{H}(A)$  is contained isometrically in the space  $\mathcal{H}(A')$  since the space  $\mathcal{H}(A)$  satisfies the identity for difference quotients. The space  $\mathcal{H}(B)$  is contained contractively in the space  $\mathcal{H}(B')$ .

Since  $T'$  is defined to agree with  $T$  on the space  $\mathcal{H}(A)$ , it remains to define  $T'$  on the orthogonal complement of the space  $\mathcal{H}(A)$  in the space  $\mathcal{H}(A')$ . The action of  $T'$  is determined within a constant coefficient since the transformation is required to take  $[f(z) - f(0)]/z$  into  $[g(z) - g(0)]/z$  whenever it takes  $f(z)$  into  $g(z)$ . A contractive transformation  $S$  is defined of the space  $\mathcal{H}(A')$  onto the orthogonal complement of  $\mathcal{C}$  in the space  $\mathcal{H}(B')$ . The transformation  $S$  is defined on the space  $\mathcal{H}(A)$  to take  $f(z)$  into  $g(z) - g(0)$  when

$T$  takes  $f(z)$  into  $g(z)$ . The transformation  $S$  is defined on the orthogonal complement of  $\mathcal{C}$  to take  $f(z)$  into  $g(z)$  with  $g(0)$  equal to zero when  $T$  takes  $[f(z) - f(0)]/z$  into  $[g(z) - g(0)]/z$ . The definition of  $S$  on  $\mathcal{H}(A)$  is consistent with the definition of  $S$  on the orthogonal complement of  $\mathcal{C}$  since  $T$  takes  $[f(z) - f(0)]/z$  into  $[g(z) - g(0)]/z$  whenever it takes  $f(z)$  into  $g(z)$ .

The transformation  $S$  acts as a partially isometric transformation of the space  $\mathcal{H}(A)$  onto a Hilbert space  $\mathcal{P}$  which is contained contractively in the orthogonal complement of  $\mathcal{C}$  in the space  $\mathcal{H}(B')$ . The transformation  $S$  acts as a partially isometric transformation of the space  $\mathcal{H}(A')$  onto a Hilbert space  $\mathcal{M}$  which is contained contractively in the orthogonal complement of  $\mathcal{C}$  in the space  $\mathcal{H}(B')$ . The space  $\mathcal{P}$  is contained contractively in  $\mathcal{M}$ . The transformation  $S$  acts as a partially isometric transformation of the orthogonal complement of the space  $\mathcal{H}(A)$  in the space  $\mathcal{H}(A')$  onto the complementary space  $\mathcal{Q}$  to  $\mathcal{P}$  in  $\mathcal{M}$ .

A Hilbert space  $\mathcal{M}'$  is defined as the set of elements  $f(z)$  of the space  $\mathcal{H}(B')$  such that  $f(z) - f(0)$  belongs to  $\mathcal{M}$  with scalar product determined by the identity

$$\|f(z)\|_{\mathcal{M}'}^2 = \|f(z) - f(0)\|_{\mathcal{M}}^2 + |f(0)|^2.$$

Since  $T$  acts as partially isometric transformation of the space  $\mathcal{H}(A)$  onto a Hilbert space  $\mathcal{P}'$  which is contained contractively in the space  $\mathcal{H}(B')$  and since  $S$  takes  $f(z)$  into  $g(z) - g(0)$  whenever  $T$  takes  $f(z)$  into  $g(z)$ , a partially isometric transformation of  $\mathcal{P}'$  onto  $\mathcal{P}$  is defined by taking  $g(z)$  into  $g(z) - g(0)$ . The space  $\mathcal{P}'$  is contained contractively in  $\mathcal{M}'$ . The complementary space  $\mathcal{Q}'$  to  $\mathcal{P}'$  in  $\mathcal{M}'$  is a Hilbert space which is contained contractively in  $\mathcal{M}'$ . A partially isometric transformation of  $\mathcal{Q}'$  into  $\mathcal{Q}$  is defined by taking  $g(z)$  into  $g(z) - g(0)$ . The transformation  $T'$  is defined on the orthogonal complement of the space  $\mathcal{H}(A)$  in the space  $\mathcal{H}(A')$  to take  $f(z)$  into  $g(z)$  when  $g(z)$  is the element of  $\mathcal{Q}'$  of least norm such that  $S$  takes  $f(z)$  into  $g(z) - g(0)$ .

This completes the proof of the theorem.

If  $A(z)$  and  $B(z)$  are power series with operator coefficients such that multiplication by  $A(z)$  and multiplication by  $B(z)$  are contractive transformations of  $\mathcal{C}(z)$  into itself and if

$$A(z)C(z) = D(z)B(z)$$

for power series  $C(z)$  and  $D(z)$  with operator coefficients such that multiplication by  $C(z)$  and multiplication by  $D(z)$  are contractive transformations of  $\mathcal{C}(z)$  into itself, then the space  $\mathcal{H}(A)$  is contained contractively in the space  $\mathcal{H}(DB)$  and a contractive transformation  $T$  of the space  $\mathcal{H}(A)$  into the space  $\mathcal{H}(B)$  which takes  $[f(z) - f(0)]/z$  into  $[g(z) - g(0)]/z$  whenever it takes  $f(z)$  into  $g(z)$  is defined by taking  $f(z)$  into  $g(z)$  when

$$f(z) - D(z)g(z)$$

belongs to the space  $\mathcal{H}(D)$  and equality holds in the inequality

$$\|f(z)\|_{\mathcal{H}(DB)}^2 \leq \|f(z) - D(z)g(z)\|_{\mathcal{H}(D)}^2 + \|g(z)\|_{\mathcal{H}(B)}^2.$$

Every contractive transformation  $T$  of the space  $\mathcal{H}(A)$  into the space  $\mathcal{H}(B)$  which takes  $[f(z) - f(0)]/z$  into  $[g(z) - g(0)]/z$  whenever it takes  $f(z)$  into  $g(z)$  is obtained when the space  $\mathcal{H}(A)$  satisfies the identity for difference quotients. The construction of the power series  $D(z)$  is an application of commutant lifting.

Spaces  $\mathcal{H}(A_r)$  and  $\mathcal{H}(B_r)$  are defined by

$$A_r(z) = z^r A(z)$$

and

$$B_r(z) = z^r B(z)$$

for every nonnegative integer  $r$ . The spaces  $\mathcal{H}(A_r)$  satisfy the identity for difference quotients since the space  $\mathcal{H}(A)$  satisfies the identity for difference quotients. A contractive transformation  $T_r$  of the space  $\mathcal{H}(A_r)$  into the space  $\mathcal{H}(B_r)$  which takes  $[f(z) - f(0)]/z$  into  $[g(z) - g(0)]/z$  whenever it takes  $f(z)$  into  $g(z)$  is defined inductively for nonnegative integers  $r$ . The transformation  $T_0$  of the space  $\mathcal{H}(A_0)$  into the space  $\mathcal{H}(B_0)$  is the transformation  $T$  of the space  $\mathcal{H}(A)$  into the space  $\mathcal{H}(B)$ . When  $T_r$  is defined with the required properties, the transformation  $T_{r+1}$  is defined with the required properties by Theorem 5 so as to extend the transformation  $T_r$ .

A transformation of the union of the spaces  $\mathcal{H}(A_r)$  into the union of the spaces  $\mathcal{H}(B_r)$  is defined which agrees with  $T_r$  on the space  $\mathcal{H}(A_r)$  for every nonnegative integer  $r$ . The space of polynomial elements of  $\mathcal{C}(z)$  of degree less than  $r$  is contained isometrically in the space  $\mathcal{H}(A_r)$  and in the space  $\mathcal{H}(B_r)$  for every nonnegative integer  $r$ . The transformation  $T_r$  of the space  $\mathcal{H}(A_r)$  into the space  $\mathcal{H}(B_r)$  takes polynomials of degree less than  $r$  into polynomials of degree less than  $r$  since it takes  $[f(z) - f(0)]/z$  into  $[g(z) - g(0)]/z$  whenever it takes  $f(z)$  into  $g(z)$ . Since the polynomial elements of  $\mathcal{C}(z)$  are dense in  $\mathcal{C}(z)$ , the transformation has a unique continuous extension as a contractive transformation of  $\mathcal{C}(z)$  into itself which takes  $[f(z) - f(0)]/z$  into  $[g(z) - g(0)]/z$  whenever it takes  $f(z)$  into  $g(z)$ . The transformation is the adjoint of multiplication by  $D(z)$  as a transformation of  $\mathcal{C}(z)$  into itself for a power series  $D(z)$  with operator coefficients such that multiplication by  $D(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself.

The transformation of the space  $\mathcal{H}(A)$  into the space  $\mathcal{H}(B)$  determines a factorization by the choice of  $D(z)$ . The identity for difference quotients is not required for the space  $\mathcal{H}(A)$ .

**Theorem 6.** *If  $A(z)$ ,  $B(z)$ , and  $D(z)$  are power series with operator coefficients such that multiplication by  $A(z)$ , multiplication by  $B(z)$ , and multiplication by  $D(z)$  are contractive transformations of  $\mathcal{C}(z)$  into itself and if the adjoint of multiplication by  $D(z)$  as a transformation of  $\mathcal{C}(z)$  into itself acts as a contractive transformation of the space  $\mathcal{H}(A)$  into the space  $\mathcal{H}(B)$ , then a power series  $C(z)$  with operator coefficients exists such that multiplication by  $C(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself, such that*

$$A(z)C(z) = D(z)B(z),$$

*and such that the range of multiplication by  $C(z)$  as a transformation of ext  $\mathcal{C}(z)$  into itself is orthogonal to the kernel of multiplication by  $A(z)$  as a transformation of ext  $\mathcal{C}(z)$  into itself.*

CHAPTER 2. RIEMANN MAPPING

A square summable power series converges in the unit disk and defines a function in the unit disk. The value

$$f(w) = \langle f(z), (1 - w^{-1}z)^{-1} \rangle$$

at  $w$  of the function represented by a square summable power series  $f(z)$  is a scalar product in the space of square summable power series with the square summable power series

$$(1 - w^{-1}z)^{-1} = 1 + (w^{-1})z + (w^{-1})^2 z^2 + \dots$$

The function represented by a square summable power series is continuous since the identity

$$f(\beta) - f(\alpha) = \langle f(z), (1 - \beta^{-1}z)^{-1} - (1 - \alpha^{-1}z)^{-1} \rangle$$

holds when  $\alpha$  and  $\beta$  are in the unit disk and since the square summable power series

$$(1 - \beta^{-1}z)^{-1} - (1 - \alpha^{-1}z)^{-1} = (\beta - \alpha)^{-1}z + (\beta^2 - \alpha^2)^{-1}z^2 + \dots$$

satisfies the inequality

$$\|(1 - \beta^{-1}z)^{-1} - (1 - \alpha^{-1}z)^{-1}\|^2 \leq |\beta - \alpha|^{-2}(1 + |\alpha + \beta| + |\alpha^2 + \alpha\beta + \beta^2| + \dots)$$

If  $f(z)$  is a square summable power series, a sequence of square summable power series  $f_n(z)$  is defined inductively by

$$f_0(z) = f(z)$$

and

$$f_{n+1}(z) = [f_n(z) - f_n(0)]/z$$

for every nonnegative integer  $n$ . Since the inequality

$$\|f_n(z)\| \leq \|f(z)\|$$

holds for every nonnegative integer  $n$ , the square summable power series

$$[f(z) - f(\alpha)]/(z - \alpha) = f_1(z) + \alpha f_2(z) + \alpha^2 f_3(z) + \dots$$

is a sum in the metric topology of the space of square summable power series when  $\alpha$  is in the unit disk. Since the power series represents a continuous function in the disk, the power series  $f(z)$  represents a differentiable function in the disk. The function

$$[f(w) - f(\alpha)]/(w - \alpha)$$

of  $w$  in the disk is continuous at  $\alpha$  when given a definition  $f'(\alpha)$  at  $\alpha$ . The concept of derivative which is applied in the complex plane is not identical with the concept of derivative applied on a line since the topology of the plane is applied in the limit.

Square summable power series which represent the same function are identical since the coefficients of a square summable power series are all zero if the function represented vanishes identically. A square summable power series is ordinarily identified with the function it represents. The reproducing kernel function

$$(1 - w^{-}z)^{-1}$$

for function values at  $w$  in the space of square summable power series is the element of the space which in a scalar product determines the value of the represented function at  $w$  when  $w$  is in the unit disk.

If  $W(z)$  is a nontrivial power series such that multiplication by  $W(z)$  is a contractive transformation of the space of square summable power series into itself, then

$$W(z)W(w)^{-}/(1 - w^{-}z)$$

is the reproducing kernel function for function values at  $w$  in the range space  $\mathcal{M}(W)$  when  $w$  is in the unit disk. For if

$$g(z) = W(z)f(z)$$

is an element of the space  $\mathcal{M}(W)$ , the identity

$$g(w) = \langle g(z), W(z)W(w)^{-}/(1 - w^{-}z) \rangle_{\mathcal{M}(W)}$$

is a consequence of the identity

$$f(w) = \langle f(z), (1 - w^{-}z)^{-1} \rangle$$

since multiplication by  $W(z)$  is an isometric transformation of the space  $\mathcal{C}(z)$  onto the space  $\mathcal{M}(W)$  and since the identity

$$g(w) = W(w)f(w)$$

is satisfied. The reproducing kernel function

$$W(z)W(w)^{-}/(1 - w^{-}z)$$

for function values at  $w$  in the space  $\mathcal{M}(W)$  is obtained from the reproducing kernel function

$$(1 - w^{-}z)^{-1}$$

for function values at  $w$  in the space of square summable power series under the adjoint of the inclusion of  $\mathcal{M}(W)$  in  $\mathcal{C}(z)$ .

The reproducing kernel function

$$[1 - W(z)W(w)^{-}]/(1 - w^{-}z)$$

for function values at  $w$  in the space  $\mathcal{H}(W)$  is obtained from the reproducing kernel function

$$(1 - w^- z)^{-1}$$

for function values at  $w$  in the space of square summable power series under the adjoint of the inclusion of the space  $\mathcal{H}(W)$  in  $\mathcal{C}(z)$ . The identity

$$f(w) = \langle f(z), [1 - W(z)W(w)^-] / (1 - w^- z) \rangle_{\mathcal{H}(W)}$$

holds for every element  $f(z)$  of the space  $\mathcal{H}(W)$ . Since the identity applies when

$$f(z) = [1 - W(z)W(w)^-] / (1 - w^- z),$$

the function represented by the power series  $W(z)$  is bounded by one in the unit disk.

Reproducing kernel functions are applied to determine the structure of a Hilbert space  $\mathcal{H}$  whose elements are functions in the unit disk. A continuous linear functional on the space is assumed to be defined for every element  $w$  of the unit disk by taking function values at  $w$ . The reproducing kernel function for function values at  $w$  is the unique element  $K(w, z)$  of the space which represents the value

$$f(w) = \langle f(z), K(w, z) \rangle_{\mathcal{H}}$$

for every element  $f(z)$  of the space. The indeterminate  $z$  is treated as a dummy variable in the notation for a function. The function

$$K(\alpha, \beta) = \langle K(\alpha, z), K(\beta, z) \rangle_{\mathcal{H}}$$

of  $\alpha$  and  $\beta$  in the unit disk is treated as an infinite matrix. The symmetry of a scalar product implies the Hermitian symmetry

$$K(\beta, \alpha) = K(\alpha, \beta)^-$$

of the matrix. The infinite matrix is nonnegative in a sense which is determined by its finite square submatrices. If  $\gamma_1, \dots, \gamma_r$  are in the unit disk, then the  $r \times r$  matrix with entry

$$K(\gamma_i, \gamma_j)$$

in the  $i$ -th row and  $j$ -th column is nonnegative. A nonnegative number results when the matrix is multiplied on the right by a column vector with  $r$  entries and on the left by the conjugate transpose row vector. The nonnegative number is a sum of products

$$c_i^- K(\gamma_i, \gamma_j) c_j$$

taken over  $i$  and  $j$  equal to  $1, \dots, r$  for complex numbers  $c_1, \dots, c_r$ .

Reproducing kernel functions are applied in interpolation. If  $\gamma_1, \dots, \gamma_r$  are distinct elements of disk, the set of elements of the Hilbert space which vanish at these elements

is a closed vector subspace whose orthogonal complement consists of functions which are determined by their values at these elements. A function on the finite set is extended to the unit disk so as to be orthogonal to functions which vanish on the finite set. The space of functions on the finite set is a Hilbert space in the scalar product inherited from the full space. Every function on the finite set is a linear combination of reproducing kernel functions which represent values taken on the set. A reproducing kernel function for values on a set is its own extrapolation to the full space. The nonnegativity of a reproducing kernel function is the condition for the existence of a scalar product for the functions on the finite set which creates a Hilbert space compatible with the reproducing property. The finite linear combinations of reproducing kernel functions form a dense vector subspace of the Hilbert space of functions defined on the unit disk. The Hilbert space is the metric completion of the dense subspace. The reproducing property permits the elements of the completion to be realized as functions defined on the unit disk.

The Jordan curve theorem states that the complex complement of a simple closed curve in the complex plane is the union of a bounded region and an unbounded region. The Cauchy formula states that the Stieltjes integral

$$\int f(z)dz = 0$$

of a continuous function over the closed curve is equal to zero if the curve has finite length, if the function has a continuous extension to the closure of the bounded region, and if the function is differentiable at all but a finite number of elements of the bounded region. An example of a simple closed curve is the unit circle, which bounds the unit disk. The Cauchy formula for the unit circle is proved by decomposing the unit disk into regions which are bounded by circles centered at the origin and straight lines through the origin.

Points of nondifferentiability are constructed for a function  $f(z)$  of  $z$  in the unit disk, which has a continuous extension to the closed disk, when the Cauchy integral

$$S(1) = \int_0^{2\pi} f(e^{i\theta})ie^{i\theta} d\theta$$

for the unit circle is nonzero. A point of nondifferentiability is constructed in the annulus

$$a < |z| < b$$

when the inequality

$$(b - a)|S(1)| \leq \left| \int_0^{2\pi} f(be^{i\theta})ibe^{i\theta} d\theta - \int_0^{2\pi} f(ae^{i\theta})iae^{i\theta} d\theta \right|$$

is satisfied. If the length of an interval  $(\alpha, \beta)$  is less than  $2\pi$ , a simple closed curve is constructed from  $ae^{i\alpha}$  to  $be^{i\alpha}$  along a radial line away from the origin, from  $be^{i\alpha}$  to  $be^{i\beta}$  counterclockwise along a circle of radius  $b$  centered at the origin, from  $be^{i\beta}$  to  $ae^{i\beta}$  along

a radial line towards the origin, and from  $ae^{i\beta}$  to  $ae^{i\alpha}$  clockwise along a circle of radius  $a$  about the origin. The Cauchy integral for the curve is

$$S(a, b; \alpha, \beta) = \int_a^b f(re^{i\alpha})e^{i\alpha} dr - \int_a^b f(re^{i\beta})e^{i\beta} dr + \int_\alpha^\beta f(be^{i\theta})ibe^{i\theta} d\theta - \int_\alpha^\beta f(ae^{i\theta})iae^{i\theta} d\theta.$$

The Cauchy integral is zero for a linear function since it is zero for a constant and for  $z$ . The nonzero nature of the integral measures the difficulty in approximating the given function by a linear function.

A point of nondifferentiability is found in the region bounded by the curve when the inequality

$$(\beta - \alpha)(b - a)|S(1)| \leq 2\pi|S(a, b; \alpha, \beta)|$$

is satisfied. A point  $w$  of nondifferentiability is obtained when the regions containing  $w$  and satisfying the inequality form a basis for the neighborhoods of  $w$ . If the inequality

$$|f(z) - g(z)| \leq \epsilon|z - w|$$

holds in the region for some linear function  $g(z)$  for a positive number  $\epsilon$ , then

$$|S(1)| \leq \epsilon$$

since the inequality

$$2\pi|S(a, b; \alpha, \beta)| \leq (\beta - \alpha)(b - a)\epsilon$$

is satisfied.

The maximum principle states that the real part of a function  $f(z)$  of  $z$  in the unit disk, which is differentiable at all but a finite number of points in the disk and which has a continuous extension to the closed disk, vanishes in the unit disk if it is nonpositive on the unit circle and nonnegative at the origin. The function  $f(z)/z$  is differentiable at all but a finite number of points in the annulus

$$a < |z| < 1$$

when  $a$  is in the interval  $(0, 1)$ . Since the identity

$$\int_0^{2\pi} f(ae^{i\theta})d\theta = \int_0^{2\pi} f(e^{i\theta})d\theta$$

holds by the proof of the Cauchy formula, the value of the function at the origin is an average

$$2\pi f(0) = \int_0^{2\pi} f(e^{i\theta})d\theta$$

of values on the boundary. If the real part of the integrand is nonpositive and real part of the integral is nonnegative, then the real part of the integral and the real part of the integrand are zero. The function is a constant since its real part vanishes in the unit disk.

An example of a function which is differentiable and bounded by one in the unit disk is

$$W(z) = (\alpha - z)/(1 - \alpha^{-1}z)$$

when  $\alpha$  is in the unit disk. A Hilbert space  $\mathcal{H}$  of functions in the unit disk exists whose reproducing kernel function for function values at  $w$  is

$$[1 - W(z)W(w)^{-1}]/(1 - w^{-1}z) = (1 - \alpha^{-1}\alpha)(1 - \alpha^{-1}z)^{-1}(1 - \alpha w^{-1})^{-1}$$

when  $w$  is in the unit disk. The space is contained isometrically in the space of square summable power series since

$$(1 - \alpha^{-1}z)^{-1}$$

is the reproducing kernel function for function values at  $\alpha$  in  $\mathcal{C}(z)$ . The orthogonal complement of  $\mathcal{H}$  in  $\mathcal{C}(z)$  is a Hilbert space  $\mathcal{M}$  which is contained isometrically in  $\mathcal{C}(z)$  and which contains the functions which vanish at  $\alpha$ . Since the reproducing kernel function for function values at  $w$  in  $\mathcal{M}$  is

$$W(z)W(w)^{-1}/(1 - w^{-1}z),$$

multiplication by  $W(z)$  is an isometric transformation of  $\mathcal{C}(z)$  onto  $\mathcal{M}$ . Since  $\mathcal{M}$  is contained isometrically in  $\mathcal{C}(z)$ , multiplication by  $W(z)$  is an isometric transformation of  $\mathcal{C}(z)$  into itself.

Applications of the maximum principle are made when a continuous function  $W(z)$  of  $z$  in the unit disk is bounded by one and differentiable at all but a finite number of points in the disk. If the inequality

$$|W(\alpha)| < 1$$

holds for some  $\alpha$  in the disk, then it holds for all  $\alpha$  in the disk. If the inequality holds for a point  $\alpha$  of differentiability, then a continuous function  $W'(z)$  of  $z$  in the unit disk, which is bounded by one and differentiable at all but a finite number of points in the disk, is defined by the identity

$$W'(z)(\alpha - z)/(1 - \alpha^{-1}z) = [W(\alpha) - W(z)]/[1 - W(\alpha)^{-1}W(z)].$$

The identity is applied as a parametrization of the continuous functions  $V(z)$ , which are bounded by one in the unit disk and differentiable at all but a finite number of points in the disk, such that

$$V(\alpha) = W(\alpha).$$

Such a function is obtained on replacing  $W(z)$  by  $V(z)$  in the identity and replacing  $W'(z)$  by a continuous function  $V'(z)$  which is bounded by one in the unit disk and differentiable at all but a finite number of points in the disk.

If a continuous function  $W(z)$  of  $z$  in the unit disk is bounded by one in the disk and is differentiable at all but a finite number of points in the disk and if a Hilbert space  $\mathcal{H}$  exists whose elements are functions of  $z$  in the disk and which has the function

$$[1 - W(z)W(w)^-]/(1 - w^-z)$$

of  $z$  as reproducing kernel function for function values at  $w$  when  $w$  is in the unit disk, then multiplication by  $W(z)$  is an isometric transformation of  $\mathcal{C}(z)$  onto a Hilbert space  $\mathcal{M}$  whose elements are functions of  $z$  in the unit disk and which has the function

$$W(z)W(w)^-/(1 - w^-z)$$

of  $z$  as reproducing kernel function for function values at  $w$  when  $w$  is in the unit disk. A Hilbert space  $\mathcal{H} \vee \mathcal{M}$  exists in which the spaces  $\mathcal{H}$  and  $\mathcal{M}$  are contained contractively as complementary spaces. The elements of the space  $\mathcal{H} \vee \mathcal{M}$  are functions defined in the unit disk. Since the reproducing kernel function for function values at  $w$  in the space  $\mathcal{H} \vee \mathcal{M}$  is the sum of the reproducing kernel functions for function values at  $w$  in the spaces  $\mathcal{H}$  and  $\mathcal{M}$ , the function

$$(1 - w^-z)^{-1}$$

of  $z$  is the reproducing kernel function for function values at  $w$  in the space  $\mathcal{H} \vee \mathcal{M}$  when  $w$  is in the unit disk. The space  $\mathcal{H} \vee \mathcal{M}$  is isometrically equal to  $\mathcal{C}(z)$  since the space of square summable power series has the same reproducing kernel functions. Since the space  $\mathcal{M}$  is contained contractively in  $\mathcal{C}(z)$ , multiplication by  $W(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself. The function  $W(z)$  is represented by a square summable power series. The space  $\mathcal{H}$  is isometrically equal to the space  $\mathcal{H}(W)$ . The space  $\mathcal{H}(W)$  is interpreted as  $\mathcal{C}(z)$  when  $W(z)$  is identically zero.

If a continuous function  $U(z)$  of  $z$  in the unit disk is bounded by one and is differentiable at all but a finite number of points in the disk and if the inequality

$$|U(\alpha)| < 1$$

holds at a point  $\alpha$  of the disk, then the continuous function

$$V(z) = [U(\alpha) - U(z)]/[1 - U(z)U(\alpha)^-]$$

of  $z$  is bounded by one in the disk and is differentiable at all but a finite number of points in the disk. Multiplication by  $U(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself if, and only if, multiplication by  $V(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself. For a Hilbert space  $\mathcal{H}(U)$  exists whose elements are functions of  $z$  in the disk and which contains the function

$$[1 - U(z)U(w)^-]/(1 - w^-z)$$

of  $z$  as reproducing kernel function for function values at  $w$  when  $w$  is in the disk if, and only if, a Hilbert space  $\mathcal{H}(V)$  exists whose elements are functions of  $z$  in the disk and which contains the function

$$[1 - V(z)V(w)^-]/(1 - w^-z)$$

of  $z$  as reproducing kernel function for function values at  $w$  when  $w$  is in the disk. Since the identity

$$\begin{aligned} & [1 - U(z)U(\alpha)^-][1 - V(z)V(w)^-][1 - U(\alpha)U(w)^-] \\ & = [1 - U(\alpha)U(\alpha)^-][1 - U(z)U(w)^-] \end{aligned}$$

is satisfied, multiplication by

$$[1 - U(\alpha)U(\alpha)^-]^{-\frac{1}{2}}[1 - U(z)U(\alpha)^-]$$

is an isometric transformation of the space  $\mathcal{H}(V)$  onto the space  $\mathcal{H}(U)$ .

If a continuous function  $U(z)$  of  $z$  in the disk is bounded by one and differentiable at all but a finite number of points in the disk and if

$$U(\alpha) = 0$$

at a point  $\alpha$  of differentiability, then the identity

$$U(z) = V(z)(\alpha - z)/(1 - \alpha^- z)$$

holds for a continuous function  $V(z)$  of  $z$  in the disk which is bounded by one and which is differentiable at all but a finite number of points in the disk. Multiplication by  $U(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself if, and only if, multiplication by  $V(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself. A space  $\mathcal{H}(U)$ , whose elements are functions of  $z$  in the unit disk and which contains the function

$$[1 - U(z)U(w)^-]/(1 - w^- z)$$

of  $z$  as reproducing kernel function for function values at  $w$  when  $w$  is in the disk, exists if, and only if, a Hilbert space  $\mathcal{H}(V)$  exists whose elements are functions of  $z$  in the disk and which contains the function

$$[1 - V(z)V(w)^-]/(1 - w^- z)$$

of  $z$  as reproducing kernel function for function values at  $w$  when  $w$  is in the disk. The space  $\mathcal{H}(V)$  is contained isometrically in the space  $\mathcal{H}(U)$  and contains the elements of the space  $\mathcal{H}(U)$  which vanish at  $\alpha$ .

If a continuous function  $W(z)$  of  $z$  in the unit disk is bounded by one in the disk and is differentiable at all but a finite number of points in the disk and if  $\alpha_1, \dots, \alpha_r$  are distinct points of differentiability in the disk, then continuous functions  $W_n(z)$  of  $z$  in the disk, which are bounded by one in the disk and which are differentiable at all but a finite number of points in the disk, are defined inductively by

$$W_0(z) = W(z)$$

and

$$W_n(z)(\alpha_n - z)/(1 - \alpha_n^- z) = [W_{n-1}(\alpha_n) - W_{n-1}(z)]/[1 - W_{n-1}(z)W_{n-1}(\alpha_n)^-]$$

when  $n$  is positive and  $W_{n-1}(z)$  is not a constant of absolute value one. A parametrization results of the continuous functions of  $z$  in the unit disk, which are bounded by one in the disk and which are differentiable at all but a finite number of points in the disk, having the same values as  $W(z)$  at the points  $\alpha_1, \dots, \alpha_r$ . Such functions are obtained on replacing  $W_r(z)$  by an arbitrary continuous function of  $z$  which is bounded by one in the unit disk and which is differentiable at all but a finite number of points in the disk. A Hilbert space  $\mathcal{H}(W)$ , whose elements are functions of  $z$  in the disk and which contains the function

$$[1 - W(z)W(w)^-]/(1 - w^-z)$$

of  $z$  as reproducing kernel function for function values at  $w$  when  $w$  is in the disk, exists if, and only if, a Hilbert space  $\mathcal{H}(W_r)$  exists whose elements are functions of  $z$  in the disk and which contains the function

$$[1 - W_r(z)W_r(w)^-]/(1 - w^-z)$$

of  $z$  as reproducing kernel function for function values at  $w$  when  $w$  is in the disk. If  $W_r(z)$  is a constant of absolute value one, the space  $\mathcal{H}(W_r)$  contains no nonzero element and the space  $\mathcal{H}(W)$  has dimension  $r$ . The condition that the space  $\mathcal{H}(W)$  has dimension at least  $r$  is necessary and sufficient for the construction of the function  $W_r(z)$ .

A theorem of Cauchy [17] states that a continuous function of  $z$  in the unit disk is represented by a power series if it is differentiable at all but a finite number of points in the disk. If a continuous function  $W(z)$  of  $z$  is bounded by one in the disk and is differentiable at all but a finite number of points in the disk, then multiplication by  $W(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself. A proof is given by showing that for every finite set of distinct points  $\alpha_1, \dots, \alpha_r$  in the disk the matrix whose entry in the  $i$ -th row and  $j$ -th column is

$$[1 - W(\alpha_i)W(\alpha_j)^-]/(1 - \alpha_j^- \alpha_i)$$

is nonnegative. The conclusion is immediate when  $\alpha_1, \dots, \alpha_r$  are points of differentiability since multiplication by  $V(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself for a power series  $V(z)$  representing a function which agrees with  $W(z)$  at the given points. The same conclusion holds by continuity when the points are not points of differentiability.

A function  $f(z)$  of  $z$  is said to be analytic in the unit disk if it is represented by a power series. The Cauchy theorem states that a function  $f(z)$  of  $z$  is analytic in the unit disk if it is continuous in the disk and is differentiable at all but a finite number of points in the disk.

A function  $\phi(z)$  of  $z$ , which is analytic and has nonnegative real part in the unit disk, admits a Poisson representation. When the function is continuous in the closed disk, the integral representation

$$2\pi \frac{\phi(z) + \phi(w)^-}{1 - w^-z} = \int_0^{2\pi} \frac{\phi(e^{i\theta}) + \phi(e^{i\theta})^-}{(1 - e^{-i\theta}z)(1 - w^-e^{i\theta})} d\theta$$

holds when  $z$  and  $w$  are in the unit disk. The Poisson representation is an application of the Cauchy integrals

$$2\pi\phi(z) = \int_0^{2\pi} \frac{\phi(e^{i\theta})d\theta}{1 - e^{-i\theta}z}$$

and

$$0 = \int_0^{2\pi} \frac{\phi(e^{i\theta})e^{i\theta}d\theta}{1 - w^{-1}e^{i\theta}}.$$

When the function  $\phi(z)$  of  $z$  is not continuous in the closed disk, a nonnegative measure  $\mu$  on the Baire subsets of the real line is constructed whose value

$$\mu(E) = \lim \int_E \frac{1}{2}[\varphi(e^{ix-y}) + \varphi(e^{ix-y})^-]dx$$

is a limit as  $y$  decreases to zero of integrals of the real part of

$$\varphi(e^{ix-y}).$$

The Poisson representation reads

$$\pi \frac{\varphi(z) + \varphi(w)^-}{1 - w^{-1}z} = \int_0^{2\pi} \frac{d\mu(e^{i\theta})}{(1 - e^{-i\theta}z)(1 - w^{-1}e^{i\theta})}$$

when  $z$  and  $w$  are in the unit disk.

A Hilbert space is constructed whose elements are equivalence classes of Baire measurable functions  $f(e^{i\theta})$  of  $e^{i\theta}$  on the unit circle for which the integral

$$2\pi\|f\|^2 = \int_0^{2\pi} |f(e^{i\theta})|^2 d\mu(e^{i\theta})$$

is finite. A partially isometric transformation of the space onto the Herglotz space  $\mathcal{L}(\phi)$  is defined by taking a function  $f(e^{i\theta})$  of  $e^{i\theta}$  on the unit circle into the function

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})d\mu(e^{i\theta})}{1 - e^{-i\theta}z}$$

of  $z$  in the unit disk. Multiplication by  $e^{-i\theta}$  in the Hilbert space of functions on the boundary corresponds to the difference-quotient transformation in the Herglotz space. A related isometric transformation exists of the Hilbert space of functions on the unit circle onto the extension space of the Herglotz space. Multiplication by  $e^{i\theta}$  in the Hilbert space of functions on the unit circle corresponds to multiplication by  $z$  in the extension space  $\mathcal{E}(\phi)$  to the Herglotz space  $\mathcal{L}(\phi)$ .

The Poisson representation makes the canonical model in the space  $\mathcal{E}(\phi)$  effective for the determination of invariant subspaces. The transformation which takes  $f(z)$  into  $f(z)/z$  in the space  $\mathcal{E}(\phi)$  is unitarily equivalent to the transformation which takes  $f(e^{i\theta})$  into

$e^{-i\theta} f(e^{i\theta})$  in the space of square integrable functions on the boundary. The determination of invariant subspaces reduces to a problem in measure theory. An invariant subspace is determined by a Baire subset of the unit circle and contains the equivalence classes of functions  $f(e^{i\theta})$  of  $e^{i\theta}$  in the unit circle which vanish almost everywhere in the set.

A convex set is formed by the Hilbert spaces which are contained contractively in the space  $\text{ext } \mathcal{C}(z)$  of square summable power series such that multiplication by  $z$  is an isometric transformation of the space onto itself. If a Hilbert space  $\mathcal{P}$  is contained contractively in  $\text{ext } \mathcal{C}(z)$  and if multiplication by  $z$  is an isometric transformation of the space onto itself, then the complementary space to  $\mathcal{P}$  in  $\text{ext } \mathcal{C}(z)$  is a Hilbert space which is contained contractively in  $\text{ext } \mathcal{C}(z)$  such that multiplication by  $z$  is an isometric transformation of the space onto itself. If Hilbert spaces  $\mathcal{P}$  and  $\mathcal{Q}$  are contained contractively in  $\text{ext } \mathcal{C}(z)$  and if multiplication by  $z$  is an isometric transformation of the space  $\mathcal{P}$  onto itself and of the space  $\mathcal{Q}$  onto itself, then multiplication by  $z$  is an isometric transformation of the space  $\mathcal{P} \vee \mathcal{Q}$  onto itself and of the space  $\mathcal{P} \wedge \mathcal{Q}$  onto itself. The extreme points are computable for the convex set of Hilbert spaces which are contained contractively in  $\text{ext } \mathcal{C}(z)$  and which admit multiplication and division by  $z$  as isometric transformations. The extreme points are the spaces which are contained isometrically in  $\text{ext } \mathcal{C}(z)$ .

If Hilbert spaces  $\mathcal{P}$  and  $\mathcal{Q}$  belong to the convex set, then the complementary space  $\mathcal{P}_-$  to  $\mathcal{P} \wedge \mathcal{Q}$  in  $\mathcal{P}$  and the complementary space  $\mathcal{Q}_-$  to  $\mathcal{P} \wedge \mathcal{Q}$  in  $\mathcal{Q}$  belong to the convex set. The complementary spaces  $\mathcal{P}_+$  to  $\mathcal{Q}_-$  in  $\mathcal{P} \vee \mathcal{Q}$  and  $\mathcal{Q}_+$  to  $\mathcal{P}_-$  in  $\mathcal{P} \vee \mathcal{Q}$  belong to the convex set. The space

$$\mathcal{P} = \frac{1}{2} \mathcal{P}_+ + \frac{1}{2} \mathcal{P}_-$$

is a convex combination of  $\mathcal{P}_+$  and  $\mathcal{P}_-$ . The space

$$\mathcal{Q} = \frac{1}{2} \mathcal{Q}_+ + \frac{1}{2} \mathcal{Q}_-$$

is a convex combination of  $\mathcal{Q}_+$  and  $\mathcal{Q}_-$ . The spaces  $\mathcal{P}_+$  and  $\mathcal{P}_-$  are isometrically equal if, and only if, the space  $\mathcal{P} \wedge \mathcal{Q}$  contains no nonzero element. The spaces  $\mathcal{Q}_+$  and  $\mathcal{Q}_-$  are isometrically equal if, and only if, the space  $\mathcal{P} \wedge \mathcal{Q}$  contains no nonzero element. The spaces  $\mathcal{P}$  and  $\mathcal{Q}$  are contained isometrically in the space  $\mathcal{P} \vee \mathcal{Q}$  if, and only if, the space  $\mathcal{P} \wedge \mathcal{Q}$  contains no nonzero element.

A partially isometric transformation of  $\text{ext } \mathcal{C}(z)$  onto  $\mathcal{C}(z)$  is defined by taking a Laurent series into the power series which has the same coefficient of  $z^n$  for every nonnegative integer  $n$ . If Hilbert spaces  $\mathcal{P}$  and  $\mathcal{Q}$  are contained contractively in  $\text{ext } \mathcal{C}(z)$  and are complementary spaces to each other in  $\text{ext } \mathcal{C}(z)$ , then the transformation acts as a partially isometric transformation of  $\mathcal{P}$  onto a Hilbert space  $\mathcal{P}'$  and of  $\mathcal{Q}$  onto a Hilbert space  $\mathcal{Q}'$  such that  $\mathcal{P}'$  and  $\mathcal{Q}'$  are contained contractively in  $\mathcal{C}(z)$  and are complementary spaces to each other in  $\mathcal{C}(z)$ . If multiplication by  $z$  is an isometric transformation of the space  $\mathcal{P}$  onto itself, then multiplication by  $z$  is an isometric transformation of the space  $\mathcal{Q}$  onto itself, and the spaces  $\mathcal{P}'$  and  $\mathcal{Q}'$  are Herglotz spaces. Since the Herglotz spaces are complementary spaces to each other in  $\mathcal{C}(z)$ , the space  $\mathcal{P}'$  is a space  $\mathcal{L}(\phi)$  and the space  $\mathcal{Q}'$  is a space  $\mathcal{L}(1 - \phi)$  for a function  $\phi(z)$  of  $z$ , which is analytic in the unit disk, such that  $\phi(z)$  and  $1 - \phi(z)$  have nonnegative real part in the disk. The space  $\mathcal{P}$  is isometrically equal to the extension

space  $\mathcal{E}(\phi)$  of the Herglotz space  $\mathcal{L}(\phi)$ . The space  $\mathcal{Q}$  is isometrically equal to the extension space  $\mathcal{E}(1 - \phi)$  of the Herglotz space  $\mathcal{L}(1 - \phi)$ .

A Herglotz space  $\mathcal{L}(\phi)$  is associated with a power series  $W(z)$  such that multiplication by  $W(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into itself. The adjoint of multiplication by  $W(z)$  as a transformation of  $\text{ext } \mathcal{C}(z)$  into itself acts as a partially isometric transformation of  $\text{ext } \mathcal{C}(z)$  onto the extension space  $\mathcal{E}(\phi)$  of a Herglotz space  $\mathcal{L}(\phi)$ . The adjoint of multiplication by  $W(z)$  as a transformation of  $\mathcal{C}(z)$  into itself acts as a partially isometric transformation of  $\mathcal{C}(z)$  onto the Herglotz space  $\mathcal{L}(\phi)$ , which is contained contractively in  $\mathcal{C}(z)$  and whose complementary space in  $\mathcal{C}(z)$  is the Herglotz space  $\mathcal{L}(1 - \phi)$ . The spaces  $\mathcal{E}(\phi)$  and  $\mathcal{E}(1 - \phi)$  are contained contractively as complementary spaces in  $\text{ext } \mathcal{C}(z)$ .

The space  $\mathcal{L}(1 - \phi)$  is the set of elements  $f(z)$  of  $\mathcal{C}(z)$  such that  $W(z)f(z)$  belongs to the space  $\mathcal{H}(W)$ . The identity

$$\|f(z)\|_{\mathcal{L}(1-\phi)}^2 = \|f(z)\|^2 + \|W(z)f(z)\|_{\mathcal{H}(W)}^2$$

holds for every element  $f(z)$  of the space  $\mathcal{L}(1 - \phi)$ . The space  $\mathcal{H}(W)$  is the set of elements  $f(z)$  of  $\mathcal{C}(z)$  which are mapped into an element  $g(z)$  of the space  $\mathcal{L}(1 - \phi)$  by the adjoint of multiplication by  $W(z)$  as a transformation of  $\mathcal{C}(z)$  into itself. The identity

$$\|f(z)\|_{\mathcal{H}(W)}^2 = \|f(z)\|^2 + \|g(z)\|_{\mathcal{L}(1-\phi)}^2$$

is then satisfied.

The Nevanlinna factorization of functions which are analytic and of bounded type in the unit disk generalizes the factorization of functions which are analytic and bounded by one in the disk. The functions factored are represented in the unit disk by power series whose multiplications define transformations with dense domain in  $\mathcal{C}(z)$  and with range in  $\mathcal{C}(z)$ . The functions are transfer functions of canonical linear systems whose state spaces have indefinite scalar products. The relationship between factorization and invariant subspaces generalizes the Hilbert space theory.

Multiplication transformations are defined in the space of square summable power series by power series. The conjugate of a power series

$$W(z) = \sum W_n z^n$$

is the power series

$$W^*(z) = \sum W_n^- z^n$$

whose coefficients are complex conjugate numbers. If  $f(z)$  is a power series,

$$g(z) = W(z)f(z)$$

is the power series obtained by Cauchy convolution of coefficients. Multiplication by  $W(z)$  in  $\mathcal{C}(z)$  is the transformation which takes  $f(z)$  into  $g(z)$  when  $f(z)$  and  $g(z)$  belongs to  $\mathcal{C}(z)$ . If multiplication by  $W(z)$  is densely defined as a transformation in  $\mathcal{C}(z)$ , then the adjoint

is a transformation whose domain contains the polynomial elements of  $\mathcal{C}(z)$ . The adjoint transformation maps a polynomial element  $f(z)$  of  $\mathcal{C}(z)$  into the polynomial element  $g(z)$  of  $\mathcal{C}(z)$  such that

$$z^{-1}g(z^{-1}) - W^*(z)z^{-1}f(z^{-1})$$

is a power series. Multiplication by  $W(z)$  in  $\mathcal{C}(z)$  is then the adjoint of its adjoint restricted to polynomial elements of  $\mathcal{C}(z)$ .

A Krein space  $\mathcal{H}(W)$ , whose elements are power series, will be constructed from a given power series  $W(z)$  when multiplication by  $W(z)$  is a densely defined transformation in  $\mathcal{C}(z)$ . The space contains

$$f(z) - W(z)g(z)$$

whenever  $f(z)$  and  $g(z)$  are elements of  $\mathcal{C}(z)$  such that the adjoint of multiplication by  $W(z)$  in  $\mathcal{C}(z)$  takes  $f(z)$  into  $g(z)$  and such that  $g(z)$  is in the domain of multiplication by  $W(z)$  in  $\mathcal{C}(z)$ . The identity

$$\langle h(z), f(z) - W(z)g(z) \rangle_{\mathcal{H}(W)} = \langle h(z), f(z) \rangle_{\mathcal{C}(z)}$$

then holds for every element  $h(z)$  of the space  $\mathcal{H}(W)$  which belongs to  $\mathcal{C}(z)$ . The series  $[f(z) - f(0)]/z$  belongs to the space  $\mathcal{H}(W)$  whenever  $f(z)$  belongs to the space. The Krein space  $\mathcal{H}(W')$  associated with the power series

$$W'(z) = zW(z)$$

is the set of power series  $f(z)$  with vector coefficients such that  $[f(z) - f(0)]/z$  belongs to the space  $\mathcal{H}(W)$ . The identity for difference quotients

$$\langle [f(z) - f(0)]/z, [f(z) - f(0)]/z \rangle_{\mathcal{H}(W)} = \langle f(z), f(z) \rangle_{\mathcal{H}(W')} - f(0)^- f(0)$$

is then satisfied. The resulting properties of the space  $\mathcal{H}(W)$  create [4] a canonical coisometric linear system with transfer function  $W(z)$ . The space  $\mathcal{H}(W)$  is the state space of the linear system. The main transformation, which maps the state space into itself, takes  $f(z)$  into

$$[f(z) - f(0)]/z.$$

The input transformation, which maps the space of complex numbers into the state space, takes  $c$  into

$$[W(z) - W(0)]c/z.$$

The output transformation, which maps the state space into the space of complex numbers, takes  $f(z)$  into  $f(0)$ . The external operator, which maps the space of complex numbers into itself, takes  $c$  into

$$W(0)c.$$

A matrix of continuous linear transformations has been constructed which maps the Cartesian product of the state space and the space of complex numbers continuously into itself. The coisometric property of the linear system states that the matrix has an isometric adjoint.

A Krein space is a vector space with scalar product which is the orthogonal sum of a Hilbert space and the anti-space of a Hilbert space. A Krein space is characterized as a vector space with scalar product which is self-dual for a norm topology.

**Theorem 1.** *A vector space with scalar product is a Krein space if it admits a norm which satisfies the convexity identity*

$$\|(1-t)a + tb\|^2 + t(1-t)\|b - a\|^2 = (1-t)\|a\|^2 + t\|b\|^2$$

for all elements  $a$  and  $b$  of the space when  $0 < t < 1$  and if the linear functionals on the space which are continuous for the metric topology defined by the norm are the linear functionals which are continuous for the weak topology induced by duality of the space with itself.

*Proof of Theorem 1.* Norms on the space are considered which satisfy the hypotheses of the theorem. The hypotheses imply that the space is complete in the metric topology defined by any such norm. If a norm  $\|c\|_+$  is given for elements  $c$  of the space, a dual norm  $\|c\|_-$  for elements  $c$  of the space is defined by the least upper bound

$$\|a\|_- = \sup |\langle a, b \rangle|$$

taken over the elements  $b$  of the space such that

$$\|b\|_+ < 1.$$

The least upper bound is finite since every linear functional which is continuous for the weak topology induced by self-duality is assumed continuous for the metric topology. Since every linear functional which is continuous for the metric topology is continuous for the weak topology induced by self-duality, the set of such elements  $b$  is a disk for the weak topology induced by self-duality. The set of elements  $a$  of the space such that

$$\|a\|_- \leq 1$$

is compact in the weak topology induced by self-duality. The set of elements  $a$  of the space such that

$$\|a\|_- < 1$$

is open for the metric topology induced by the plus norm. Since the set is a disk for the weak topology induced by self-duality, the set of elements  $b$  of the space such that

$$\|b\|_+ \leq 1$$

is compact in the weak topology induced by self-duality.

The convexity identity

$$\|(1-t)a + tb\|_+^2 + t(1-t)\|b - a\|_+^2 = (1-t)\|a\|_+^2 + t\|b\|_+^2$$

holds by hypothesis for all elements  $a$  and  $b$  of the space when  $0 < t < 1$ . It will be shown that the convexity identity

$$\|(1-t)u + tv\|_-^2 + t(1-t)\|v - u\|_-^2 = (1-t)\|u\|_-^2 + t\|v\|_-^2$$

holds for all elements  $u$  and  $v$  of the space when  $0 < t < 1$ . Use is made of the convexity identity

$$\begin{aligned} & \langle (1-t)a + tb, (1-t)u + tv \rangle + t(1-t)\langle b-a, v-u \rangle \\ & = (1-t)\langle a, u \rangle + t\langle b, v \rangle \end{aligned}$$

for elements  $a, b, u$ , and  $v$  of the space when  $0 < t < 1$ . Since the inequality

$$\begin{aligned} & |(1-t)\langle a, u \rangle + t\langle b, v \rangle| \\ & \leq \|(1-t)a + tb\|_+ \|(1-t)u + tv\|_- + t(1-t)\|b-a\|_+ \|v-u\|_- \end{aligned}$$

holds by the definition of the minus norm, the inequality

$$\begin{aligned} |(1-t)\langle a, u \rangle + t\langle b, v \rangle|^2 & \leq [\|(1-t)a + tb\|_+^2 + t(1-t)\|b-a\|_+^2] \\ & \times [\|(1-t)u + tv\|_-^2 + t(1-t)\|v-u\|_-^2] \end{aligned}$$

is satisfied. The inequality

$$\begin{aligned} |(1-t)\langle a, u \rangle + t\langle b, v \rangle|^2 & \leq [(1-t)\|a\|_+^2 + t\|b\|_+^2] \\ & \times [\|(1-t)u + tv\|_-^2 + t(1-t)\|v-u\|_-^2] \end{aligned}$$

holds by the convexity identity for the plus norm. The inequality is applied for all elements  $a$  and  $b$  of the space such that the inequalities

$$\|a\|_+ \leq \|u\|_-$$

and

$$\|b\|_+ \leq \|v\|_-$$

are satisfied. The inequality

$$(1-t)\|u\|_-^2 + t\|v\|_-^2 \leq \|(1-t)u + tv\|_-^2 + t(1-t)\|v-u\|_-^2$$

follows by the definition of the minus norm. Equality holds since the reverse inequality is a consequence of the identities

$$(1-t)[(1-t)u + tv] + t[(1-t)u - (1-t)v] = (1-t)u$$

and

$$[(1-t)u + tv] - [(1-t)u - (1-t)v] = v.$$

It has been verified that the minus norm satisfies the hypotheses of the theorem. The dual norm to the minus norm is the plus norm. Another norm which satisfies the hypotheses of the theorem is defined by

$$\|c\|_t^2 = (1-t)\|c\|_+^2 + t\|c\|_-^2$$

when  $0 < t < 1$ . Since the inequalities

$$|\langle a, b \rangle| \leq \|a\|_+ \|b\|_-$$

and

$$|\langle a, b \rangle| \leq \|a\|_- \|b\|_+$$

hold for all elements  $a$  and  $b$  of the space, the inequality

$$|\langle a, b \rangle| \leq (1-t)\|a\|_+ \|b\|_- + t\|a\|_- \|b\|_+$$

holds when  $0 < t < 1$ . The inequality

$$|\langle a, b \rangle| \leq \|a\|_t \|b\|_{1-t}$$

follows for all elements  $a$  and  $b$  of the space when  $0 < t < 1$ . The inequality implies that the dual norm of the  $t$  norm is dominated by the  $1-t$  norm. A norm which dominates its dual norm is obtained when  $t = \frac{1}{2}$ .

Consider the norms which satisfy the hypotheses of the theorem and which dominate their dual norms. Since a nonempty totally ordered set of such norms has a greatest lower bound, which is again such a norm, a minimal such norm exists by the Zorn lemma. If a minimal norm is chosen as the plus norm, it is equal to the  $t$ -norm obtained when  $t = \frac{1}{2}$ . It follows that a minimal norm is equal to its dual norm.

If a norm satisfies the hypotheses of the theorem and is equal to its dual norm, a related scalar product is introduced on the space which may be different from the given scalar product. Since the given scalar product assumes a subsidiary role in the subsequent argument, it is distinguished by a prime. A new scalar product is defined by the identity

$$4\langle a, b \rangle = \|a + b\|^2 - \|a - b\|^2 + i\|a + ib\|^2 - i\|a - ib\|^2.$$

The symmetry of a scalar product is immediate. Linearity will be verified.

The identity

$$\langle wa, wb \rangle = w^- w \langle a, b \rangle$$

holds for all elements  $a$  and  $b$  of the space if  $w$  is a complex number. The identity

$$\langle ia, b \rangle = i \langle a, b \rangle$$

holds for all elements  $a$  and  $b$  of the space. The identity

$$\langle ta, b \rangle = t \langle a, b \rangle$$

will be verified for all elements  $a$  and  $b$  of the space when  $t$  is a positive number. It is sufficient to verify the identity

$$\|ta + b\|^2 - \|ta - b\|^2 = t\|a + b\|^2 - t\|a - b\|^2$$

since a similar identity follows with  $b$  replaced by  $ib$ . The identity holds since

$$\|ta + b\|^2 + t\|a - b\|^2 = t(1 + t)\|a\|^2 + (1 + t)\|b\|^2$$

and

$$\|ta - b\|^2 + t\|a + b\|^2 = t(1 + t)\|a\|^2 + (1 + t)\|b\|^2$$

by the convexity identity.

If  $a, b$ , and  $c$  are elements of the space and if  $0 < t < 1$ , the identity

$$\begin{aligned} 4\langle(1 - t)a + tb, c\rangle &= \|(1 + t)(a + c) + t(b + c)\|^2 \\ &- \|(1 - t)(a - c) + t(b - c)\|^2 + i\|(1 - t)(a + ic) + t(b + ic)\|^2 \\ &- i\|(1 - t)(a - ic) + t(b - ic)\|^2 \end{aligned}$$

is satisfied with the right side equal to

$$\begin{aligned} &(1 - t)\|a + c\|^2 + t\|b + c\|^2 - (1 - t)\|a - c\|^2 - t\|b - c\|^2 \\ &+ i(1 - t)\|a + ic\|^2 + it\|b + ic\|^2 - i(1 - t)\|a - ic\|^2 - it\|b - ic\|^2 \\ &= 4(1 - t)\langle a, c\rangle + 4t\langle b, c\rangle. \end{aligned}$$

The identity

$$\langle(1 - t)a + tb, c\rangle = (1 - t)\langle a, c\rangle + t\langle b, c\rangle$$

follows.

Linearity of a scalar product is now easily verified. Scalar self-products are nonnegative since the identity

$$\langle c, c\rangle = \|c\|^2$$

holds a for every element  $c$  of the space. A Hilbert space is obtained whose norm is the minimal norm. Since the inequality

$$|\langle a, b\rangle'| \leq \|a\|\|b\|$$

holds for all elements  $a$  and  $b$  of the space, a contractive transformation  $J$  of the Hilbert space into itself exists such that the identity

$$\langle a, b\rangle' = \langle Ja, b\rangle$$

holds for all elements  $a$  and  $b$  of the space. The symmetry of the given scalar product implies that the transformation  $J$  is self-adjoint. Since the Hilbert space norm is self-dual with respect to the given scalar product, the transformation  $J$  is also isometric with respect to the Hilbert space scalar product. The space is the orthogonal sum of the space of eigenvectors of  $J$  for the eigenvalue one and the space of eigenvectors of  $J$  for the eigenvalue minus one. These spaces are also orthogonal with respect to the given scalar

product. They are the required Hilbert space and anti-space of a Hilbert space for the orthogonal decomposition of the vector space with scalar product to form a Krein space.

This completes the proof of the theorem.

The orthogonal decomposition of a Krein space is not unique since equivalent norms can be used. The dimension of the anti-space of a Hilbert space in the decomposition is however an invariant called the Pontryagin index of the Krein space. Krein spaces are a natural context for a complementation theory which was discovered in Hilbert spaces [3].

A generalization of the concept of orthogonal complement applies when a Krein space  $\mathcal{P}$  is contained continuously and contractively in a Krein space  $\mathcal{H}$ . The contractive property of the inclusion means that the inequality

$$\langle a, a \rangle_{\mathcal{H}} \leq \langle a, a \rangle_{\mathcal{P}}$$

holds for every element  $a$  of  $\mathcal{P}$ . Continuity of the inclusion means that an adjoint transformation of  $\mathcal{H}$  into  $\mathcal{P}$  exists. A self-adjoint transformation  $P$  of  $\mathcal{H}$  into  $\mathcal{H}$  is obtained on composing the inclusion with the adjoint. The inequality

$$\langle Pc, Pc \rangle_{\mathcal{H}} \leq \langle Pc, Pc \rangle_{\mathcal{P}}$$

for elements  $c$  of  $\mathcal{H}$  implies the inequality

$$\langle P^2c, c \rangle_{\mathcal{H}} \leq \langle Pc, c \rangle_{\mathcal{H}}$$

for elements  $c$  of  $\mathcal{H}$ , which is restated as an inequality

$$P^2 \leq P$$

for self-adjoint transformations in  $\mathcal{H}$ .

The properties of adjoint transformations are used in the construction of a complementary space  $\mathcal{Q}$  to  $\mathcal{P}$  in  $\mathcal{H}$ .

**Theorem 2.** *If a Krein space  $\mathcal{P}$  is contained continuously and contractively in a Krein space  $\mathcal{H}$ , then a unique Krein space  $\mathcal{Q}$  exists, which is contained continuously and contractively in  $\mathcal{H}$ , such that the inequality*

$$\langle c, c \rangle_{\mathcal{H}} \leq \langle a, a \rangle_{\mathcal{P}} + \langle b, b \rangle_{\mathcal{Q}}$$

*holds whenever  $c = a + b$  with  $a$  in  $\mathcal{P}$  and  $b$  in  $\mathcal{Q}$  and such that every element  $c$  of  $\mathcal{H}$  admits some such decomposition for which equality holds.*

*Proof of Theorem 2.* Define  $\mathcal{Q}$  to be the set of elements  $b$  of  $\mathcal{H}$  such that the least upper bound

$$\langle b, b \rangle_{\mathcal{Q}} = \sup[\langle a + b, a + b \rangle_{\mathcal{H}} - \langle a, a \rangle_{\mathcal{P}}]$$

taken over all elements  $a$  of  $\mathcal{P}$  is finite. It will be shown that  $\mathcal{Q}$  is a vector space with scalar product having the desired properties. Since the origin belongs to  $\mathcal{P}$ , the inequality

$$\langle b, b \rangle_{\mathcal{H}} \leq \langle b, b \rangle_{\mathcal{Q}}$$

holds for every element  $b$  of  $\mathcal{Q}$ . Since the inclusion of  $\mathcal{P}$  in  $\mathcal{H}$  is contractive, the origin belongs to  $\mathcal{Q}$  and has self-product zero. If  $b$  belongs to  $\mathcal{Q}$  and if  $w$  is a complex number, then  $wb$  is an element of  $\mathcal{Q}$  which satisfies the identity

$$\langle wb, wb \rangle_{\mathcal{Q}} = w^{-1}w \langle b, b \rangle_{\mathcal{Q}}.$$

The set  $\mathcal{Q}$  is invariant under multiplication by complex numbers. The set  $\mathcal{Q}$  is shown to be a vector space by showing that it is closed under convex combinations.

It will be shown that  $(1-t)a + tb$  belongs to  $\mathcal{Q}$  whenever  $a$  and  $b$  are elements of  $\mathcal{Q}$  and  $t$  is a number,  $0 < t < 1$ . Since an arbitrary pair of elements of  $\mathcal{P}$  can be written in the form  $(1-t)a + tv$  and  $v - u$  for elements  $u$  and  $v$  of  $\mathcal{P}$ , the identity

$$\begin{aligned} & \langle (1-t)a + tb, (1-t)a + tb \rangle_{\mathcal{Q}} + t(1-t) \langle b - a, b - a \rangle_{\mathcal{Q}} \\ &= \sup[\langle (1-t)(a + u) + t(b + v), (1-t)(a + u) + t(b + v) \rangle_{\mathcal{H}} \\ & \quad + t(1-t) \langle (b + v) - (a + u), (b + v) - (a + u) \rangle_{\mathcal{H}} \\ & \quad - \langle (1-t)u + tv, (1-t)u + tv \rangle_{\mathcal{P}} - t(1-t) \langle v - u, v - u \rangle_{\mathcal{P}}] \end{aligned}$$

holds with the least upper bound taken over all elements  $u$  and  $v$  of  $\mathcal{P}$ . By the convexity identity the least upper bound

$$\begin{aligned} & \langle (1-t)a + tb, (1-t)a + tb \rangle_{\mathcal{Q}} + t(1-t) \langle b - a, b - a \rangle_{\mathcal{Q}} \\ &= \sup[\langle a + u, a + u \rangle_{\mathcal{H}} - \langle u, u \rangle_{\mathcal{P}}] + \sup[\langle b + v, b + v \rangle_{\mathcal{H}} - \langle v, v \rangle_{\mathcal{P}}] \end{aligned}$$

holds over all elements  $u$  and  $v$  of  $\mathcal{P}$ . It follows that the identity

$$\begin{aligned} & \langle (1-t)a + tb, (1-t)a + tb \rangle_{\mathcal{Q}} + t(1-t) \langle b - a, b - a \rangle_{\mathcal{Q}} \\ &= (1-t) \langle a, a \rangle_{\mathcal{Q}} + t \langle b, b \rangle_{\mathcal{Q}} \end{aligned}$$

is satisfied.

This completes the verification that  $\mathcal{Q}$  is a vector space. It will be shown that a scalar product is defined on the space by the identity

$$4 \langle a, b \rangle_{\mathcal{Q}} = \langle a + b, a + b \rangle_{\mathcal{Q}} - \langle a - b, a - b \rangle_{\mathcal{Q}} + i \langle a + ib, a + ib \rangle_{\mathcal{Q}} - i \langle a - ib, a - ib \rangle_{\mathcal{Q}}.$$

Linearity and symmetry of a scalar product are verified as in the characterization of Krein spaces. The nondegeneracy of a scalar product remains to be verified.

Since the inclusion of  $\mathcal{P}$  in  $\mathcal{H}$  is continuous, a self-adjoint transformation  $P$  of  $\mathcal{H}$  into itself exists which coincides with the adjoint of the inclusion of  $\mathcal{P}$  in  $\mathcal{H}$ . If  $c$  is an element of  $\mathcal{H}$  and if  $a$  is an element of  $\mathcal{P}$ , the inequality

$$\langle a - Pc, a - Pc \rangle_{\mathcal{H}} \leq \langle a - Pc, a - Pc \rangle_{\mathcal{P}}$$

implies the inequality

$$\langle (1-P)c, (1-P)c \rangle_{\mathcal{Q}} \leq \langle c, c \rangle_{\mathcal{H}} - \langle Pc, Pc \rangle_{\mathcal{P}}.$$

Equality holds since the reverse inequality follows from the definition of the self-product in  $\mathcal{Q}$ . If  $b$  is an element of  $\mathcal{Q}$  and if  $c$  is an element of  $\mathcal{H}$ , the inequality

$$\langle b-c, b-c \rangle_{\mathcal{H}} \leq \langle Pc, Pc \rangle_{\mathcal{P}} + \langle b-(1-P)c, b-(1-P)c \rangle_{\mathcal{Q}}$$

can be written

$$\langle b, b \rangle_{\mathcal{H}} - \langle b, c \rangle_{\mathcal{H}} - \langle c, b \rangle_{\mathcal{H}} \leq \langle b, b \rangle_{\mathcal{Q}} - \langle b, (1-P)c \rangle_{\mathcal{Q}} - \langle (1-P)c, b \rangle_{\mathcal{Q}}.$$

Since  $b$  can be replaced by  $wb$  for every complex number  $w$ , the identity

$$\langle b, c \rangle_{\mathcal{H}} = \langle b, (1-P)c \rangle_{\mathcal{Q}}$$

is satisfied. The nondegeneracy of a scalar product follows in the space  $\mathcal{Q}$ . The space  $\mathcal{Q}$  is contained continuously in the space  $\mathcal{H}$  since  $1-P$  coincides with the adjoint of the inclusion of  $\mathcal{Q}$  in the space  $\mathcal{H}$ .

The intersection of  $\mathcal{P}$  and  $\mathcal{Q}$  is considered as a vector space  $\mathcal{P} \wedge \mathcal{Q}$  with scalar product

$$\langle a, b \rangle_{\mathcal{P} \wedge \mathcal{Q}} = \langle a, b \rangle_{\mathcal{P}} + \langle a, b \rangle_{\mathcal{Q}}.$$

Linearity and symmetry of a scalar product are immediate, but nondegeneracy requires verification. If  $c$  is an element of  $\mathcal{H}$ ,

$$P(1-P)c = (1-P)Pc$$

is an element of  $\mathcal{P} \wedge \mathcal{Q}$  which satisfies the identity

$$\langle a, P(1-P)c \rangle_{\mathcal{P} \wedge \mathcal{Q}} = \langle a, c \rangle_{\mathcal{H}}$$

for every element  $a$  of  $\mathcal{P} \wedge \mathcal{Q}$ . Nondegeneracy of a scalar product in  $\mathcal{P} \wedge \mathcal{Q}$  follows from nondegeneracy of the scalar product in  $\mathcal{H}$ . The space  $\mathcal{P} \wedge \mathcal{Q}$  is contained continuously in the space  $\mathcal{H}$ . The self-adjoint transformation  $P(1-P)$  in  $\mathcal{H}$  coincides with the adjoint of the inclusion of  $\mathcal{P} \wedge \mathcal{Q}$  in  $\mathcal{H}$ . The inequality

$$0 \leq \langle c, c \rangle_{\mathcal{P} \wedge \mathcal{Q}}$$

holds for every element  $c$  of  $\mathcal{P} \wedge \mathcal{Q}$  since the identity

$$0 = c - c$$

with  $c$  in  $\mathcal{P}$  and  $-c$  in  $\mathcal{Q}$  implies the inequality

$$0 \leq \langle c, c \rangle_{\mathcal{P}} + \langle c, c \rangle_{\mathcal{Q}}.$$

It will be shown that the space  $\mathcal{P} \wedge \mathcal{Q}$  is a Hilbert space. The metric topology of the space is the disk topology resulting from duality of the space with itself. Since the inclusion of  $\mathcal{P} \wedge \mathcal{Q}$  in  $\mathcal{P}$  is continuous from the weak topology induced by  $\mathcal{P} \wedge \mathcal{Q}$  into the weak topology induced by  $\mathcal{P}$ , it is continuous from the disk topology induced by  $\mathcal{P} \wedge \mathcal{Q}$  into the disk topology induced by  $\mathcal{P}$ . Since  $\mathcal{P}$  is a Krein space, it is complete in its disk topology. A Cauchy sequence of elements  $c_n$  of  $\mathcal{P} \wedge \mathcal{Q}$  is then a convergent sequence of elements of  $\mathcal{P}$ . The limit is an element  $c$  of  $\mathcal{P}$  such that the identity

$$\langle c, a \rangle_{\mathcal{P}} = \lim \langle c_n, a \rangle_{\mathcal{P}}$$

holds for every element  $a$  of  $\mathcal{P}$  and such that the identity

$$\langle c, c \rangle_{\mathcal{P}} = \lim \langle c_n, c_n \rangle_{\mathcal{P}}$$

is satisfied. Since the inclusion of  $\mathcal{P}$  in  $\mathcal{H}$  is continuous from the disk topology of  $\mathcal{P}$  into the disk topology of  $\mathcal{H}$ , the identity

$$\langle c, a \rangle_{\mathcal{H}} = \lim \langle c_n, a \rangle_{\mathcal{H}}$$

holds for every element  $a$  of  $\mathcal{H}$  and the identity

$$\langle c, c \rangle_{\mathcal{H}} = \lim \langle c_n, c_n \rangle_{\mathcal{H}}$$

is satisfied.

If  $b$  is an element of  $\mathcal{Q}$ , the limits

$$\lim \langle c_n, b \rangle_{\mathcal{Q}}$$

and

$$\lim \langle c_n, c_n \rangle_{\mathcal{Q}}$$

exist since the inclusion of  $\mathcal{P} \wedge \mathcal{Q}$  in  $\mathcal{Q}$  is continuous from the disk topology of  $\mathcal{P} \wedge \mathcal{Q}$  into the disk topology of  $\mathcal{Q}$ . The sequence of elements  $c_n$  of  $\mathcal{Q}$  is Cauchy in the disk topology of  $\mathcal{Q}$ . If  $a$  is an element of  $\mathcal{P}$ , the identity

$$\langle a + c, a + c \rangle_{\mathcal{H}} = \lim \langle a + c_n, a + c_n \rangle_{\mathcal{H}}$$

is satisfied. Since the inequality

$$\langle a + c_n, a + c_n \rangle_{\mathcal{H}} - \langle a, a \rangle_{\mathcal{P}} \leq \langle c_n, c_n \rangle_{\mathcal{Q}}$$

holds for every index  $n$ , the inequality

$$\langle a + c, a + c \rangle_{\mathcal{H}} - \langle a, a \rangle_{\mathcal{P}} \leq \lim \langle c_n, c_n \rangle_{\mathcal{Q}}$$

is satisfied. It follows that  $c$  belongs to  $\mathcal{Q}$  and that

$$\langle c, c \rangle_{\mathcal{Q}} \leq \lim \langle c_n, c_n \rangle_{\mathcal{Q}}.$$

Since the inequality

$$\langle c - c_m, c - c_m \rangle_{\mathcal{Q}} \leq \lim \langle c_n - c_m, c_n - c_m \rangle_{\mathcal{Q}}$$

holds for every index  $m$  and since the elements  $c_n$  of  $\mathcal{Q}$  form a Cauchy sequence in the disk topology of  $\mathcal{Q}$ , the limit of the elements  $c_n$  of  $\mathcal{Q}$  is equal to  $c$ . This completes the proof that  $\mathcal{P} \wedge \mathcal{Q}$  is a Hilbert space.

The Cartesian product of  $\mathcal{P}$  and  $\mathcal{Q}$  is isomorphic to the Cartesian product of  $\mathcal{H}$  and  $\mathcal{P} \wedge \mathcal{Q}$ . If  $a$  is an element of  $\mathcal{P}$  and if  $b$  is an element of  $\mathcal{Q}$ , a unique element  $c$  of  $\mathcal{P} \wedge \mathcal{Q}$  exists such that the identity

$$\langle a - c, a - c \rangle_{\mathcal{P}} + \langle b + c, b + c \rangle_{\mathcal{Q}} = \langle a + b, a + b \rangle_{\mathcal{H}} + \langle c, c \rangle_{\mathcal{P} \wedge \mathcal{Q}}$$

is satisfied. Every element of the Cartesian product of  $\mathcal{H}$  and  $\mathcal{P} \wedge \mathcal{Q}$  is a pair  $(a + b, c)$  for elements  $a$  of  $\mathcal{P}$  and  $b$  of  $\mathcal{Q}$  for such an element  $c$  of  $\mathcal{P} \wedge \mathcal{Q}$ . Since  $\mathcal{H}$  is a Krein space and since  $\mathcal{P} \wedge \mathcal{Q}$  is a Hilbert space, the Cartesian product of  $\mathcal{P}$  and  $\mathcal{Q}$  is a Krein space. Since  $\mathcal{P}$  is a Krein space, it follows that  $\mathcal{Q}$  is a Krein space.

The existence of a Krein space  $\mathcal{Q}$  with the desired properties has now been verified. Uniqueness is proved by showing that a Krein space  $\mathcal{Q}'$  with these properties is isometrically equal to the space  $\mathcal{Q}$  constructed. Such a space  $\mathcal{Q}'$  is contained contractively in the space  $\mathcal{Q}$ . The self-adjoint transformation  $1 - P$  in  $\mathcal{H}$  coincides with the adjoint of the inclusion of  $\mathcal{Q}'$  in  $\mathcal{H}$ . The space  $\mathcal{P} \wedge \mathcal{Q}'$  is a Hilbert space which is contained contractively in the Hilbert space  $\mathcal{P} \wedge \mathcal{Q}$ . Since the inclusion is isometric on the range of  $P(1 - P)$ , which is dense in both spaces, the space  $\mathcal{P} \wedge \mathcal{Q}'$  is isometrically equal to the space  $\mathcal{P} \wedge \mathcal{Q}$ . Since the Cartesian product of  $\mathcal{P}$  and  $\mathcal{Q}'$  is isomorphic to the Cartesian product of  $\mathcal{P}$  and  $\mathcal{Q}$ , the spaces  $\mathcal{Q}$  and  $\mathcal{Q}'$  are isometrically equal.

This completes the proof of the theorem.

The space  $\mathcal{Q}$  is called the complementary space to  $\mathcal{P}$  in  $\mathcal{H}$ . The space  $\mathcal{P}$  is recovered as the complementary space to the space  $\mathcal{Q}$  in  $\mathcal{H}$ . The decomposition of an element  $c$  of  $\mathcal{H}$  as  $c = a + b$  with  $a$  an element of  $\mathcal{P}$  and  $b$  an element of  $\mathcal{Q}$  such that equality hold in the inequality

$$\langle c, c \rangle_{\mathcal{H}} \leq \langle a, a \rangle_{\mathcal{P}} + \langle b, b \rangle_{\mathcal{Q}}$$

is unique. The minimal decomposition results when  $a$  is obtained from  $c$  under the adjoint of the inclusion of  $\mathcal{P}$  in  $\mathcal{H}$  and  $b$  is obtained from  $c$  under the adjoint of the inclusion of  $\mathcal{Q}$  in  $\mathcal{H}$ .

A construction is made of complementary subspaces whose inclusion in the full space have adjoints coinciding with given self-adjoint transformations.

**Theorem 3.** *If a self-adjoint transformation  $P$  of a Krein space into itself satisfies the inequality*

$$P^2 \leq P,$$

then unique Krein spaces  $\mathcal{P}$  and  $\mathcal{Q}$  exist, which are contained continuously and contractively in  $\mathcal{H}$  and which are complementary spaces in  $\mathcal{H}$ , such that  $P$  coincides with the adjoint of the inclusion of  $\mathcal{P}$  in  $\mathcal{H}$  and  $1 - P$  coincides with the adjoint of the inclusion of  $\mathcal{Q}$  in  $\mathcal{H}$ .

*Proof of Theorem 3.* The proof repeats the construction of a complementary space under a weaker hypothesis. The range of  $P$  is considered as a vector space  $\mathcal{P}'$  with scalar product determined by the identity

$$\langle Pc, Pc \rangle_{\mathcal{P}'} = \langle Pc, c \rangle_{\mathcal{H}},$$

for every element  $c$  of  $\mathcal{H}$ . The space  $\mathcal{P}'$  is contained continuously and contractively in the space  $\mathcal{H}$ . The transformation  $P$  coincides with the adjoint of the inclusion of  $\mathcal{P}'$  in  $\mathcal{H}$ . A Krein space  $\mathcal{Q}$ , which is contained continuously and contractively in  $\mathcal{H}$ , is defined as the set of elements  $b$  of  $\mathcal{H}$  such that the least upper bound

$$\langle b, b \rangle_{\mathcal{Q}} = \sup[\langle a + b, a + b \rangle_{\mathcal{H}} - \langle a, a \rangle_{\mathcal{P}'}]$$

taken over all elements  $a$  of  $\mathcal{P}'$  is finite. The adjoint of the inclusion of  $\mathcal{Q}$  in  $\mathcal{H}$  coincides with  $1 - P$ . The complementary space to  $\mathcal{Q}$  in  $\mathcal{H}$  is a Krein space  $\mathcal{P}$  which contains the space  $\mathcal{P}'$  isometrically and which is contained continuously and contractively in  $\mathcal{H}$ . The adjoint of the inclusion of  $\mathcal{P}$  in  $\mathcal{H}$  coincides with  $1 - P$ .

This completes the proof of the theorem.

A factorization of continuous and contractive transformations in Krein spaces is an application of complementation theory.

**Theorem 4.** *The kernel of a continuous and contractive transformation  $T$  of a Krein space  $\mathcal{P}$  into a Krein space  $\mathcal{Q}$  is a Hilbert space which is contained continuously and isometrically in  $\mathcal{P}$  and whose orthogonal complement in  $\mathcal{P}$  is mapped isometrically onto a Krein space which is contained continuously and contractively in  $\mathcal{Q}$ .*

*Proof of Theorem 4.* Since the transformation  $T$  of  $\mathcal{P}$  into  $\mathcal{Q}$  is continuous and contractive, the self-adjoint transformation  $P = TT^*$  in  $\mathcal{Q}$  satisfies the inequality  $P^2 \leq P$ . A unique Krein space  $\mathcal{M}$ , which is contained continuously and contractively in  $\mathcal{Q}$ , exists such that  $\mathcal{P}$  coincides with the adjoint of the inclusion of  $\mathcal{M}$  in  $\mathcal{Q}$ . It will be shown that  $T$  maps  $\mathcal{P}$  contractively into  $\mathcal{M}$ .

If  $a$  is an element of  $\mathcal{P}$  and if  $b$  is an element of  $\mathcal{Q}$ , then

$$\begin{aligned} & \langle Ta + (1 - P)b, Ta + (1 - P)b \rangle_{\mathcal{Q}} \\ &= \langle T(a - T^*b), T(a - T^*b) \rangle_{\mathcal{Q}} + \langle b, b \rangle_{\mathcal{Q}} + \langle b, T(a - T^*b) \rangle_{\mathcal{Q}} + \langle T(a - T^*b), b \rangle_{\mathcal{Q}} \end{aligned}$$

is less than or equal to

$$\begin{aligned} & \langle a - T^*b, a - T^*b \rangle_{\mathcal{P}} + \langle b, b \rangle_{\mathcal{Q}} + \langle T^*b, a - T^*b \rangle_{\mathcal{P}} + \langle a - T^*b, T^*b \rangle_{\mathcal{P}} \\ &= \langle a, a \rangle_{\mathcal{P}} + \langle (1 - TT^*)b, b \rangle_{\mathcal{Q}}. \end{aligned}$$

Since  $b$  is an arbitrary element of  $\mathcal{Q}$ ,  $Ta$  is an element of  $\mathcal{M}$  which satisfies the inequality

$$\langle Ta, Ta \rangle_{\mathcal{M}} \leq \langle a, a \rangle_{\mathcal{P}}.$$

Equality holds when  $a = T^*b$  for an element  $b$  of  $\mathcal{Q}$  since

$$\langle TT^*b, TT^*b \rangle_{\mathcal{M}} = \langle TT^*b, b \rangle_{\mathcal{Q}} = \langle T^*b, T^*b \rangle_{\mathcal{P}}.$$

Since the transformation of  $\mathcal{P}$  into  $\mathcal{M}$  is continuous by the closed graph theorem, the adjoint transformation is an isometry. The range of the adjoint transformation is a Krein space which is contained continuously and isometrically in  $\mathcal{P}$  and whose orthogonal complement is the kernel of  $T$ . Since  $T$  is contractive, the kernel of  $T$  is a Hilbert space.

This completes the proof of the theorem.

A continuous transformation of a Krein space  $\mathcal{P}$  into a Krein space  $\mathcal{Q}$  is said to be a partial isometry if its kernel is a Krein space which is contained continuously and isometrically in  $\mathcal{P}$  and whose orthogonal complement is mapped isometrically into  $\mathcal{Q}$ . A partially isometric transformation of a Krein space into a Krein space is contractive if, and only if, its kernel is a Hilbert space. Complementation is preserved under contractive partially isometric transformations of a Krein space onto a Krein space.

**Theorem 5.** *If a contractive partially isometric transformation  $T$  maps a Krein space  $\mathcal{H}$  onto a Krein space  $\mathcal{H}'$  and if Krein spaces  $\mathcal{P}$  and  $\mathcal{Q}$  are contained continuously and contractively as complementary subspaces of  $\mathcal{H}$ , then Krein spaces  $\mathcal{P}'$  and  $\mathcal{Q}'$ , which are contained continuously and contractively as complementary subspaces of  $\mathcal{H}'$ , exist such that  $T$  acts as a contractive partially isometric transformation of  $\mathcal{P}$  onto  $\mathcal{P}'$  and of  $\mathcal{Q}$  onto  $\mathcal{Q}'$ .*

*Proof of Theorem 5.* Since the Krein spaces  $\mathcal{P}$  and  $\mathcal{Q}$  are contained continuously and contractively in  $\mathcal{H}$  and since  $T$  is a continuous and contractive transformation of  $\mathcal{H}$  into  $\mathcal{H}'$ ,  $T$  acts as a continuous and contractive transformation of  $\mathcal{P}$  into  $\mathcal{H}'$  and of  $\mathcal{Q}$  into  $\mathcal{H}'$ . Krein spaces  $\mathcal{P}'$  and  $\mathcal{Q}'$ , which are contained continuously and contractively in  $\mathcal{H}'$ , exist such that  $T$  acts as a contractive partially isometric transformation of  $\mathcal{P}$  onto  $\mathcal{P}'$  and of  $\mathcal{Q}$  onto  $\mathcal{Q}'$ . It will be shown that  $\mathcal{P}'$  and  $\mathcal{Q}'$  are complementary subspaces of  $\mathcal{H}'$ .

An element  $a$  of  $\mathcal{P}'$  is of the form  $Ta$  for an element  $a$  of  $\mathcal{P}$  such that

$$\langle Ta, Ta \rangle_{\mathcal{P}'} = \langle a, a \rangle_{\mathcal{P}}.$$

An element  $b$  of  $\mathcal{Q}'$  is of the form  $Tb$  for an element  $b$  of  $\mathcal{Q}$  such that

$$\langle Tb, Tb \rangle_{\mathcal{Q}'} = \langle b, b \rangle_{\mathcal{Q}}.$$

The element  $c = a + b$  of  $\mathcal{H}$  satisfies the inequalities

$$\langle c, c \rangle_{\mathcal{H}} \leq \langle a, a \rangle_{\mathcal{P}} + \langle b, b \rangle_{\mathcal{Q}}$$

and

$$\langle Tc, Tc \rangle_{\mathcal{H}'} \leq \langle c, c \rangle_{\mathcal{H}}.$$

The element  $Tc = Ta + Tb$  of  $\mathcal{H}'$  satisfies the inequality

$$\langle Tc, Tc \rangle_{\mathcal{H}'} \leq \langle Ta, Ta \rangle_{\mathcal{P}'} + \langle Tb, Tb \rangle_{\mathcal{Q}'}$$

An element of  $\mathcal{H}'$  is of the form  $Tc$  for an element  $c$  of  $\mathcal{H}$  such that

$$\langle Tc, Tc \rangle_{\mathcal{H}'} = \langle c, c \rangle_{\mathcal{H}}.$$

An element  $a$  of  $\mathcal{P}$  and an element  $b$  of  $\mathcal{Q}$  exist such that  $c = a + b$  and

$$\langle c, c \rangle_{\mathcal{H}} = \langle a, a \rangle_{\mathcal{P}} + \langle b, b \rangle_{\mathcal{Q}}.$$

Since the element  $Ta$  of  $\mathcal{P}'$  satisfies the inequality

$$\langle Ta, Tb \rangle_{\mathcal{P}'} \leq \langle a, a \rangle_{\mathcal{P}}$$

and since the element  $Tb$  of  $\mathcal{Q}'$  satisfies the inequality

$$\langle Tb, Tb \rangle_{\mathcal{Q}'} \leq \langle b, b \rangle_{\mathcal{Q}},$$

the element  $Tc$  of  $\mathcal{H}$  satisfies the inequality

$$\langle Tc, Tc \rangle_{\mathcal{H}'} \geq \langle Ta, Ta \rangle_{\mathcal{P}'} + \langle Tb, Tb \rangle_{\mathcal{Q}'}$$

Equality holds since the reverse inequality is satisfied.

This completes the proof of the theorem.

A canonical coisometric linear system whose state space is a Hilbert space is constructed when multiplication by  $W(z)$  is a contractive transformation in  $\mathcal{C}(z)$ . The range of multiplication by  $W(z)$  in  $\mathcal{C}(z)$  is a Hilbert space which is contained contractively in  $\mathcal{C}(z)$  when considered with the unique scalar product such that multiplication by  $W(z)$  acts as a partially isometric transformation of  $\mathcal{C}(z)$  onto the range. The complementary space in  $\mathcal{C}(z)$  to the range is the state space  $\mathcal{H}(W)$  of a canonical coisometric linear system with transfer function  $W(z)$ . Every Hilbert space which is the state space of a canonical coisometric linear system is so obtained.

A spectral subspace of contractivity is constructed for a closed relation  $T$  whose domain is contained in a Hilbert space  $\mathcal{P}$  and whose range is contained in a Hilbert space  $\mathcal{Q}$ . The relation  $T$  is then the adjoint of the adjoint relation  $T^*$  which has its domain contained in the Hilbert space  $\mathcal{Q}$  and its range contained in the Hilbert space  $\mathcal{P}$ . A self-adjoint relation  $H$  in the Cartesian product Hilbert space  $\mathcal{P} \times \mathcal{Q}$  is defined by taking  $(a, b)$  into  $(T^*b, Ta)$  when  $a$  is in the domain of  $T$  and  $b$  is in the domain of  $T^*$ . The spectral subspace of contractivity for  $H$  is a closed subspace of the Cartesian product such that  $H$  acts as a

contractive transformation of the subspace into itself. The orthogonal complement is a closed subspace such that the inverse of  $H$  acts as a contractive transformation of the subspace into itself. Eigenvectors of  $H$  for eigenvalues of absolute value one belong to the spectral subspace of contractivity for  $H$ . The square of  $H$  is a self-adjoint relation in the Cartesian product space which has the same spectral subspace of contractivity. Since the transformation which takes  $(a, b)$  into  $(a, -b)$  commutes with the square of  $H$ , the spectral subspace of contractivity for  $H$  is the Cartesian product of a closed subspace of  $\mathcal{P}$  and a closed subspace of  $\mathcal{Q}$ . The spectral subspace of contractivity for  $T$  is the closed subspace of  $\mathcal{P}$ . The spectral subspace of contractivity for  $T^*$  is the closed subspace of  $\mathcal{Q}$ . The relation  $T$  acts as a contractive transformation of the spectral subspace of contractivity for  $T$  into the spectral subspace of contractivity for  $T^*$ . The relation  $T^*$  acts as a contractive transformation of the spectral subspace of contractivity for  $T^*$  into the spectral subspace of contractivity for  $T$ . The inverse of  $T$  acts as a contractive transformation of the orthogonal complement of the spectral subspace of contractivity for  $T^*$  into the orthogonal complement of the spectral subspace of contractivity for  $T$ . The inverse of  $T^*$  acts as a contractive transformation of the orthogonal complement of the spectral subspace of contractivity for  $T$  into the orthogonal complement of the spectral subspace of contractivity for  $T^*$ . If  $a$  is an element of  $\mathcal{P}$  and if  $b$  is an element of  $\mathcal{Q}$  such that the identities

$$Ta = b$$

and

$$T^*b = a$$

are satisfied, then  $a$  belongs to the spectral subspace of contractivity for  $T$  and  $b$  belongs to the spectral subspace of contractivity for  $T^*$ .

A Herglotz space is associated with the transfer function  $W(z)$  of a canonical coisometric linear system whose state space is a Hilbert space. Since multiplication by  $W(z)$  is an everywhere defined and contractive transformation in  $\mathcal{C}(z)$ , the adjoint of multiplication by  $W(z)$  is an everywhere defined and contractive transformation in  $\mathcal{C}(z)$ . The range of the adjoint is a Hilbert space which is contained contractively in  $\mathcal{C}(z)$  when it is considered with the scalar product such that the adjoint acts as a partially isometric transformation of  $\mathcal{C}(z)$  onto the range. The space is a Herglotz space. The space  $\mathcal{C}(z)$  is a Herglotz space  $\mathcal{L}(1)$  whose Herglotz function is identically one. The complementary space to the range Herglotz space is a Herglotz space whose Herglotz function  $\phi(z)$  is determined within an added constant, which is a skew-conjugate operator, by the identity

$$\phi(z) + \phi^*(z^{-1}) = 2 - 2W^*(z^{-1})W(z).$$

The space  $\mathcal{L}(\phi)$  is the set of elements  $f(z)$  of  $\mathcal{C}(z)$  such that  $W(z)f(z)$  belongs to the space  $\mathcal{H}(W)$ . The scalar product in the space  $\mathcal{L}(\phi)$  is determined by the identity

$$\langle f(z), f(z) \rangle_{\mathcal{L}(\phi)} = \langle f(z), f(z) \rangle_{\mathcal{C}(z)} + \langle W(z)f(z), W(z)f(z) \rangle_{\mathcal{H}(W)}.$$

The adjoint of multiplication by  $W(z)$  as a transformation in  $\mathcal{C}(z)$  acts as a partially isometric transformation of  $\mathcal{C}(z)$  onto the complementary space  $\mathcal{L}(1 - \phi)$  to  $\mathcal{L}(\phi)$  in  $\mathcal{C}(z)$ .

Since the polynomial elements of  $\mathcal{C}(z)$  are dense in  $\mathcal{C}(z)$ , the polynomial elements of the space  $\mathcal{L}(1 - \phi)$  are dense in the space  $\mathcal{L}(1 - \phi)$ .

If a Herglotz space  $\mathcal{L}$  is contained contractively in  $\mathcal{C}(z)$  and if the polynomial elements of the space are dense in the space, then a partially isometric transformation of  $\mathcal{C}(z)$  onto  $\mathcal{L}$  exists which commutes with the difference-quotient transformation and whose kernel is invariant under multiplication by  $z$ . The resulting contractive transformation of  $\mathcal{C}(z)$  into itself coincides with the adjoint of multiplication by  $V(z)$  for a power series  $V(z)$  with complex coefficients and that multiplication by  $V(z)$  is an everywhere defined and contractive transformation in  $\mathcal{C}(z)$  and such that the orthogonal complement of the range of multiplication by  $V(z)$  in  $\mathcal{C}(z)$  is invariant under multiplication by  $z$ .

The construction of  $V(z)$  is supplied when  $W(z)$  is a power series such that multiplication by  $W(z)$  is an everywhere defined and contractive transformation in  $\mathcal{C}(z)$ . The associated Herglotz space  $\mathcal{L}(\phi)$  contains the elements  $f(z)$  of  $\mathcal{C}(z)$  such that  $W(z)f(z)$  belongs to the space  $\mathcal{H}(W)$ . The identity

$$\|f(z)\|_{\mathcal{L}(\phi)}^2 = \|f(z)\|_{\mathcal{C}(z)}^2 + \|W(z)f(z)\|_{\mathcal{H}(W)}^2$$

holds for every element  $f(z)$  of the space  $\mathcal{L}(\phi)$ . The space  $\mathcal{L}(\phi)$  is contained contractively in  $\mathcal{C}(z)$ . The complementary space to  $\mathcal{L}(\phi)$  in  $\mathcal{C}(z)$  is a Herglotz space  $\mathcal{L}(1 - \phi)$ , which is contained contractively in  $\mathcal{C}(z)$ , such that the adjoint of multiplication by  $W(z)$  in  $\mathcal{C}(z)$  acts as a partially isometric transformation of  $\mathcal{C}(z)$  onto the space  $\mathcal{L}(1 - \phi)$ . The polynomial elements of the space  $\mathcal{L}(1 - \phi)$  are dense in the space. A power series  $V(z)$ , such that multiplication by  $V(z)$  is an everywhere defined and contractive transformation in  $\mathcal{C}(z)$ , exists such that the adjoint of multiplication by  $V(z)$  as a transformation in  $\mathcal{C}(z)$  acts as a partially isometric transformation of  $\mathcal{C}(z)$  onto the space  $\mathcal{L}(1 - \phi)$  and such that the kernel of the adjoint transformation is invariant under multiplication by  $z$ . A power series  $U(z)$  exists such that multiplication by  $U(z)$  is an everywhere defined and contractive transformation in  $\mathcal{C}(z)$ , such that the adjoint of multiplication by  $W(z)$  in  $\mathcal{C}(z)$  is the composition of the adjoint of multiplication by  $U(z)$  in  $\mathcal{C}(z)$  and the adjoint of multiplication by  $V(z)$  in  $\mathcal{C}(z)$ , and such that the range of the adjoint of multiplication by  $U(z)$  in  $\mathcal{C}(z)$  is orthogonal to the kernel of the adjoint of multiplication by  $V(z)$  in  $\mathcal{C}(z)$ .

The Beurling factorization

$$W(z) = V(z)U(z)$$

results of a power series  $W(z)$  such that multiplication by  $W(z)$  is an everywhere defined and contractive transformation in  $\mathcal{C}(z)$ . The outer function  $V(z)$  is a power series such that multiplication by  $V(z)$  is an everywhere defined and contractive transformation in  $\mathcal{C}(z)$  with range dense in  $\mathcal{C}(z)$ . The inner function  $U(z)$  is a power series such that multiplication by  $U(z)$  is a partially isometric transformation in  $\mathcal{C}(z)$  and such that the range of multiplication by  $U(z)$  in  $\mathcal{C}(z)$  is orthogonal to the kernel of multiplication by  $V(z)$ .

The Nevanlinna factorization of a power series  $W(z)$ , such that multiplication by  $W(z)$  is densely defined as a transformation in  $\mathcal{C}(z)$ , is a variant of the Beurling factorization. An outer function is again a power series  $V(z)$  such that multiplication by  $V(z)$  is an everywhere defined and contractive transformation in  $\mathcal{C}(z)$  with range dense in  $\mathcal{C}(z)$ .

**Theorem 6.** *If  $W(z)$  is a power series such that multiplication by  $W(z)$  is densely defined as a transformation in  $\mathcal{C}(z)$ , then an outer function  $V(z)$  exists such that multiplication by*

$$U(z) = W(z)V(z)$$

*is an everywhere defined and contractive transformation in  $\mathcal{C}(z)$  and such that no nonzero element  $f(z)$  of the space  $\mathcal{H}(V)$  exists such that  $W(z)f(z)$  belongs to the space  $\mathcal{H}(U)$ .*

*Proof of Theorem 6.* Multiplication by  $W(z)$  is extended to a transformation with domain and range in  $\text{ext } \mathcal{C}(z)$  which commutes with multiplication by  $z$ . An element  $f(z)$  of the space  $\text{ext } \mathcal{C}(z)$  belongs to the domain of multiplication by  $W(z)$  as transformation in  $\text{ext } \mathcal{C}(z)$  if  $z^r f(z)$  belongs to the domain of multiplication by  $W(z)$  in  $\mathcal{C}(z)$  for some nonnegative integer  $r$ . Multiplication by  $W(z)$  in  $\text{ext } \mathcal{C}(z)$  takes  $z^r f(z)$  into  $z^r g(z)$  when multiplication by  $W(z)$  in  $\mathcal{C}(z)$  takes  $f(z)$  into  $g(z)$ . The definition is independent of  $r$ . Multiplication by  $W(z)$  in  $\text{ext } \mathcal{C}(z)$  is a densely defined transformation in  $\text{ext } \mathcal{C}(z)$ , whose closure is however not assumed to be a transformation. The adjoint of multiplication by  $W(z)$  as a transformation in  $\text{ext } \mathcal{C}(z)$  is a transformation. The spectral subspace of contractivity for the adjoint of multiplication by  $W(z)$  in  $\text{ext } \mathcal{C}(z)$  and its orthogonal complement are invariant subspaces for multiplication and division by  $z$ .

The space  $\text{ext } \mathcal{C}(z)$  is contained contractively in a Hilbert space  $\text{ext } \mathcal{P}$  such that a dense set of elements of the complementary space to  $\text{ext } \mathcal{C}(z)$  in  $\text{ext } \mathcal{P}$  belong to  $\text{ext } \mathcal{C}(z)$  and such that the adjoint of multiplication by  $W(z)$  as a transformation in  $\text{ext } \mathcal{C}(z)$  maps the intersection of its domain with the orthogonal complement of its spectral subspace of contractivity onto the intersection of  $\text{ext } \mathcal{C}(z)$  with the complementary space to  $\text{ext } \mathcal{C}(z)$  in  $\text{ext } \mathcal{P}$ . The intersection of  $\text{ext } \mathcal{C}(z)$  with the complementary space to  $\text{ext } \mathcal{C}(z)$  in  $\text{ext } \mathcal{P}$  is invariant under division by  $z$ . Division by  $z$  is an isometric transformation with respect to the scalar product of  $\mathcal{P}$  as well as with respect to the scalar product of the complementary space to  $\text{ext } \mathcal{C}(z)$  in  $\text{ext } \mathcal{P}$ .

The canonical projection of  $\text{ext } \mathcal{C}(z)$  onto  $\mathcal{C}(z)$  determines a partially isometric transformation of  $\text{ext } \mathcal{P}$  onto a Hilbert space  $\mathcal{P}$ , a dense set of whose elements belong to  $\mathcal{C}(z)$ . The space  $\mathcal{C}(z)$  is contained contractively in the space  $\mathcal{P}$ . The partially isometric transformation of  $\text{ext } \mathcal{P}$  onto  $\mathcal{P}$  acts as a partially isometric transformation of the complementary space to  $\text{ext } \mathcal{C}(z)$  in  $\text{ext } \mathcal{P}$  onto the complementary space to  $\mathcal{C}(z)$  in  $\mathcal{P}$ . The intersection of  $\mathcal{C}(z)$  with  $\mathcal{P}$  and the intersection of  $\mathcal{C}(z)$  with the complementary space to  $\mathcal{C}(z)$  in  $\mathcal{P}$  are invariant subspaces for the difference-quotient transformation. The continuous extension of the difference-quotient transformation has an isometric adjoint in  $\mathcal{P}$  as well as in the complementary space to  $\mathcal{C}(z)$  in  $\mathcal{P}$ .

Since the polynomial elements of  $\mathcal{C}(z)$  are dense in  $\mathcal{P}$ , an isometric transformation of  $\mathcal{P}$  onto  $\mathcal{C}(z)$  exists which intertwines the continuous extension of the difference-quotient transformation in  $\mathcal{P}$  with the difference-quotient transformation in  $\mathcal{C}(z)$ . Since  $\mathcal{C}(z)$  is contained contractively in  $\mathcal{P}$ , a contractive transformation of  $\mathcal{C}(z)$  into itself is obtained which commutes with the difference-quotient transformation. The transformation is the adjoint of multiplication by  $V(z)$  for a power series  $V(z)$  such that multiplication by  $V(z)$  is everywhere defined and contractive as a transformation in  $\mathcal{C}(z)$ . A Hilbert space

$\mathcal{H}(V)$  exists which is the state space of a canonical coisometric linear system with transfer function  $V(z)$ . The Herglotz space  $\mathcal{L}(\phi)$  associated with the space  $\mathcal{H}(V)$  is contained contractively in  $\mathcal{C}(z)$ . The continuous extension of the adjoint of multiplication by  $V(z)$  acts as an isometric transformation of  $\mathcal{P}$  onto  $\mathcal{C}(z)$ . The adjoint of multiplication by  $V(z)$  as a transformation in  $\mathcal{C}(z)$  acts as an isometric transformation of  $\mathcal{C}(z)$  onto the complementary space  $\mathcal{L}(1 - \psi)$  to the space  $\mathcal{L}(\psi)$  in  $\mathcal{C}(z)$ . The continuous extension of the adjoint of multiplication by  $V(z)$  as a transformation in  $\mathcal{C}(z)$  acts as an isometric transformation of the complementary space to  $\mathcal{C}(z)$  in  $\mathcal{P}$  onto the space  $\mathcal{L}(\psi)$ .

A Hilbert space ext  $\mathcal{Q}$ , which is contained contractively in the space ext  $\mathcal{P}$ , exists such that the intersection of ext  $\mathcal{Q}$  with ext  $\mathcal{C}(z)$  is the range of the adjoint of multiplication by  $W(z)$  as a transformation in ext  $\mathcal{C}(z)$ . The space ext  $\mathcal{Q}$  is the orthogonal sum of its intersection with the spectral subspace of contractivity for multiplication by  $W(z)$  in ext  $\mathcal{C}(z)$  and the closure of its intersection with the orthogonal complement in ext  $\mathcal{C}(z)$  of the spectral subspace. The complementary space to ext  $\mathcal{C}(z)$  in ext  $\mathcal{Q}$  is isometrically equal to the closure in ext  $\mathcal{Q}$  of its intersection with the orthogonal complement of the spectral subspace. The adjoint of multiplication by  $W(z)$  as a transformation in ext  $\mathcal{C}(z)$  acts as a partially isometric transformation of its spectral subspace of contractivity onto the intersection of ext  $\mathcal{Q}$  with the spectral subspace of contractivity for multiplication by  $W(z)$  as a transformation in ext  $\mathcal{C}(z)$ . The space ext  $\mathcal{Q}$  and its complementary space in the space ext  $\mathcal{P}$  are invariant subspaces for the continuous extension of division by  $z$ . The continuous extension of division by  $z$  is an isometric transformation in ext  $\mathcal{Q}$  and its complementary space in ext  $\mathcal{P}$ .

The partially isometric transformation of ext  $\mathcal{P}$  onto  $\mathcal{P}$ , which is determined by the canonical projection of ext  $\mathcal{C}(z)$  onto  $\mathcal{C}(z)$ , acts as a partially isometric transformation of ext  $\mathcal{Q}$  onto a Hilbert space which is contained contractively in  $\mathcal{P}$ . The canonical projection of ext  $\mathcal{C}(z)$  onto  $\mathcal{C}(z)$  acts as a partially isometric transformation of the complementary space to ext  $\mathcal{Q}$  in ext  $\mathcal{P}$  onto the complementary space to  $\mathcal{Q}$  in  $\mathcal{P}$ . The space  $\mathcal{Q}$  and its complementary space in  $\mathcal{P}$  are invariant subspaces for the continuous extension of the difference-quotient transformation. The continuous extension of the difference-quotient transformation has an isometric adjoint in  $\mathcal{Q}$  and in its complementary space in  $\mathcal{P}$ .

The power series

$$U(z) = W(z)V(z)$$

has properties which are derived from adjoints of multiplication transformations. Since multiplication by  $W(z)$  is a densely defined transformation in  $\mathcal{C}(z)$  by hypothesis, the adjoint of multiplication by  $W(z)$  as a transformation in  $\mathcal{C}(z)$  is contained in the closure of the composition of the adjoint of multiplication by  $W(z)$  as a transformation in ext  $\mathcal{C}(z)$  with the canonical projection of ext  $\mathcal{C}(z)$  onto  $\mathcal{C}(z)$ . The range of the adjoint of multiplication by  $W(z)$  as a transformation in  $\mathcal{C}(z)$  is contained in  $\mathcal{Q}$ . The continuous extension of the adjoint of multiplication by  $W(z)$  as a transformation in  $\mathcal{C}(z)$  is a contractive transformation of  $\mathcal{C}(z)$  into  $\mathcal{Q}$ . The continuous extension of the adjoint of multiplication by  $V(z)$  as a transformation in  $\mathcal{C}(z)$  acts as a contractive transformation of  $\mathcal{Q}$  into  $\mathcal{C}(z)$ . Since the adjoint of multiplication by  $U(z)$  in  $\mathcal{C}(z)$  is contained in the composition of the continuous extension of the adjoint of multiplicative by  $W(z)$  in  $\mathcal{C}(z)$  with the continuous extension of

the adjoint of multiplication by  $V(z)$  in  $\mathcal{C}(z)$  and since  $\mathcal{C}(z)$  is contained contractively in  $\mathcal{Q}$ , the adjoint of multiplication by  $U(z)$  is an everywhere defined and contractive transformation in  $\mathcal{C}(z)$ . The contractive property is first verified on polynomial elements of  $\mathcal{C}(z)$ . It then follows for all elements of  $\mathcal{C}(z)$ . Multiplication by  $U(z)$  is everywhere defined and contractive as a transformation in  $\mathcal{C}(z)$ .

A Hilbert space  $\mathcal{H}(U)$  exists which is the state space of a canonical coisometric linear system with transfer function  $U(z)$ . The adjoint of multiplication by  $U(z)$  as a transformation in  $\mathcal{C}(z)$  acts as a partially isometric transformation of  $\mathcal{C}(z)$  onto the complementary space  $\mathcal{L}(1 - \phi)$  in  $\mathcal{C}(z)$  to the Herglotz space  $\mathcal{L}(\phi)$  associated with the space  $\mathcal{H}(0)$ .

Since the complementary space to  $\mathcal{C}(z)$  in the space  $\mathcal{P}$  is contained isometrically in the space  $\mathcal{Q}$ , no nonzero element of the complementary space to  $\mathcal{C}(z)$  in the space  $\mathcal{P}$  belongs to the complementary space to the space  $\mathcal{Q}$  in the space  $\mathcal{P}$ . Since the continuous extension of the adjoint of multiplication by  $V(z)$  in  $\mathcal{C}(z)$  acts as an isometric transformation of the complementary space to  $\mathcal{C}(z)$  in  $\mathcal{P}$  onto the space  $\mathcal{L}(\psi)$  and of the complementary space to  $\mathcal{Q}$  in  $\mathcal{P}$  onto  $\mathcal{L}(\phi)$ , the intersection of the spaces  $\mathcal{L}(\phi)$  and  $\mathcal{L}(\psi)$  contains no nonzero element.

The space  $\mathcal{H}(V)$  is the set of elements  $f(z)$  of  $\mathcal{C}(z)$  such that the adjoint of multiplication by  $V(z)$  in  $\mathcal{C}(z)$  maps  $f(z)$  into an element  $g(z)$  of the space  $\mathcal{L}(\psi)$ . The identity

$$\|f(z)\|_{\mathcal{H}(V)}^2 = \|f(z)\|_{\mathcal{C}(z)}^2 + \|g(z)\|_{\mathcal{L}(\psi)}^2$$

is then satisfied. Since the continuous extension of the adjoint of multiplication by  $V(z)$  as a transformation in  $\mathcal{C}(z)$  acts as an isometric transformation of the complementary space to  $\mathcal{C}(z)$  in the space  $\mathcal{P}$  onto the space  $\mathcal{L}(\psi)$ , the space  $\mathcal{H}(V)$  is the intersection of  $\mathcal{C}(z)$  with the complementary space to  $\mathcal{C}(z)$  in the space  $\mathcal{P}$ . The square of the norm of an element of the space  $\mathcal{H}(V)$  is the sum of the square of its norm as an element of  $\mathcal{C}(z)$  and the square of its norm as an element of the complementary space to  $\mathcal{C}(z)$  in the space  $\mathcal{P}$ .

The space  $\mathcal{H}(U)$  is the set of element  $f(z)$  of  $\mathcal{C}(z)$  such that the adjoint of multiplication by  $U(z)$  in  $\mathcal{C}(z)$  maps  $f(z)$  into an element  $g(z)$  of the space  $\mathcal{L}(\phi)$ . The identity

$$\|f(z)\|_{\mathcal{H}(U)}^2 = \|f(z)\|_{\mathcal{C}(z)}^2 + \|g(z)\|_{\mathcal{L}(\phi)}^2$$

is then satisfied. Since the continuous extension of the adjoint of multiplication by  $V(z)$  as a transformation in  $\mathcal{C}(z)$  acts as an isometric transformation of the complementary space to  $\mathcal{Q}$  in  $\mathcal{P}$  onto the space  $\mathcal{L}(\phi)$ , the space  $\mathcal{H}(U)$  is the set of elements  $f(z)$  of  $\mathcal{C}(z)$  such that the adjoint of multiplication by  $W(z)$  in  $\mathcal{C}(z)$  maps  $f(z)$  into an element  $h(z)$  of the complementary space to  $\mathcal{Q}$  in  $\mathcal{P}$ . Since  $h(z)$  then belongs to the spectral subspace of contractivity for multiplication by  $W(z)$  as a transformation in  $\text{ext } \mathcal{C}(z)$ , the element  $f(z)$  of  $\mathcal{C}(z)$  is the projection of an element of  $\text{ext } \mathcal{C}(z)$  which belongs to the spectral subspace of contractivity for the adjoint of multiplication by  $W(z)$  as a transformation in  $\text{ext } \mathcal{C}(z)$ .

If  $f(z)$  is an element of the space  $\mathcal{H}(V)$  such that  $W(z)f(z)$  belongs to the space  $\mathcal{H}(U)$ , then  $W(z)f(z)$  belongs to the orthogonal complement of the spectral subspace of contractivity for multiplication by  $W(z)$  as a transformation in  $\text{ext } \mathcal{C}(z)$  since  $f(z)$  belongs

to the orthogonal complement of the spectral subspace of contractivity for multiplication by  $W(z)$  as a transformation in  $\text{ext } \mathcal{C}(z)$ . An element  $g(z)$  of the spectral subspace of contractivity for the adjoint of multiplication by  $W(z)$  as a transformation in  $\text{ext } \mathcal{C}(z)$  exists which has  $W(z)f(z)$  as its orthogonal projection in  $\mathcal{C}(z)$ . The product  $W(z)f(z)$  is equal to zero since it is orthogonal to  $g(z)$ . The element  $f(z)$  of  $\mathcal{C}(z)$  is equal to zero since it is orthogonal to the spectral subspace of contractivity for multiplication by  $W(z)$  as a transformation in  $\text{ext } \mathcal{C}(z)$ .

This completes the proof of the theorem.

The Hilbert space  $\mathcal{H}(U)$  is contained continuously and isometrically in a Krein space  $\mathcal{H}(W)$ , whose elements are power series, such that multiplication by  $W(z)$  acts as an isometric transformation of the anti-space of the Hilbert space  $\mathcal{H}(V)$  onto the orthogonal complement of the space  $\mathcal{H}(U)$  in the space  $\mathcal{H}(W)$ . The space  $\mathcal{H}(U')$  corresponding to the power series

$$U'(z) = zU(z)$$

with complex coefficients is the set of power series  $f(z)$  with complex coefficients such that  $[f(z) - f(0)]/z$  belongs to the space  $\mathcal{H}(U)$ . The identity for difference quotients

$$\|[f(z) - f(0)]/z\|_{\mathcal{H}(U)}^2 = \|f(z)\|_{\mathcal{H}(U')}^2 - f(0)^- f(0)$$

is satisfied. The space  $\mathcal{H}(U)$  is contained contractively in the space  $\mathcal{H}(U')$ . The space  $\mathcal{H}(W')$  corresponding to the power series

$$W'(z) = zW(z)$$

with complex coefficients is the set of power series  $f(z)$  with complex coefficients such that  $[f(z) - f(0)]/z$  belongs to the space  $\mathcal{H}(W)$ . The identity for difference quotients

$$\langle [f(z) - f(0)]/z, [f(z) - f(0)]/z \rangle_{\mathcal{H}(W)} = \langle f(z), f(z) \rangle_{\mathcal{H}(W')} - f(0)^- f(0)$$

is satisfied. The space  $\mathcal{H}(U')$  is contained continuously and isometrically in the space  $\mathcal{H}(W')$ . Multiplication by  $W'(z)$  is an isometric transformation of the anti-space of the Hilbert space  $\mathcal{H}(V)$  onto the orthogonal complement of the space  $\mathcal{H}(U')$  in the space  $\mathcal{H}(W')$ . The space  $\mathcal{H}(W)$  is contained continuously and contractively in the space  $\mathcal{H}(W')$ . Multiplication by  $W(z)$  is a partially isometric transformation of the space of complex numbers onto the complementary space to the space  $\mathcal{H}(W)$  in the space  $\mathcal{H}(W')$ . The space  $\mathcal{H}(W)$  is the state space of a canonical coisometric linear system with transfer function  $W(z)$ .

A canonical unitary linear is constructed from a canonical coisometric linear system with state space  $\mathcal{H}(W)$  and transfer function  $W(z)$  when the inequality for difference quotients

$$\langle [f(z) - f(0)]/z, [f(z) - f(0)]/z \rangle_{\mathcal{H}(W)} \leq \langle f(z), f(z) \rangle_{\mathcal{H}(W)} - f(0)^- f(0)$$

holds for every element  $f(z)$  of the space. The inequality is satisfied by the canonical coisometric linear system constructed when multiplication by  $W(z)$  is densely defined as

a transformation in  $\mathcal{C}(z)$ . The elements of the state space  $\mathcal{D}(W)$  of the canonical unitary linear system are pairs  $(f(z), g(z))$  of power series. Power series  $f(z)$  and

$$g(z) = \sum a_n z^n$$

with complex coefficients determine an element of the space  $\mathcal{D}(W)$  if  $f(z)$  is an element of the space  $\mathcal{H}(W)$  such that

$$z^{r+1}f(z) - W(z)(a_0 z^r + \dots + a_r)$$

belongs to the space  $\mathcal{H}(W)$  for every nonnegative integer  $r$  and such that the sequence of numbers

$$\begin{aligned} &\langle z^{r+1}f(z) - W(z)(a_0 z^r + \dots + a_r), z^{r+1}f(z) - W(z)(a_0 z^r + \dots + a_r) \rangle_{\mathcal{H}(W)} \\ &+ a_0^- a_0 + \dots + a_r^- a_r \end{aligned}$$

is bounded. The inequality for difference quotients in the space  $\mathcal{H}(W)$  implies that the sequence is nondecreasing. The limit of the sequence is taken as the definition of the scalar self-product

$$\langle f(z), g(z) \rangle, \langle f(z), g(z) \rangle_{\mathcal{D}(W)}.$$

The space  $\mathcal{D}(W)$  is a Krein space. A contractive partially isometric transformation of the space  $\mathcal{D}(W)$  onto the space  $\mathcal{H}(W)$  is defined by taking  $(f(z), g(z))$  into  $f(z)$ . A continuous transformation of the space  $\mathcal{D}(W)$  into itself is defined by taking  $(f(z), g(z))$  into

$$([f(z) - f(0)]/z, zg(z) - W^*(z)f(0)).$$

The identity for difference quotients

$$\begin{aligned} &\langle ([f(z) - f(0)]/z, zg(z) - W^*(z)f(0)), ([f(z) - f(0)]/z, zg(z) - W^*(z)f(z)) \rangle_{\mathcal{D}(W)} \\ &= \langle (f(z), g(z)), (f(z), g(z)) \rangle_{\mathcal{D}(W)} - f(0)^- f(0) \end{aligned}$$

is satisfied. The adjoint transformation of the space  $\mathcal{D}(W)$  into itself takes  $(f(z), g(z))$  into

$$(zf(z) - W(z)g(0), [g(z) - g(0)]/z).$$

The identity for difference quotients

$$\begin{aligned} &\langle (zf(z) - W(z)g(0), [g(z) - g(0)]/z), (zf(z) - W(z)g(0), [g(z) - g(0)]/z) \rangle_{\mathcal{D}(W)} \\ &= \langle (f(z), g(z)), (f(z), g(z)) \rangle_{\mathcal{D}(W)} - g(0)^- g(0) \end{aligned}$$

is satisfied.

A construction has been made of the state space  $\mathcal{D}(W)$  of a canonical unitary linear system with transfer function  $W(z)$ . The main transformation takes  $(f(z), g(z))$  into

$$([f(z) - f(0)]/z, zg(z) - W^*(z)f(0)).$$

The input transformation takes  $c$  into

$$([W(z) - W(0)]c/z, [1 - W^*(z)W(0)]c).$$

The output transformation takes  $(f(z), g(z))$  into  $f(0)$ . The external operator is  $W(0)$ . The unitary property of the linear system is a consequence of the two identities for difference quotients. The transformation which takes  $(f(z), g(z))$  into  $(g(z), f(z))$  maps the space  $\mathcal{D}(W)$  isometrically onto the state space  $\mathcal{D}(W^*)$  of a canonical unitary linear system with transfer function  $W^*(z)$ .

Uniqueness of a canonical unitary linear system with transfer function  $W(z)$  is obtained when multiplication by  $W(z)$  is densely defined as a transformation in  $\mathcal{C}(z)$ .

**Theorem 7.** *If  $W(z)$  is a power series such that multiplication by  $W(z)$  is densely defined as a transformation in  $\mathcal{C}(z)$ , if  $V(z)$  and*

$$U(z) = W(z)V(z)$$

*are power series such that multiplication by  $U(z)$  and multiplication by  $V(z)$  are everywhere defined and contractive as transformations in  $\mathcal{C}(z)$ , if no nonzero element  $f(z)$  of the space  $\mathcal{H}(V)$  exists such that  $W(z)f(z)$  belongs to the space  $\mathcal{H}(U)$ , and if  $\mathcal{D}(W)$  is the state space of a canonical unitary linear system with transfer function  $W(z)$ , then a contractive partially isometric transformation of the space  $\mathcal{D}(W)$  onto a Krein space  $\mathcal{H}(W)$ , which is the state space of a canonical coisometric linear system with transfer function  $W(z)$ , is defined by taking  $(f(z), g(z))$  into  $f(z)$ . The space  $\mathcal{H}(U)$  is contained continuously and isometrically in the space  $\mathcal{H}(W)$ . Multiplication by  $W(z)$  is a partially isometric transformation of the anti-space of the space  $\mathcal{H}(V)$  onto the orthogonal complement of the space  $\mathcal{H}(U)$  in the space  $\mathcal{H}(W)$ .*

*Proof of Theorem 7.* A transformation of the Cartesian product of the space  $\mathcal{D}(W)$  and the space  $\mathcal{D}(V)$  onto a vector space  $\mathcal{D}$ , whose elements are pairs of power series with complex coefficients, is defined by taking an element  $(f(z), g(z))$  of the space  $\mathcal{D}(W)$  and an element  $(h(z), k(z))$  of the space  $\mathcal{D}(V)$  into the element  $(u(z), v(z))$  of the space  $\mathcal{D}$  defined by

$$u(z) = f(z) + W(z)h(z)$$

and

$$v(z) = k(z) + V^*(z)g(z).$$

The space  $\mathcal{D}$  is the state space of a linear system with transfer function  $U(z)$ . The main transformation takes  $(u(z), v(z))$  into

$$([u(z) - u(0)]/z, zv(z) - U^*(z)u(0)).$$

If

$$u(z) = f(z) + W(z)h(z)$$

and

$$v(z) = k(z) + V^*(z)g(z),$$

then

$$[u(z) - u(0)]/z = f'(z) + W(z)h'(z)$$

and

$$zv(z) - U^*(z)u(0) = k'(z) + V^*(z)g'(z)$$

with  $(f'(z), g'(z))$  the element of the space  $\mathcal{D}(W)$  defined by

$$f'(z) = [f(z) - f(0)]/z + [W(z) - W(0)]h(0)/z$$

and

$$g'(z) = zg(z) - W^*(z)h(0) + [1 - W^*(z)W(0)]h(0)$$

and with  $(h'(z), k'(z))$  the element of the space  $\mathcal{D}(V)$  defined by

$$h'(z) = [h(z) - h(0)]/z$$

and

$$k'(z) = zk(z) - V^*(z)h(0).$$

The input transformation takes a complex number  $c$  into

$$([1 - U(z)U(0)^-]c, [U^*(z) - U^*(0)]c/z)$$

where

$$[1 - U(z)U(0)^-]c = [1 - W(z)W(0)^-]c + W(z)[1 - V(z)V(0)^-]W(0)^-c$$

and

$$[U^*(z) - U^*(0)]c/z = [V^*(z) - V^*(0)]W^*(0)c/z + V^*(z)[U^*(z) - U^*(0)]c/z$$

with

$$([1 - W(z)W(0)^-]c, [W^*(z) - W^*(0)]c/z)$$

an element of the space  $\mathcal{D}(W)$  and

$$([1 - V(z)V(0)^-]W(0)^-c, [V^*(z) - V^*(0)]W^*(0)c/z)$$

an element of the space  $\mathcal{D}(W)$ . The output transformation takes  $(u(z), v(z))$  into  $u(0)$ . If

$$u(z) = f(z) + W(z)h(z)$$

and

$$v(z) = k(z) + V^*(z)g(z).$$

The Bieberbach conjecture is an estimate of coefficients of a power series

$$W(z) = W_1z + W_2z^2 + \dots$$

with constant coefficient zero which represents an injective mapping of the unit disk. The inequality

$$|W_n| \leq n|W_1|$$

is conjectured for every positive integer  $n$ . The proof of the conjecture applies the Löwner parametrization of Riemann mapping functions.

The members of a Löwner family are power series  $W(t, z)$  in  $z$  with constant coefficient zero which represent injective mappings of the unit disk. The members of the family are parametrized by the positive numbers  $t$ . The region onto which  $W(a, z)$  maps the unit disk is contained in the region onto which  $W(b, z)$  maps the unit disk when  $a$  is less than or equal to  $b$ . The identity

$$W(a, z) = W(b, W(b, a, z))$$

then holds for a power series  $W(b, a, z)$  with constant coefficient zero and with coefficient of  $z$  equal to  $a/b$  which represents an injective mapping of the unit disk into itself.

An essential case of a Löwner family occurs when the identity

$$W(t, z) = tW(z)$$

holds for every positive number  $t$ . The region onto which  $W(z)$  maps the unit disk is then star-like with respect to the origin. A simply connected region which contains the origin and which is not the full plane is shown to be the image of the unit disk under a member of a Löwner family. A region which is star-like with respect to the origin is simply connected. The construction of Riemann mapping functions for star-like regions prepares the construction of Riemann mapping functions for simply connected regions. Estimates of coefficients are obtained in the star-like case which permit the transition to general mapping functions.

When the identity

$$W(t, z) = tW(z)$$

is satisfied, the identity

$$W(a, z) = W(b, W(b, a, z))$$

when  $a$  is less than or equal to  $b$  reads

$$tW(z) = W(W(1, t, z))$$

when  $t$  is less than or equal to one. The identity mapping function  $z$  is approximated by the mapping function

$$W(1, t, z)$$

when  $t$  is close to one. The ratio

$$\frac{z - W(1, t, z)}{z(1 - t)}$$

has nonnegative real part in the limit as  $t$  increases to one. The limit of the ratio is an analytic function  $\phi(z)$  of  $z$  with constant coefficient one which has nonnegative real part in the unit disk. The differential equation

$$W(z) = \phi(z)zW'(z)$$

is satisfied. The equation admits a solution  $W(z)$  which is a power series with constant coefficient zero and coefficient of  $z$  equal to one which converges in the unit disk.

### CHAPTER 3. FOUNDATIONS OF ANALYSIS

The cardinality of set  $A$  is said to be less than or equal to the cardinality of set  $B$  if an injective transformation of set  $A$  into set  $B$  exists. If the cardinality of set  $A$  is less than or equal to the cardinality of set  $B$  and if the cardinality of set  $B$  is less than or equal to the cardinality of set  $A$ , then an injective transformation exists of set  $A$  onto set  $B$ . Sets  $A$  and  $B$  are said to have the same cardinality. The cardinality of set  $A$  is said to be less than the cardinality of set  $B$  if  $A$  and  $B$  are sets of unequal cardinality such that the cardinality of set  $A$  is less than or equal to the cardinality of set  $B$ .

Experience with finite sets creates the expectation that any two sets are comparable in cardinality. If  $A$  and  $B$  are sets of unequal cardinality, then either the cardinality of set  $A$  is less than the cardinality of set  $B$  or the cardinality of set  $B$  is less than the cardinality of set  $A$ . The desired conclusion, or its equivalent, is accepted as a hypothesis in the axiomatic definition of sets.

The axiom of choice is the most plausible of the hypotheses which are equivalent to the desired comparability of cardinalities of sets. If a transformation  $T$  takes set  $A$  onto set  $B$ , then a transformation  $S$  of set  $B$  into set  $A$  exists such that the composed transformation  $TS$  is the inclusion transformation of set  $B$  in itself.

The axiom of choice displaces the previously favored hypothesis which is equivalent to the comparability of cardinalities of sets. A partial ordering of a set  $S$  is determined by distinguished pairs  $(a, b)$  of elements  $a$  and  $b$  of  $S$ . The inequality  $a \leq b$  is written when  $(a, b)$  is a distinguished pair. It is assumed that the inequality  $a \leq c$  holds whenever  $a$  and  $c$  are elements of the set for which the inequalities  $a \leq b$  and  $b \leq c$  hold for some element  $b$  of the set. The inequality  $c \leq c$  is assumed for every element  $c$  of the set. Elements  $a$  and  $b$  of the set are assumed to be equal if the inequalities  $a \leq b$  and  $b \leq a$  are satisfied. A set is said to be well-ordered if every nonempty subset contains a least element. An equivalent of the axiom of choice is the hypothesis that every set admits a well-ordering.

The Kuratowski-Zorn lemma is a flexible reformulation of the principle of induction implicit in well-ordering. A partially ordered set admits a maximal element if every well-ordered subset has an upper bound in the set.

The proof of the Kuratowski-Zorn lemma from the axiom of choice is an application of induction. Assume that  $S$  is a partially ordered set in which every well-ordered subset has an upper bound. An augmentation of a well-ordered subset  $A$  is a well-ordered subset  $B$  whose elements are the elements of  $A$  and some upper bound of  $A$  which does not

belong to  $A$ . The axiom of choice is applied to a set whose elements are the pairs  $(A, B)$  consisting of an augmentable well-ordered subset  $A$  and an augmentation  $B$  of  $A$ . The set is mapped onto the set of augmentable well-ordered subsets by taking  $(A, B)$  into  $A$ . The axiom of choice asserts the existence of a transformation which takes every augmentable well-ordered subset  $A$  into an augmentation  $(A, A')$  of  $A$ .

The proof of the Kuratowski–Zorn lemma is facilitated by the introduction of notation. A ladder is well-ordered subset  $A$  which is constructed by the chosen augmentation procedure. For every element  $b$  of  $A$  the augmentation of the set of elements of  $A$  which are less than  $b$  is the set of elements of  $A$  which are less than or equal to  $b$ . The intersection of ladders  $A$  and  $B$  is a ladder which is either equal to  $A$  or equal to  $B$ . If  $A$  and  $B$  are ladders, then either  $A$  is contained in  $B$  or  $B$  is contained in  $A$ . The union of all ladders is a ladder which contains every ladder. Since the greatest ladder is assumed to have an upper bound, it has a greatest element. The greatest element of the greatest ladder is a maximal element of the given partially ordered set  $S$ .

Cardinal numbers are constructed by a theorem of Cantor which states that no transformation maps a set onto the class of all its subsets. If a transformation  $T$  maps a set  $\mathcal{S}$  into the subsets of  $\mathcal{S}$ , then a subset  $\mathcal{S}_\infty$  of  $\mathcal{S}$  is constructed which does not belong to the range of  $T$ . The set  $\mathcal{S}_\infty$  is the set of elements  $s$  of  $\mathcal{S}$  for which no elements  $s_n$  of  $\mathcal{S}$  can be chosen for every nonnegative integer  $n$  so that  $s_0$  is equal to  $s$  and so that  $s_n$  belongs to  $Ts_{n-1}$  when  $n$  is positive. An element  $s$  of  $\mathcal{S}$  belongs to  $\mathcal{S}_\infty$  if, and only if,  $Ts$  is contained in  $\mathcal{S}_\infty$ . This property implies that  $\mathcal{S}_\infty$  is not equal to  $Ts$  for an element  $s$  of  $\mathcal{S}$ .

If  $\gamma$  is a cardinal number, a continuum of order  $\gamma$  is defined as a set of least cardinality which has the same cardinality as the class of its subsets which are continua of order less than  $\gamma$ . The empty set is a continuum of order equal to its cardinality. A set with one element is a continuum of order equal to its cardinality. No other finite set is a continuum of order  $\gamma$  for a cardinal number  $\gamma$ . A countably infinite set is a continuum of order equal to its cardinality.

A parametrization of a continuum  $\mathcal{S}$  of order  $\gamma$  is an injective transformation  $J$  of  $\mathcal{S}$  onto the class of its subsets which are continua of order less than  $\gamma$  such that no elements  $s_n$  of  $\mathcal{S}$  can be chosen for every nonnegative integer  $n$  so that  $s_n$  belongs to  $Js_{n-1}$  when  $n$  is positive. A continuum of order  $\gamma$  admits a parametrization since an injective transformation  $T$  exists of  $\mathcal{S}$  onto the class of its subsets which are continua of order less than  $\gamma$ . Since  $\mathcal{S}_\infty$  is then a continuum of order  $\gamma$ , it has the same cardinality as  $\mathcal{S}$ . The restriction of  $T$  to  $\mathcal{S}_\infty$  is a parametrization of  $\mathcal{S}_\infty$ . If  $W$  is an injective transformation of  $\mathcal{S}$  onto  $\mathcal{S}_\infty$ , then a parametrization  $J$  of  $\mathcal{S}$  is defined so that  $Ja$  is the set of elements  $b$  of  $\mathcal{S}$  such that  $Wb$  belongs to  $TWa$ .

A parametrization  $J$  of a continuum  $\mathcal{S}$  of order  $\gamma$  is essentially unique. If an injective transformation  $T$  maps  $\mathcal{S}$  onto the class of its subsets which are continua of order less than  $\gamma$ , then an injective transformation  $W$  of  $\mathcal{S}$  onto  $\mathcal{S}_\infty$  exists such that  $Ja$  is always the set of elements  $b$  such that  $Wb$  belongs to  $TWa$ . The construction of  $T$  is an application of the Kuratowski–Zorn lemma. Consider the class  $\mathcal{C}$  of injective transformations  $W$  with domain contained in  $\mathcal{S}$  and with range contained in  $\mathcal{S}_\infty$  such that every element of  $Ja$

belongs to the domain of  $W$  whenever  $a$  belongs to the domain of  $W$  and such that  $Ja$  is always the set of elements  $b$  of  $\mathcal{S}$  such that  $Wb$  belongs to  $JWa$ . The class  $\mathcal{C}$  is partially ordered by the inclusion ordering of the graph. A well-ordered subclass of  $\mathcal{C}$  has an upper bound in  $\mathcal{C}$  whose graph is a union of graphs. A maximal member of the class  $\mathcal{C}$  has  $\mathcal{S}$  as its domain.

A nonempty set of cardinal numbers contains a least element since a ladder of well-ordered sets can be constructed with these cardinalities.

A continuum of order  $\gamma$  exists when  $\gamma$  is the cardinality of an uncountable set. It is sufficient to construct a set which has the same cardinality as the class of its subsets which are continua of cardinality less than  $\gamma$ . If a cardinal number  $\alpha$  is greater than the cardinality of every continuum of order less than  $\gamma$ , it is sufficient to construct a set which has the same cardinality as the class of its subsets of cardinality less than  $\alpha$ . Such a set is constructed when  $\alpha$  is the least cardinality greater than the cardinality of an infinite set  $\mathcal{S}$ . The class  $\mathcal{C}$  of all subsets of  $\mathcal{S}$  is a set which has the same cardinality as the class of its subsets of cardinality less than  $\alpha$ . The cardinality of the class of all subsets of  $\mathcal{C}$  of cardinality less than  $\alpha$  is less than or equal to the cardinality of all transformations of  $\mathcal{S}$  into the set of functions defined on  $\mathcal{S}$  with values zero or one. The cardinality of the class of all subsets of  $\mathcal{C}$  with values zero or one is less than or equal to the cardinality of the set of all functions defined on the Cartesian product  $\mathcal{S} \times \mathcal{S}$  with values zero or one. Since  $\mathcal{S}$  is an infinite set, the cardinality of  $\mathcal{S} \times \mathcal{S}$  is equal to the cardinality of  $\mathcal{S}$ . The cardinality of the class of all subsets of  $\mathcal{C}$  of cardinality less than  $\alpha$  is less than or equal to the cardinality of  $\mathcal{C}$ .

The continuum hypothesis originates in the conjecture that the class of all subsets of a countably infinite set has the least cardinality of uncountable sets. The continuum hypothesis states that the cardinality of every continuum is equal to its order. An equivalent conjecture is that every infinite set is a continuum whose order is equal to its cardinality. The continuum hypothesis is compatible with the axioms of set theory including the axiom of choice. A continuum is said to be inaccessible if no greatest cardinality exists less than its order and if the continuum is not a union of smaller cardinality of continua of smaller order. The continuum of least order and the continuum of least infinite order is inaccessible. The hypothesis that no other continuum is inaccessible is compatible with the axioms of set theory.

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